Fractal Analysis Applied to the Precarious Nature of the Financial Markets

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Thesis Statement

The conventional investment theories assume that the behavior of a stock’s price is random, but the data shows otherwise. A stock’s price is more complex than a random walk, as extreme movements occur in the markets far more often than traditional theory predicts. The behavior of the markets is instead better modeled by using fractal analysis.

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1. Introduction

At the turn of the 20th century, a promising French mathematician by the name Louis Bachelier was studying the fluctuations of commodity prices for his doctoral thesis. From his analysis of commodity prices over time, he conjectured that past price movements have no effect on future price movements. He also noted that as one looks at prices over an extended time horizon, the prices tend to have larger fluctuations. Bachelier developed a formula to model the price of a commodity over time. This formula strongly resembled the formula for Brownian motion, which describes the random collisions of particles suspended in a liquid solution. Bachelier included these discoveries in his thesis “The Theory of Speculation.” In the beginning Bachelier received little attention for his work, and it was not until after his death when economists began using his work as a central concept in the development of investment theories during the mid 20th century (Smith, 2001).

The random walk hypothesis also became known as the drunkard’s walk. A drunkard does not walk in a straight or predictable path. A drunkard may stumble two steps to the left or maybe a few steps to the right, but whichever way the drunkard stumbles it does not have any relation to the direction he previously stumbled. Thus drunkard’s walk became synonymous with random walk, because the analogy of the drunkard mimics the idea of completely random movement.

To better understand a random walk, one can use the toss of a coin to model a basic random walk. Say you start off at the position (0,0), the first number represents the number of coin tosses completed and the second number represents the current position up or down from the original position of 0. The coin has two sides, heads and tails, each occurring with equal probability (.5). Imagine that if the coin lands on heads then you move up 1, and if the coin lands on tails you move down 1. So for example, after your second toss your positions can be: (2,2), (2,0), or (2,-2). The position (2,2) means that you got two heads, the position (2,-2) means that you got two tails, and the position of (2,0) means that you got one heads and one tails. After your third toss your positions can be: (3,3), (3,2), (3,1), (3,0),
(3,-1), (3,-2), or (3,-3). Since both tails and heads are of equal probability, after n number of tosses you are expected to be at a position relatively near (n,0), whereas positions of (n,n) and (n,-n) are highly unlikely for large values of n. So say, for example, that n=99; if you were at the position (99,99) that means out of 99 coin tosses all 99 were heads, which is extraordinarily unlikely. So according to the random walk hypothesis extreme movements are very unlikely, and most movements will be nearby the average (Spitzer, 1964).

In the mid 1900s academia showed an increasing interest in the financial markets in an attempt to explain their behavior. Finance was a growing industry at this time with the aid of the booming post-war American economy, and, in response, professors sought to uncover the formulas to an investor's success in the market. Professors began utilizing the computational power of computers, enabling them to perform statistical analysis on the stock market that was previously nearly impossible to calculate. University of Chicago professor Eugene Fama developed the efficient market hypothesis (EMH) 1960s, which became widely accepted and taught throughout business schools, receiving little criticism until the 1990s. The EMH claims that a stock's price will reflect all current public information on it, and if new information becomes available it is immediately taken into account by investors and the price properly increases or decreases based on that information. The simple rule to investing is to buy low and sell high, but if no stocks are undervalued then there is no opportunity for an investor to buy low. The EMH predicts that investor success is due to pure chance, not a result of a surefire, professional investor trading strategy. According to the EMH, whenever an investor buys a stock they are buying it at its fair value. Once a stock is purchased the direction of that stock's price is governed by pure chance. Under the EMH if one were to enter the stock market, they should only anticipate a return equal to that of the overall stock market. All of the professional investors who charge exorbitant fees for their expertise do not actually have any special knowledge, affording them an advantage over other investors (Malkiel, 2011).
There exists an enormous variety of information available for a stock; some of the information is less widely available than other information. Academics noted this issue of different levels of information availability for a stock. Academics wanted to know what level of information was available to the trading community in order to figure out what degree of a company’s information would affect its stock price. They divided up the EMH into three subdivisions: strong, semi-strong, and weak. Each subdivision represents a specific degree of information that is available to traders and is thus factored into the stock’s price (Malkiel, 2011).

The “strong” form of the EMH implies that all publically available information and all insider information are factored into a stock’s price. The “semi-strong” form of the EMH states that only publically available information is accounted for in a stock’s price. The “weak” form of the EMH says that past price information has no effect on future price information. What exactly is considered publically available information and insider information (Malkiel, 2011)?

Insider information is any information that comes from the “inside.” The “inside” meaning people who actually work for the company passing on information to others that is not publically available. Although insider trading is highly illegal, there exists the temptation for someone on the inside to pass on some information that has yet to have been released. Once that information is released, the people involved in the insider trading will have either made a hefty profit or avoided a detrimental loss. This is exactly the legal case with Martha Stewart in 2002, who avoided a loss of $43,673 through insider information from her broker at Merrill Lynch (Shiller, 2005).

A public company issues securities such as stock shares and bonds in order to raise capital to invest in the company. A public company is required by the government to release financial information so that investors can make educated decisions with their investment capital. Providing financial information on the company lets investors see how the company has progressed up to the current time,
and from there investors can make predictions on how the company will perform in the future (Malkiel, 2011).

Most all economists and institutional investors disregard the “strong” form of the EMH, since the “strong” form implies that no one, not even someone who trades on insider information, can profit off of a stock. Insider trading has the potential of being very profitable, but is highly illegal. Economists and institutional investors believe that even if an individual(s) did gain inside information and subsequently took a position on a stock, their personal amount of trading on the stock would pale in comparison to the overall amount of trading on the stock. An inside trader would not trade enough of the stock to cause a noticeable difference in its price. (Malkiel, 2011).

As a young economics student at UCLA during the 1950s, William Sharpe was very interested in both computers and the financial markets. While pursuing his PhD, Sharpe came to know Harry Markowitz, who had been doing work on portfolio optimization. The main goal of portfolio optimization was to own a number of stocks that generated high returns with little risk. Through guidance by Markowitz, Sharpe was able to determine that the “basic underlying factor” that drove the prices of stocks was the actual movement of the overall market. The majority of fund managers during this time would select stocks based on their individual investment criteria; simply if a company looked profitable one would invest. An underlying flaw was embedded in this approach; when just a few of one’s stocks drop sharply it drastically drives down the value of one’s overall portfolio. Sharpe believed that one should not concentrate their funds within a few individual stocks. He argued that if one bought shares in a large variety of stocks, then one’s portfolio would not suffer from the plummeting of a few stocks. From his observations, Sharpe developed a model for selecting stocks to incorporate into one’s portfolio. The model took into account the risk associated with a stock due only to movements in the overall market and also the rate of return on that stock. The risk associated with a stock due only to the movements of the entire market is known as systematic risk. Each stock has a certain amount of
systematic risk, which is measured as a figure known as beta. Using beta, investors can figure out a reward-to-risk ratio for any stock. Sharpe’s model examines the reward-to-risk ratio of all stocks, and then selects the combination of numerous stocks that lead to the highest expected return. With the advent of computers, it was now possible for someone to go through and find the risk and reward associated with each stock, and then find the most advantageous portfolio from that information. Sharpe’s work became known as the Capital Asset Pricing Model (CAPM), for which he would go on to win the Nobel Prize in Economics in 1990. Like the EMH, CAPM took into account the random behavior of the stock market. CAPM incorporates the underlying principles of the EMH, and, as such, can be viewed as an investment strategy theory stemming from assumptions made in the EMH. CAPM countered the random behavior of the stock market by maintaining a large number of stocks. A large variety of stocks “diversified” one’s portfolio to the extent that the swings of individual stocks had a miniscule effect on the overall portfolio (Smith, 2001).

The general method used to determine whether an investor was successful for the year or not is to compare the investor’s annual return to that of the S&P500. The S&P500 is a stock market index composed of 500 leading companies traded on the U.S. stock market. Consequently it serves as a good estimate of the overall performance of U.S. stocks. If an investor’s annual return is greater than that of the S&P500, it shows that the investor was generally able to outperform the market. This signal would seemingly indicate that the investor’s trading strategy was successful, but under the EMH that investor was simply just viewed as having been lucky.

Bill Miller served as the sole portfolio manager of Legg Mason Value Trust. For fifteen years straight, Miller’s fund outperformed the S&P500. Miller was lionized for his unparalleled foresight in pinning down profitable investments over such an extended period of time. Miller’s performance seems to defy the EMH, for surely Miller could not have done so well year after year by pure chance. Caltech physics Professor Leonard Mlodinow, who specializes in probability theory, states that the probability of
an investor performing as well as Miller is 1 in 32,768. Mlodinow says though, that if one looks at the sheer number of fund managers over a 40 year period, chances are actually 3 in 4 that at least one of the fund managers would outperform the S&P500 for fifteen straight years or more. According to Mlodinow’s analysis, it appears the EMH still holds. Miller did not have any ultimate trading wisdom that allowed him to succeed, as others had believed (Mlodinow, 2008).

Over the years there have been countless approaches to investing in the financial markets. Some methods have been met with success, while others have not, but no method of investing has been found to be foolproof. The most successful investment strategies have always failed in the end. There is an incredible amount of investors in the financial markets. Each of them has their own investment strategy, but they can generally be placed in one of three camps of investing: fundamental, technical, and quantitative.

Fundamental investors are interested in the core aspects of a company. Fundamentalists believe, as economist and popular investing author Burton Malkiel puts it, “the market is 90% logical and only 10% psychological.” A fundamentalist seeks to find companies that have solid balance sheets and talented management that have yet to be discovered and profited on by other investors. Any trends in past price movements a fundamentalist is not interested in, for they believe past price movements have nothing to do with the future direction of a company. A fundamentalist works to determine an underlying value for a company’s stock by examining its financial data, competitors, risk, dividends and market demand. They believe that the future prospects of a company are heavily influenced by both the financial health of the company and the global, economic position of the company’s industry. If the underlying value is less than the company’s current stock price, then the fundamentalist would invest.
Fundamental analysis is in agreement with the random walk hypothesis that past price information has no influence on a stock’s future price path. CAPM is an example of fundamental analysis. Longtime experienced investor and founder/chairman of Dreman Value Management, David N. Dreman says, “the
proper prediction of the stream of future earnings and dividends, including an assessment of the risk factor, is considered to be the core of this method (fundamental investing).” As you can see, the CAPM follows those same principles of fundamental investing, decision making based on the predicted risk and reward calculated by using a company’s financial information (Dreman, 1977).

Fundamental analysis would be useless if the “strong” or “semi-strong” forms of the efficient market hypothesis were true, because then all available information is already taken into account leaving no opportunity to buy low and sell high. Fundamentalists believe the “strong” and “semi-strong” forms to be untrue. They believe that every investor interprets the financial data on a company differently. What one investor may read as a buy signal another may read as a sell signal. There is much ambiguity as to how certain pieces of information may affect a stock’s price. What kind of effect does a rise in the price of silver have on General Motors? How does a change in management affect the share price of a company? What does it mean for a company when the earnings report of a competitor surpasses expectations? There are no agreed upon answers to questions such as these; each investor will respond differently to the information. Even though many of the early studies showed support for the EMH, David N. Drenan responded to these studies in his 1977 book *Psychology and the Stock Market*:

To use these findings as proof of market efficiency is the equivalent to saying that because millions of chess players understand how to properly move the pieces on a chess board, they all play the game with equal ability. Millions of chess players, then, play as well as Masters, Grandmasters, or even Bobby Fischer himself. Obviously, this statement is untrue. Unfortunately the enormity of the interpretive problem has not been examined by market researchers to date. (236)

The amount of information pumped out by a company for investors is enormous. Certain fundamentalists place emphasis on different aspects of the earnings report than other fundamentalists.
An incredible variety of diagnostics can be calculated using financial data on a company. To put it simply, investors do not all agree. Stock analysts will often have different earnings estimates and give different ratings to the same stock. So if investors are interpreting earnings reports differently, then there is a discrepancy over a stock’s intrinsic value. One man’s trash is another man’s treasure. The information companies release provide no universally accepted calculation for a stock’s true price (Malkiel, 2011).

Fundamentalists believe that if one is methodical and sapient in their interpretation of company information, then one can consistently outperform the market. There have been numerous fundamentalists who have achieved great success in the markets like Bill Miller did. Proponents of the EMH say otherwise; prices just move too quickly when information is released for investors to make any profit. Benjamin Graham, considered the father of fundamental analysis, shortly before he died, reluctantly said in an interview “I am no longer an advocate of elaborate techniques of security analysis in order to find superior value opportunities...I doubt whether such extensive efforts will generate sufficiently superior selections to justify their cost.” A good number of fundamentalists have abandoned fundamental analysis as the markets have changed over the years. They have adopted a simpler method of investing in index funds, which are funds that buy and hold a large number of stocks indefinitely in order to mimic the returns of an overall specific financial market. Those who believe in index funds praise their low investment costs and little risk. Although index funds may not be as alluring as performing elaborate analysis and picking individual stocks, they provide one of the most safe and consistently profitable investment strategies without doing much of any research (Malkiel, 2011).

Technical investors are not concerned with the fundamentals of a company. However odd that may seem, that an investor could have little concern about the company’s financial health, technicians have much logic behind their investment philosophy. Technicians are primarily concerned with the supply and demand for a stock. If a technician predicts investors are going to start heavily investing in a stock, they can purchase that stock before the other investors. When the other investors demand for
the stock kicks into effect, leading to an increase in price, the technician can sell the stock at the now inflated price. Technicians believe that by looking at past price trends and numerous diagnostics they can successfully predict the future price path of a stock. Technicians go directly against the random walk hypothesis and EMH; they believe that the past does influence the future. Technicians endlessly scrutinize charts to try and determine how the investor crowd will move on a stock ahead of time. They look not only at the chart of the stock’s price, but also the trading volume graph and countless charting tools (Dreman, 1977).

Technical investing did not come until several decades after fundamental investing. Fundamental investing had become integral on Wall Street, and so by the time technical investing came around there was a vehement rejection of technical analysis by Wall Street veterans. Academia also rejected technical investing. Research performed by Eugene Fama and others like him concluded that prices in the stock market were indeed random. Academics analyzed many of the technicians charting and trend finding tools, only to find little to no validity in them. They showed that even if a stock were to go up seven days in a row, it is just as likely to go up the next day as it is to go down. The fact that academic studies at that time showed past price trends had no effect on future price undermined technical investment strategy. Technicians paid no heed to the criticisms of the fundamentalists and academics. They continued to invest and some technicians succeeded in making noteworthy profits, just as some fundamentalists did, though it is unclear whether these successes were due to sheer luck (Dreman, 1977).

Quantitative investing often incorporates ideas of both technical and fundamental investing. Quantitative investing is one of the most recent approaches to investing that places a heavy influence on computerized trading through advanced mathematics. Wall Street became interested in this new approach to investing and began hiring physicists and statisticians to create complex trading algorithms. These algorithms would then be used in a computerized trading program that would analyze
unfathomable amounts of data, and then make investment decisions based on how the algorithm interprets the data. There is no conventional method of quantitative investing; each firm builds its own trading algorithm based on defined criteria for decision making. These criteria can incorporate both technical and fundamental trading signals. The new age of powerful computers and high speed online trading has enabled these complex algorithms to react to new market information almost instantly. Quantitative investing is still in its youth, and many further developments lie ahead in the near future. There have been some very successful investment funds that have employed quantitative investing, but it has yet to been proven if there is any surefire, successful strategy (Leinweber, 2009).

II. Volatility of financial markets

From its inception the stock market has gone through vicious swings causing booms and busts, leading to a more capricious market than theory would predict. The work done by Fama, Sharpe, and other leading economists during the mid 20th century would lead us to believe that the market would run a stable course over time. According to them the market would move in a general uptrend with small deviations from that uptrend. They believed in investor rationality that would lead to efficient markets where securities would be priced accurately and immediately in response to new pertinent information. Their theories predict that large swings should occur in the market only if detrimental information was released for a company. For example, if a company’s earnings report falls significantly below analysts’ expectations then the company’s stock price would take a hefty dive. Enron is a great example of this (Shiller, 1988).

Enron was an American energy company that began in the mid 1980s and grew throughout the years into a massive corporation with nearly 20,000 employees. In 1999 Enron’s stock price grew 56 percent and then 87 percent the following year. Fortune magazine rated Enron as the most innovative large company in America for six consecutive years. Enron was able to attract large amounts of investment through its alluring profits and unparalleled success. What investors did not know, was that
Enron was fabricating much of its supposed success through accounting manipulations. The company grew and grew as more money flowed in, but all that came to a screeching halt during the fall of 2001. The Securities and Exchange Commission (SEC) decided to investigate into Enron’s financial success. The SEC discovered the egregious accounting fraud by the top level management, exposing the company’s false sense of success. The share price for Enron collapsed. In less than two years time, the price of Enron’s stock plummeted from $90 a share to $0.12. Shareholders and employees retirement plans that included shares of Enron had lost basically all the money they had invested in the company. Not only that, but Enron’s auditor Arthur Andersen, once one of the largest accounting firms in the world, endured a complete collapse. In December of 2001, Enron declared bankruptcy (Benston, Policy Analysis No. 497).

The downfall of Enron and its share price, despite being an incredibly large decline, was justified by the news that came from the SEC’s report on the company. Events like this should be rare according to the random walk hypothesis and EMH, but are justified when detrimental information arises leading to such a collapse in share price. There is a multitude of instances when the share price of a company plummeted, yet the fundamental information provided no indication that such a sharp decline should have occurred. For example, the dot-com bubble of 2000-2001 was created by a boom in tech stocks during the 1990s. Prices for many tech stocks became highly inflated, because of their lauded profitability by investors that lead to an ever increasing investment of capital. After some of these companies, the most notable example being Pets.com, collapsed under poor management, the tech stocks all took a major dive. Companies such as Cisco Systems and Amazon.com had their share prices take a serious dive, Cisco declining by 86% and Amazon.com going from over $100 a share to less than $10 a share. Yet the financial health of Cisco and Amazon was strong as ever and as everyone knows these companies still run today with great success. If the fundamental information of Cisco and Amazon
showed little signs of financial instability, then why did their share prices face such enormous downturns?

On December 5, 1996, Alan Greenspan was giving a speech on the current state of central banking and the financial markets. Greenspan warned that the market was overvalued due to, what he termed, the “irrational exuberance” of investors. After Greenspan’s speech, the Asian financial markets immediately responded by falling several percentage points. The European markets soon followed with significant slumps of their own, and the American markets opened the following day also down several percentage points. The reaction to Greenspan’s speech by investors, gave validation to his words addressing investor irrationality. This comical, yet important, incident made “irrational exuberance” a common catch phrase among the investment community (Shiller, 2005).

The irrationality of investors has created bubbles in the markets as early as the 1600s. The famous example of early investor irrationality is the Dutch tulip mania in the 1630s. Tulips were highly sought-after by wealthy Europeans in the 17th century. Certain tulips that possessed very rare colors could fetch a substantial sum in the tulip markets. Traders became excited over the profitability of buying and selling tulips. Most of the time the traders would never actually physically possess the tulips, but instead the market’s secretary would mark down each transaction in a record book. In the beginning traders made excellent profits that were envied by others who were tempted to enter into the tulip market. Traders continued to buy and buy tulips with the expectation that they would always be able to sell them for more at a later time. Traders’ demand for tulips inflated the price beyond what collectors were actually willing to pay for them. Soon traders’ found that there were no buyers for their tulips. During January 1637, tulip prices increased twenty-fold, but then dropped by even more the following month. Wild fluctuations, such as the tulip mania, show how investor behavior can drive the price of a commodity far beyond its actual worth (Bookstaber, 2007).
Despite the apparent consequences of investor mania driven by fear and greed, investors have not learned from the past. Investors continue, to this day, to inflate the price of securities far beyond their worth, building castles in the air that come to collapse decimating their own portfolios. What actually causes bubbles in financial markets is a complex issue that goes beyond the scope of this paper. Numerous books have been written on the subject, investigating what factors precipitate the various bubbles that have occurred throughout history. Like the most recent recession, the story is never plain and simple as to how such extreme situations come into being. Many economists, most notably Yale economist Robert Shiller, have done extensive studies on the issue of market volatility. Some, such as Shiller, have come to argue that market volatility has only increased over the years. Shiller, in his book *Irrational Exuberance*, gives twelve different factors that are possible culprits to the increase in market volatility over the years. He explains that online trading has increased the speed of market transactions and allowed more people to have access to the markets. He also notes the overall growth in the financial industry. More and more of people’s “savings” are tied up in the markets through pension plans or employer stock offers. The reporting of the financial markets by the media networks has only increased over the years. Individuals are exposed to information about the markets everyday and often hear of the fantastic profits they can make in the market. Shiller admits though that it is difficult to determine what exactly has lead to this huge inflow of money into the markets. He comments on the recent inflows of investing by saying, “they are thus difficult, if not impossible, to capture in predictive scientific explanations.”

The random walk hypothesis and EMH agree that the movements in the stock market are independent of previous price information and normally distributed. Statisticians term this type of behavior of a collection of random variables as independent and identically distributed (i.i.d.). A variable being i.i.d. implies that the random variables are independent of one another, meaning the value of one random variable does not affect the value of another. It also implies that each random variable follows
the same probability distribution. A tremendous number of random variables in the sciences and social sciences are normally distributed, so for early financial theorists, such as Fama, to believe that market returns were normally distributed was not such a farfetched assumption. What exactly does it mean that a random variable is normally distributed? In a normal distribution, the values of a random variable are concentrated around the mean of those values, and the farther a value is from the mean the less likely that value is to appear. For example, the height of people follows a normal distribution. Say the mean height of all adult males is 5'10" and anyone can tell you that finding an adult male who is 6'1" is pretty common, just as finding an adult male 5'7" is also pretty common. On the other hand, it's extremely uncommon for one to come across an adult male less than 4ft or an adult male more than 7ft.

Figure 1, seen below, shows a normal distribution. In the figure, standard deviation represents a degree of magnitude away from the mean in relation to the overall variation in the data set. From the figure it can be seen that for a data set that is normally distributed, a majority of the observations (68.26%) lie within one standard deviation away from the mean. If you have a normally distributed data set and decide to pick one observation at random, from that data set, then it is highly unlikely that the observation you pick will lie three standard deviations away from the mean (.13%+.13%=.26%). As you can see, the further away an observation is away from the mean the rarer it is. The EMH claims that market returns follow a normal distribution. The average daily percentage change on the Dow Jones may be centered about a mean of say about 0.7%. According to the EMH daily percentage changes should in most cases lay fairly close to the mean of 0.7%. Sure you will have some down days and some up days, but if the returns are normally distributed, one should not have to worry about wild fluctuations in the market. If there are not any wild fluctuations in the market, then how does one account for the fluctuations caused by growing and bursting of market bubbles?
Nassim Nicholas Taleb is a mathematics professor at New York University. Prior to his professorship, Taleb had worked on Wall Street for approximately two decades before retiring in 2004. He learned quite a bit from his experiences working in the chaotic, fast-paced world of investing. These experiences provided inspiration for his book *The Black Swan*. In his book he explains that individuals ignore the possibility of highly improbable events occurring, and investigates why these highly improbable events occur far more often than we would expect. In his book, Taleb recounts his experience on Black Monday (October, 19, 1987), a day on which the Dow Jones fell an astonishing 22.61%, a drop far surpassing the previous record of a single day loss, 12.82%. He explains how shell-shocked his colleagues were that day, some even taking their own lives. He describes his feelings of being “vindicated intellectually, but I was afraid of being too right and seeing the system crumble under my feet.” Taleb had been an early embracer of quantitative trading. He studied quantitative models that took into account the highly improbable, which he claims many other investors failed to take into account. Since the market had never had even close to such a dramatic one day drop, most all investors assumed a drop like that on Black Monday would be impossible. How could stocks be devalued so quickly? Wouldn’t other investors have seen these greatly reduced prices as a perfect buying opportunity? A rational investor would have seen this as an excellent buying opportunity, but barely any
investors viewed the market this way. The results of that single day changed investors’ outlook of the financial markets as an abysmal situation. Seeing something they had never thought imaginable before triggered within them a fear for the state of the economy (Taleb, 2007).

Taleb claims that it is difficult for people to imagine an event occurring that goes beyond anything that has ever occurred before. He takes the example of WWI. In the beginning of 1914 no one would have ever imagined the events that would transpire over the next several years. He claims that after a highly improbably event occurs, we often take the information leading up to that event and paint a picture of why it is not so strange to have happened. Few people predicted the most recent recession, but now that we look back on it, it does not seem so improbable given the multitude of precipitating factors. The same goes for major historical events such as 9/11. On 9/11 around the world people were in complete shock at what had transpired that morning. Since 9/11 has occurred, we see that there is a real possibility of terrorists hijacking planes and crashing them into buildings. He likes to compare peoples’ view of the highly improbable to that of a turkey on a farm. A turkey is raised on a farm given shelter and food to eat consistently throughout its life. It views the farmer as a generous man that provides it with everything it needs throughout its life never imagining the farmer would harm it, but then one day the farmer chops off the head of the turkey (Taleb, 2007).

The ideas that Taleb laid out in his book has now become known as black swan theory. He defines a black swan to be an event with three distinct properties. “First, it is an outlier...nothing in the past can convincingly point to its possibility. Second, it carries an extreme impact. Third, in spite of its outlier status, human nature makes us concoct explanations for its occurrence after the fact, making it explainable and predictable.” In black swan theory, Taleb divides events into two separate categories, those events that belong in the world of Mediocristan and those that belong in the world Extremistan. The events of Mediocristan follow a normal distribution, are easy to predict, and are initiated by a multitude of explanatory factors. Examples of such events are a person’s height, IQ, car accidents, and
SAT scores. Events that belong in Extremistan follow a Pareto or Lévy distribution where the "unusual" events occur more often. These events are very hard to predict, and are determined by one or only a few extreme occurrences. Examples of matters that belong in Extremistan are income of a drug dealer, damage caused by a tsunami, book sales, and wars. Taleb claims that market returns belong under Extremistan. He believes the empirical evidence on market returns is significant enough to support his claim. As we will see, when one examines the distribution of market returns there is significant evidence that they follow a Pareto or Lévy distribution, a central attribute of matters belonging to Extremistan (Taleb, 2007).

As was explained previously a normal distribution has a majority of values centered around the mean of those values, and the further a value is from the mean the less likely such a value is to occur. It was noted that the early investment theories such as the random walk and the EMH assumed that the returns on stocks were normally distributed. Edgar E. Peters, a risk analyst and quantitative investor for several decades, states in his book Fractal Market Analysis that there was empirical evidence available in the 1960s and 70s discrediting the idea that market returns were normally distributed. Financial theorists, however, maintained the assumption that returns were normally distributed because of its ease of use in developing trading strategies and their skepticism in the contradictory evidence (Peters, 1994).

The empirical evidence rejecting the assumption of market returns following a normal distribution has grown over the years. Some this empirical evidence will be presented later in this paper. The information on past market returns has shown that instead of following a normal distribution market returns more closely follow a stable Pareto or stable Levy distribution. These distributions have infinite variance, leading to a distribution with fatter tails and a larger peak around the mean. Fatter tails and a larger peak around the mean translate, respectively, to a more frequent occurrence of extreme values and many values being very close to the mean. These distributions are considered fractal
distributions (Peters, 1994). More will be explained later in this paper as to what exactly a fractal is, why these are considered fractal distributions, and what exactly is meant by infinite variance.

Over the past few decades the growing use of computer technology has transformed the landscape of the financial markets. The internet has given rise to the at home trader and who can buy and sell stocks instantly without the need for a broker. In the past, executing a trade in the financial markets took dramatically longer. If one wished to buy or sell shares of a stock they would have to contact someone working on the trading floor, who would then go and match up a buyer or seller who is willing to trade for the shares. This long process was superseded by the internet that used computer servers to nearly instantaneously match buyers and sellers, this lead to the birth of high frequencies traders (HFT) who would use computer algorithms to perform their trades. The advantage of using a computer algorithm for trading was that a computer would receive new market information and be able to make trading decisions based on that information nearly instantaneously. By using a computer to making trading decisions one would not have to worry about human delay in reacting to market information; essentially one could beat other traders to the punch when it came to financial news. The downfall to this is that a computer program is now making major investment decisions without constant human oversight. If the trading algorithm contains flaws, then it may lead to fallacious trades (Zubulake and Lee, 2011).

In the first several months of 2010 the U.S. stock market was experiencing consistently strong gains with having only a few down days. Near the end of April the stock market began to have some significant down days. Institutional investors took this a sign of a possible pullback in the market occurring soon. The bearish outlook of institutional investors led them to set up safeguards in their portfolios. If there was a significant downturn in the market, their stocks would automatically sell before the price dropped too low leading them to lose their profits they had made over the past several months. On May 6th of 2010 the Dow Jones Industrial Average was down 300 points at 2:42pm (EST),
over the next five minutes the Dow Jones Industrial Average plunged 600 points. The market was at a staggering loss of over 900 points (about 9%) for the day. Trading algorithms of some of the quantitative investors recognized that stocks had become undervalued, and the automated trading programs quickly reacted by purchasing the stocks at a bargain price. The stock market was able to rally back, recovering most of its 600 point loss. The events that occurred within that brief span of time in the afternoon of May 6th remain as one of the most shocking examples of the effects of automated trading. The automated sell orders caused the market to spiral down and shortly after automated trading algorithms saw this as a buy opportunity, leading to a market recovery (Zubulake and Lee, 2011). The automated trading that now proliferates on Wall Street created one of the most volatile days in trading history.

Some academics theorize that technological changes have caused the market to become more volatile over the years. This increase in volatility leads to a proliferation in the mispricing of equities. The mispricing of equities is the central issue leading to the discrediting of the random walk hypothesis and EMH. If equities are mispriced then market bubbles are created, leading to a more frequent occurrence of large swings in market returns than predicted by the normal distribution of market returns assumption (Shiller, 2003).

World renowned Yale economist Robert Shiller has published a number of books on financial market volatility, and has become one of the foremost scholars in this field. In his book *The New Financial Order: Risk in the 21st Century*, he examines the influence technology has had on market volatility and what impact it may have in the future. He outlines several aspects of technology that will pose a serious threat to the stability of our financial markets. One of those aspects is what Shiller refers to as "the consummation of cybernetics." Cybernetics is the study of human capabilities and the development of technology with the similar capabilities, for the purpose of replacing the labor of humans. Since the days of the industrial revolution, unskilled labor has experienced an ever growing implementation of new technologies to replace their jobs. The capabilities of technology have been
expanding at an exponential rate, and since the advent of the new millennium unskilled laborers have
not been the only ones at risk of being replaced by a machine. We have seen the skills of individuals
such as translators, bookkeepers, and even lawyers being compromised by new technology. Although
translators, bookkeepers, and lawyers have not been completely replaced by machines, machines can
perform certain aspects of their work, making a need for less people in such positions. Computers can
process information at incredible rates that no human can match. Shiller claims that “workers whose
particular talents are in learning complex but routinizable tasks are clearly at risk.” He says that it is
important to note that although technology does replace jobs it creates new jobs. The issue of economic
volatility stems from the fact that no one knows for sure what jobs will be replaced. Industries can be
dramatically reformed by technology, leaving the laborers of that industry jobless. Some workers will
adapt to the new technologies and find opportunities for new careers, but not all are able to change
careers so easily. More industries will rise and fall over time with the growth of technological
advancement. The unpredictability of the future health of companies that stems from these
advancements will generate more frequent upheaval within the financial markets (Shiller, 2003).

Another aspect of technology Shiller argues will lead to more fragile financial markets is the
“breakdown of geographic barriers.” Advances in technology have enabled companies to become less
geographically bound. The ease of transportation has led to companies being spread all throughout the
globe. To ship its products and raw materials from one location to the next, or even move entire
production lines, can now be done swiftly and efficiently. This allows companies to take advantage of
the resources of certain geographic locations for a relatively low cost. This is most prominently noted in
the economic boom of China over the past couple decades. Companies moved much of their
manufacturing to China, where cheap labor cost made relocating a financially intelligent decision.
Advancements in communication technology allowed companies to now hold business meetings,
between offices spread all throughout the globe. Two offices on opposite sides of the globe can work in
together as if they were working side by side. Shiller theorizes the rapidity of economic booms and busts geographically will only increase. He believes companies’ decreasing limitation of physical location will lead to increased crises in labor and real estate markets (Shiller, 2003).

One last aspect of technological growth that Shiller argues has led to, and will continue to lead to greater market instability, is the advancements made in military and terrorist technology. In the words of Shiller “advances in control and information technology often have the consequence that the control of large amounts of power is concentrated in a very small, and often fragile, center, making possible sudden shifts in power and misuses of this power.” Shiller cites 9/11 as an example of how technology led to the control of large amounts of power concentrated in a very small, fragile center. Terrorists were able to gain access to advanced aviation technology, that being massive Boeing 767 aircrafts. Planes of the past contained much smaller fuel tanks, but the 767s used in the 9/11 attacks had such large fuel tanks that they packed serious explosive power. To be able to exact that concentration of power and inflict serious detriment to a nation poses an unstable market environment. After the 9/11 attacks the S&P 500 fell over 10% in a single week. The growth of military and terrorist technology goes beyond physical destruction. The proliferation of cyber warfare over the past decade has led to much destruction that eludes the eyes of the public. Whether the individuals engaged in cyber warfare are rogue terrorists or military personnel, their attacks are shown to be on the rise and economically malicious in their nature. Cyber attacks are often targeted at important infrastructural industries such as banking, transportation, military, medical, education, energy, and government. These attacks when successful can be crippling to these sorts of industries, often leading to a sizable economic loss. The future potency of these attacks is unknown, but one can only imagine the economic consequences of a cyber attack that shuts down our entire power grid (Shiller, 2003).

As you can see, multiple facets of the technological age have led to greater volatility in the financial markets. The unpredictability of these expeditious market moves leads to the random walk
theory and EMH being less applicable to the current, and future, market environment. The random walk theory and EMH imply the normality of market returns, and therefore do not allow for this frequency of these expeditious and potent market swings. Since 1987, the twenty largest intraday point swings have all occurred since the year 2000.

III. Irrationality of investors

Traditional finance theories, such as the EMH, assume the participants in the market are rational individuals who base their decisions on achieving the maximum utility possible. These rational individuals are able to process all known information, and then use this information in order to come up with the most financially advantageous decision. This creates a market where assets are priced correctly according to all available information. This view of markets can be appealing in ways for its simplicity and neatness in creating financial models/theories. The underlying issue with this view is its unrealistic assumptions of human behavior. In the mid 1970s academic work began to surface showing that humans are not as rational when it comes to economic decisions as traditional finance predicted. This work that began in the mid 1970s has led to the development of the fields known as behavioral economics and behavioral finance. These fields take into account the emotional, social, and neurochemical factors that contribute to an individual’s economic decision (Baker and Nofsinger, 2010).

Although behavioral finance concludes that investors are susceptible to their own emotions, professional investors are still very rational and methodical in their analysis of a stock. When performing an analysis on a stock, a professional investor will examine an enormous quantity of available information that influences their investment decision. Whenever professional investors take a position on a stock, they have a multitude of facts and figures to defend their position. A professional investor paints a picture of the financial position of a company, and can often become overconfident in the accuracy of the picture they paint. Since the professional investor had performed such thorough analysis on a company, they have convinced themselves of the certainty of their position. If a professional
investor takes a position, whether negative (short) or positive (long), on a stock, and information later comes out that undermines their position, they are often reluctant to go back on their position. So for example, a professional investor decides that Apple has excellent prospects for growth in the future and therefore he or she invests in the stock. If a month later news comes out that iPhone sales are falling behind, then it is unlikely that investor will give up on their position right away. In the words of finance and accounting professor Richard J. Taffler and psychoanalysis professor David A. Tucket, “market underreaction to bad news is one of the best established and seemingly most robust of all stock market anomalies.” The stress of receiving bad news can lead to an investor using mental defenses that initially put them in denial, leading to a delayed response in pricing to the bad news. When it comes to responding to good news, studies have shown that investors do not have much delay in reacting to this news, leading the pricing of the equity to be fairly accurate. The positive emotions of investors lead them to enthusiastically accept such good news. In Tucket and Taffler’s words:

Bad news is also associated with anxiety and stress, which people seek to avoid. Good news provokes the opposite emotions of excitement or pleasure, which people constantly seek. This can possibly explain why markets tend to respond immediately and appropriately to good news (Emotional Finance: The Role of the Unconscious in Financial Decisions).

The overconfidence of investors leads them to behave irrationally at times, and they can be willing to take on more risk whilst paying little heed to the possible downside (Baker and Nofsinger, 2010).

Nothing is ever guaranteed when investing in the stock market. Investing is an extremely competitive industry, where one is always seeking to outperform all other investors. Investors who reign in returns that far outperform those of the average investor are met with praise and a greater inflow of capital to invest with. This temptation of an investor to outperform his peers often involves taking on greater risk. These aggressive investors have been found to have abnormally large trading volumes.
Researchers have found that when modeling overconfidence in the market, a high trading volume persists when more aggressive traders exist in the market. The high trading volume does not match the behavior of what would seem to be a rational investor. The constant change of an investor's position on a stock despite the lack of new information brings into question the rational foundations behind the investor's trades. Behavioral finance professor Werner DeBondt and behavioral economics professor Richard Thaler, refer to this issue of abnormally high trading volume as “perhaps the single most embarrassing fact to the standard finance paradigm.” Beyond the aspect of high trading volume, overconfident investors also show less reluctance in purchasing a riskier stock, since riskier stocks generally have higher expected return. If an investor becomes confident in the future of a risky stock, they will often invest in it, concentrating on the possible upside and ignoring the possible downside which they feel is unlikely to occur. This mentality is similar to that which psychologists see in gamblers. A gambler can become so set on the rewards of winning that they pay little, to no, heed to the penalty of losing (Baker and Nofsinger, 2010).

One of the most famous examples of investor ignorance to risk is the story of Long Term Capital Management (LTCM). LTCM was a hedge fund formed in 1993; it hired some of the biggest names on Wall Street at the time. At this time the behavioral financiers and economists had presented many strong arguments against the efficiency of the financial markets. Believers of the EMH still had strong faith in the EMH, and thought that irrational investors only created small and ephemeral price discrepancies in the market. Some of the EMH supporters believed one could capitalize on these small price discrepancies created by irrational investor behavior; this is what is referred to as arbitrage. Theoretically as long as an investor knows the proper price of equity, then they can easily profit off of any mispricing that occurs. LTCM’s central investing strategy was arbitrage within the bond markets. Their team of Nobel Prize winning economists and other Wall Street all-stars created mathematical trading models that determined bond prices throughout the world and exploited any price discrepancies
according to their formulas. For the first few years LTCM had made exceptional returns on the market, 40% in each of its first two years. The price discrepancies calculated by their bond pricing formulas were minute, so in order to make any profits they needed to invest a large amount of money on a single bond (Smith, 2001). For example, if a bond is only mispriced by 5 cents they’d need to invest $100 million in order to just make a $5 million. This often led LTCM to leverage their bets, in other words borrow money for investing, so that they could get the large returns they needed. In 1998 LTCM was a $3 billion hedge fund with more than $100 billion at risk in the bond markets. One of the bond markets it was most heavily invested in was the Russian bond market. Russia was believed to be safe investment, because it was believed to be “too big to fail” and the United States would bail the nation out due to its nuclear armaments. This however proved to be wrong when Russia defaulted on its debt in August of 1998, and subsequently its bond prices collapsed. LTCM completely collapsed and the Federal Reserve Bank of New York organized a bailout for the fund to pay its creditors. The story of LTCM, serves as a powerful example of how ignorance of risk can even occur amongst the brightest investors on Wall Street. Their confidence in their models and the efficiency of markets led to their demise. Some of the former principals of LTCM claim the collapse was unforeseeable. The Russian financial crises was a black swan, and the principals of LTCM simply ignored the possibility of the highly improbably; it was a freak incident in their eyes. In the words of Richard Bookstaber, a PhD economist and veteran risk manager for hedge funds:

We take comfort in ascribing the potential for fantastic losses to the forces of nature and unavoidable economic uncertainty. But that is not the case. More often than not, crises aren’t the result of sudden economic downturns or natural disasters. Virtually all mishaps over the past decades had their roots in the complex structure of the financial markets themselves.
One of the fundamental observations on human society is the influence of individuals upon one another. Constant communication between humans leads to a likeness in thinking and has been the bases for the formation of cultural, ethnic, religious and geographic identities. Investors as a group of people are persistently in communication with one another, and whether that communication is direct or indirect it fosters a likeness in thinking. The reason for any likeness in thinking amongst investors is that investors are all reacting to the same information, so long as that information is publically available.

It has been noted in social psychology studies that individuals fear rejection from their peers, and often can be timid when taking an opposing position from most others. This is the basis of what is called groupthink, a psychological phenomenon where individuals working in a group seek harmony within a common consensus, often blocking out alternative interpretations (Montier, 2007). In 1952 social psychologist Solomon Asch reported some very interesting findings on the power of social pressures to be in agreement with others. In Asch’s experiment each subject under study was placed into a group of seven to nine people. Asch would then ask the group to answer twelve questions about the lengths of line segments that were shown to them. The correct answers to the questions were obvious, but unbeknownst to the subject everyone else in the group was told by Asch to give the same incorrect answer to seven of the twelve questions. Under the pressure of all the others giving the wrong answers, the subject also gave the same wrong answer a third of the time despite the correct answer being trivial. The subjects would show signs of distress and angst when it came to answering the questions. Asch’s study showed the fear of being rejected by one’s peers. Further studies have shown that this same phenomenon exists even when one is not face to face with one’s peers. Even if the subjects do not have to fear social rejection, they often begin to doubt their own logical conclusion when they find it disagrees with everyone else’s (Shiller, 2005).

As has been stated earlier in this paper, all investors interpret information differently. Often there exists some disagreement, even amongst professionals, as to whether a certain stock is a good
investment or not. However, it is seen often that a certain stock or even an overall market can gain a general consensus amongst investors. They can become very enthusiastic about a particular investment, and that investment soon becomes a hot topic on Wall Street. This enthusiasm can be self-fulfilling. The more an investment is lauded for its great profitability, the more money will be put into that investment continually inflating its price. There may be some investors who disagree, but they are a small enough minority that they do little to influence the price. This is precisely the prime driving factor behind stock bubbles. It is what happened in the earlier examples of the tech stocks of the late 1990s and the tulip market of The Netherlands in 17th century. The zeitgeist amongst investors on a particular investment is usually driven by rational interpretation of the information available, but at times the zeitgeist can get out of hand leading to pricing that is not in agreement with the public information (Shiller, 2005).

After market bubbles grow to a critical point of saturation in the market, more often than not, a pivotal piece of news comes out that sends investors into panic. The news can come in a variety of forms such as disappointing earnings reports or defaulting on debt. The market related to the bubble will experience such a severe decline over a short period, sometimes within only a day, that the entire atmosphere of investors is in a panic. Those who held positions in the equities affected by the bubble often completely sell off their positions to avoid incurring any further loses. Those who did not hold positions in the equities affected are ambivalent about buying them up due to uncertainty in the future of the market. The lack of buyers to match the sellers sends prices down to historically low levels. At an eventual point prices fall so low that investors now see the prices as a gross undervaluation. Investors buy up the affected equities; prices quickly recover to levels similar to the beginning of the bubble’s mania. Over time the returns on the equities begin to experience more stable fluctuations (Montier, 2007).

In the traditional theories of economics and finance, it was assumed that humans were rational players in the economy who based their decisions on maximizing their self-interest. Any individuals who
behaved irrationally were such a small minority of the population that they had no effect on market prices. A compelling amount of research has been published to show otherwise. It has been shown that investors are susceptible to their emotions and neurological framework, leading them to often make similar cognitive errors. The phenomena of investor overconfidence, ignorance of risk and panic have all induced consistent mispricing of financial assets. Some of the most comprehensive research in behavioral finance has been done on these phenomena. Many other phenomena have been noted and discussed by those in the academic community, but much of that goes beyond the scope of this paper. What is important to be noted is that there is substantial evidence on investor behavior contradicting the assumptions of the EMH. The EMH believes that even if investors do behave irrationally, any of their effects of mispricing will be counteracted by other investors who view the mispricing as a profit opportunity. Incidents such as the collapse of LTCM and the persistent market bubbles throughout history have shown that irrational investor behavior has a powerful affect on mispricing. Investors' susceptibility to emotions causes more extreme fluctuations in the stock market. Earlier in the paper, it was mentioned that empirical evidence showed that market returns did not follow a normal distribution, but instead followed a distribution with "fatter tails," the fatter tails being the incidence of extreme market returns. It was also noted earlier that the distributions of market returns were fractal in their nature. The issues of market volatility are embedded in the framework of the market and the psychology of investors themselves. The causes of market volatility give doubt to the random walk hypothesis and EMH, and invoke the investigation of other possible explanations for the observed market behavior.

IV. Applications of fractals to financial modeling

From a young age we learn the very basics of geometry. We are exposed to the concepts of lines and simple shapes such as squares, triangles, and circles. During one's high school education they encounter more advanced concepts of geometry and more complex shapes, yet these further studies do
not nearly encompass all that is geometry. The geometry one learns in k-12 education is what is referred to as Euclidean geometry, named after the ancient Greek mathematician Euclid who deduced the principles of Euclidean geometry through a set of basic axioms. Euclidean geometry is very neat and elegant in ways, but it fails to accurately describe many of the shapes encountered in nature. The pulchritude of the Chilean coastline, the Milky Way, and leaves of a Christmas tree could not be captured within the geometric tools Euclid presented us with. The geometry of these natural structures was the inspiration for Benoit Mandelbrot’s conception of fractal geometry, which sought to provide the tools necessary to describe this geometry of nature (Mandelbrot, 1983).

Mandelbrot coined the word fractal from the Latin word *fractus*, meaning broken. He chose this word, because he noticed that these fractal shapes were, in a sense, infinitely jagged. When examining the Chilean coastline, one would find a persistent roughness in its outline. If one were to zoom in on their view of the coastal outline, they would find a continual roughness to its shape no matter how much they magnified their view. This roughness is the reason why the smooth objects of Euclidean geometry fail to explain the detail of these natural shapes. In the words of Benoit Mandelbrot, “clouds are not spheres, and bark is not smooth, nor does lightning travel in a straight line.” By consequence of the infinite jaggedness of these natural shapes, there exists no tangent line to any point of the shape, making it distinctly non-differentiable. Another important aspect of fractal geometry is that of “self-similarity.” If one were to split a fractal shape up into pieces, they would see that each piece follows the same pattern. No matter how greatly one magnifies or widens their view of a fractal, they will always find the shape to be of the same pattern (Mandelbrot, 1983).
Figure 2: The follow two diagrams above were generated using a set of conditions. Believe it or not, the top diagram began as only a triangle with horizontal lines extending outward from the base. At the next step, four triangles half the size of the original triangle are added to the diagram. Two of these triangles bases lie along the sides of the original triangle, and the other two lie on the two horizontal lines extending outward. The triangles have a 50% likelihood of facing in, creating an indentation, or a 50% likelihood of facing out, creating an extension. This process is now repeated on the four triangles just produced; this step adding eight more triangles. After the process continues out to a degree, we are left with the complex, jagged diagram above. The self similarity of the triangles and persistence of pattern under various magnifications shows how it is a fractal, a fractal that reminiscent a coastline. The bottom diagram used a much more complex fractal process, but as you can see it looks like any coastline you would find in an atlas. This is a consummate example of how nature contains fractals. No person looking at the bottom diagram would have any reason to believe that is not a real coastline.


The development of fractal geometry easily has been one of the greatest accomplishments in mathematics during the 20th century. Fractals have been applied to all areas of science. From the study of acid rain dispersion, to telecommunication antennas, to the dynamics of ecosystems, to heartbeat patterns, to modeling internet traffic, fractals have found an incredible array of applications. Many observable phenomena of nature contain the properties of infinite jaggedness and self-similarity. One of
the applications of fractals that even Mandelbrot himself has strongly advocated for is their usefulness in modeling the stock market. Mandelbrot had noticed that the traditional theories of investing allowed for some roughness in the price charts. Random walk hypothesis and EMH accounted for the roughness in price charts, and referred to it as the volatility of the price. According to the random walk hypothesis and EMH, the volatility in price is assumed to follow a normal distribution. We’ve come to understand now through the empirical evidence on price volatility that it does not follow a normal distribution. As has been discussed the fluctuations in the financial markets are much more wild than that. In a sense, volatility itself is volatile. Mandelbrot was aware of the empirical evidence on market volatility, and saw the opportunity for the application of his concept of fractals. Fractals could model this extraordinary roughness in the financial markets (Mandelbrot and Hudson, 2004).

Imagine you are in put in charge of measuring the length of the coastline of Maine. At first to save time you use a massive mile long ruler; ignore any of the physical limitations of carrying such gigantic a ruler. You measure the distances from one promontory of the coast to the next, and total up all your distances. You realize that your measurement of the coastal length is off from the actual length. You failed to capture all the jaggedness of the coast. You go back and use a smaller ruler and notice that your total calculation has now become more accurate, but there still remains some discrepancy in your measurement and the actual value. So you continue to go back and use smaller and smaller rulers. You will begin to notice an interesting phenomenon: the total coastal length you calculate is growing at a faster rate than the rulers are shrinking. This phenomenon is what is referred to as fractal dimension. The fractal dimension of a shape is dependent on the amount of jaggedness pertaining to that shape. So the fractal dimension of Maine’s coastline depends on how much roughness it contains in its shape. The relatively smooth coastline of South Africa has a fractal dimension of 1.02, while the jagged British coastline has a fractal dimension of 1.25. Fractal dimensions of whole numbers describe objects belonging to Euclidean geometry: 0 describes a set of points, 1 describes a line, 2 describes a plane, and
3 describes a volume. The coastlines of South Africa and Britain have a fractal dimension between 1 and 2, so, in a sense, their shapes fall somewhere between a line and a plane. This concept of fractal dimension gives us a new tool of measurement, what Mandelbrot refers to as a “yardstick for roughness” (Mandelbrot and Hudson, 2004).

In the section on Black Swan Theory, it was explained that Taleb divided events into those that belonged to Mediocristan and those that belong to Extremistan. The events of Extremistan are often social phenomena that are difficult to model and are often dictated by one or a few extreme occurrences. It was also said that these events most closely follow Pareto or Levy distributions.

Vilifredo Pareto was a 19th century Italian economist and engineer who, amongst many things, was interested in the distribution of wealth and power. He gathered historical data on wealth and income throughout Europe and used it to create a diagram that showed the number of people within each income level. In his diagram he noticed there were the fewest people at the highest levels of income, and as you moved towards the mean level of income the number of people increased at an exponential rate. The mean level of income though was much further on the lower end of the income spectrum. A majority of the people were concentrated in the lower end of the range of wealth. Figure 3 is the original drawing of Pareto’s 1909 diagram on wealth distribution. You can see from the figure how the majority of individuals were on the bottom of the wealth distribution, while the number of wealthy individuals diminishes exponentially. The statistical nature this distribution was the basis for what became known as the Pareto distribution (Mandelbrot and Hudson, 2004).
Figure 3: The y-axis is the level of income. The x-axis is the number of individuals approximated by his distribution.


French mathematician Paul Lévy, who happened to be one of Mandelbrot's professors, was responsible for the development of the Lévy stable distribution family. He classified a set of distributions based off the normal distribution. A normal distribution implies a finite variance, but Lévy was interested in how the distribution would change if the variance was not finite. The class of distributions that resulted was the stable distribution family; also known as the Lévy alpha-stable distribution family.

The family of distributions varies based on the parameter known as alpha. Alpha in a sense controls the "fatness of the tails" of the distribution. Pareto's distribution of wealth happens to fall under this family of stable distributions with an alpha of approximately 1.5. The empirical evidence gathered by Mandelbrot and many other researches over time has shown that changes in market prices follow a distribution belonging to this stable distribution family. Different alpha values have been attached to the distribution of market prices depending on the data set and the specific market related to that data set.

For example, Mandelbrot found cotton prices to follow a stable distribution with an alpha of 1.7. The research has shown clearly that market prices, regardless of the specific market under consideration, follow a stable distribution with fat tails, not a normal distribution. Figure 4 shows the normal
distribution (in gray) juxtaposed with the distribution of market returns on the Dow Jones Industrial Average (in black). The x-axis shows the number of standard deviations. As you notice the normal distribution tapers off as the number of standard deviations increases. The distribution of market returns on the Dow Jones Industrial Average continues to have returns many standard deviations away from the mean. The extreme changes in market prices caused by underlying framework of the market and the emotional biases of investors lead to these fat tailed distributions (Mandelbrot and Hudson, 2004).

![Graph comparing the Dow and the Standard Model](image)

**Figure 4:** The y-axis is the log(number of changes). The x-axis is the number of standard deviations away from the mean.


In figure 4, the returns on the Dow Jones Industrial Average over various investment horizons are shown with the normal frequency subtracted. From looking at these figures, it can be seen that when compared to the normal distribution, the actual market returns are grouped more towards the mean and the extreme values near the ends. It was mentioned that fractals have a self-similar nature to them. So, much like examining a coastline at various magnifications, these figures show market returns at different magnifications. Whether looking at 1 day returns or 90 day returns, you still get the same pattern of a large peak at the mean and fat tails at ends. Figure 5 graphs the frequency difference...
between the Dow Jones Industrial Average market returns across different investment horizons and the normal distribution. The figures use historical data on market returns from January 2, 1888, through December 31, 1991. Note the similar characteristics, such as the fat tails and large peak, across the different time periods. The fact that the distributions follow the same pattern at different investment horizons shows how distributions of market returns are fractal in nature. We are limited by the scope of historical market data to show the distributions of market returns much beyond 90 days. As more market data is recorded over time we will have the ability to analyze the distributions of market returns over greater time horizons (Peters, 1994).

**Figure 5.1:** Dow Jones Industrials, 1-day returns – normal frequency

**Figure 5.2:** Dow Jones Industrials, 5-day returns – normal frequency
Figure 5.3: Dow Jones Industrials, 10-day returns – normal frequency

Figure 5.4: Dow Jones Industrials, 20-day returns – normal frequency

Figure 5.5: Dow Jones Industrials, 30-day returns – normal frequency
Figure 5.6: Dow Jones Industrials, 90-day returns – normal frequency


It is believed the self-similarity of market returns will continue on for greater time horizons, but at some point the distribution will change. The longer the time horizon the more likely any mispricing in the markets will be corrected for. What is known as the “Sharpe ratio,” is a measurement of market return in relation to risk. Quantitative investor and author Edgar E. Peters looked at the Sharpe ratio of the Dow Jones Industrial Average, using over 100 years of market data. According to his analysis the market reaches a point of mean-reversion once the investment horizon is stretched to approximately 4 years. What is meant by this point of mean-reversion is that, in the most basic sense, the market corrects for any mispricing over a period of 4 years or greater. According to his analysis the market returns no longer will follow the same fractal distribution over any period of at least 4 years. This determination of 4 years is only a rough estimation. It is a number that can be worked with and investigated over time. Much more time will have to pass, quite possibly even several centuries, before enough market data becomes available to see when exactly market returns become mean-reverting (Peters, 1994).

In the early 20th century, a hydrologist by the name of H.E. Hurst was working on the Nile River Dam Project. In the design of a dam, a hydrologist must consider the amount of water inflow and in response adjust the outflow of water in order to maintain proper capacity of the dam’s water reservoirs.
The inflow of water was originally assumed to be a random process. Hurst did extensive research on the historical data of the Nile's overflows and believed that the inflow of water was indeed not random. He noticed that there were cycles where the Nile would have consecutive periods of lower-than-average overflow and later consecutive periods of high-than-average overflow. He realized there did appear to be a correlation in the overflows of the Nile, but standard statistical analysis failed to pick up on any significant correlation. Hurst looked at the range of discharges from the Nile; discharge is the volume rate of water flow. He then rescaled the range of discharges depending on the time horizon examined. He developed the following equation: \((R/S)_n = c*n^H\). The letter \(c\) is a constant based on the observed system, \((R/S)_n\) represents the rescaled range of Nile discharges based on the time increment \(n\). The rescaling of the range \((R)\) is performed by dividing the range by the standard deviation \((S)\) of the original observations. The rescaling allows the formula to be consistent across different time series data sets. \(H\) is what is now referred to as the Hurst exponent (Peters, 1994).

If \(H\) has a value equal to 0.5, then the system is a random process. For 0≤\(H\)<0.5, the process is considered antipersistent, meaning that if the system experiences a period of high values it is likely to be followed by a period of low values. For 0.5<\(H\)≤1.0, the process is considered persistent, so a period of high values is likely to be followed by a period of high values again. The closer the Hurst exponent is to 0.5, the lesser that prevalence of trends will be observed in a system. Hurst found the Nile to have a Hurst exponent of 0.9, making it a very persistent system strongly influenced by trends. Hurst examined other rivers and also found the Hurst exponent to indicate persistence. He was surprised by this phenomenon, and so he decided to test his theory through simulation. He used a deck of cards to mimic a random process and found a Hurst exponent of 0.5. Then he started using two decks of cards and would switch cards from one deck to the other based on what card came up when he cut the first deck. He noticed that when he preformed this simulation and calculated the Hurst exponent, it had a value of 0.72. The bias of switching cards from one deck to the other depending on the card that came up when
he cut the deck created a memory to the process. This memory was captured in the Hurst exponent, and so to his surprise he was correct in his formula (Peters, 1996).

Mandelbrot became interested in the work of Hurst. He noticed that one could use Hurst’s development of R/S analysis to determine a correlation between events across different periods of time. If a correlation existed, making $0.5 < H \leq 1.0$, then the system exhibited the fractal nature of what is referred to as fractional Brownian motion. Mandelbrot was able to show that the inverse of $H$ was a system’s fractal dimension. So a completely random process, meaning $H = 0.5$, has a fractal dimension of $(0.5)^{-1} = (1/0.5) = 2$, which is representative of a random walk in fractal dimension. The results match as expected (Peters, 1996).

If the stock market does follow a random walk, then we’d expect to find a Hurst exponent of, or close to, 0.5. A multitude of R/S analyses have been performed on the financial markets. They have found that market returns of all varieties have Hurst exponents $0.5 < H \leq 1.0$, meaning that previous price changes influence price changes. This directly contradicts the assumptions of the random walk hypothesis and EMH, which claim that previous price changes have no effect on future price changes. According to the numerous R/S analyses performed, the market does exhibit trends (Peters, 1996).

Some of the most extensive R/S analyses published are those by Edgar E. Peters. The following data is some of Peters’ analyses R/S analyses from his book *Chaos and Order in the Capital Markets*. Figure 6 shows the monthly returns of the S&P500 from January 1950 to July 1988. The graph is a log/log plot with log(R/S) as the y-axis and log(number of months) as the x-axis. The log function is commonly used to compare prices changes across different periods of time. In essence, the log function enables market returns in periods such as the 1930s to be comparable to market returns in the 2000s. The figure also includes two lines representing a Hurst exponent of 0.78 and 0.5. As you can see the returns of the S&P500 follow the Hurst exponent line of 0.78, meaning a positive trend occurs in the monthly returns. According the Mandelbrot’s work, since we now know the Hurst exponent of the
monthly returns we can calculate the fractal dimension. Fractal dimension is equal to the inverse of the Hurst exponent, so in this case of monthly returns we have a fractal dimension of \((0.78)^{-1} = (1/0.78) \approx 1.28\). At a certain point in figure 6, the monthly returns drift from the Hurst exponent line of 0.78 and begin to follow the line of \(H=0.5\). This implies that after a certain length of time the returns of the S&P500 follow a random walk. This behavior was mentioned earlier when talking about Peters’ analysis of the Sharpe ratio. It is assumed that over a long enough period of time any mispricing in the market is accounted for. Mispricing being accounted for can occur in a variety of forms, a simple example would be the bursting of a market bubble. Peters’ looked at the Sharpe ratio and concluded that the Dow Jones Industrial Average corrected for any mispricing after approximately a 4 year time frame. From figure 6, Peters’ concluded that the S&P500 has mispricing corrections after approximately a 4 year span. This means that monthly returns more than 48 months apart have little to no correlation (Peters, 1996).

![Graph of R/S analysis of S&P500 monthly returns](image)

**Figure 6**: R/S analysis of S&P500 monthly returns (January 1950-July 1988). Estimated \(H=0.78\).


Peters’ R/S analysis on the S&P500 found the monthly returns have a Hurst exponent of 0.78, significantly different from a random walk Hurst exponent of 0.5. He wanted to also show that the Hurst
exponent is actually a valid estimate of the correlation in market returns. As a test he decided to perform the same R/S analysis, but this time he scrambled the data on market returns so that there was no longer a chronological order to it. Figure 7 shows the results of his R/S analysis with the data scrambled. The figure shows the two Hurst exponent lines of 0.78 and 0.5 along with the scrambled and unscrambled market returns. As you can see, scrambling the market returns produces dramatically different results from the unscrambled market returns. The scrambled data produced a Hurst exponent of 0.51, showing that there is virtually no correlation between the market returns now that they are out of chronological order. In the scrambled data there is also no drop in the Hurst exponent after 4 years, or any period of time for that matter. The results of the R/S analysis on the scrambled data are as expected; it shows a consistent random behavior (Peters, 1996).

![Figure 7: R/S analysis scrambling test of S&P 500 monthly returns (January 1950-July 1988). Estimated unscrambled H=0.78; estimated scrambled H=0.51.](image)


From Peters’ analysis it is clear that the S&P500 does have an underlying trend. Peters looked at other varieties of market returns and calculated the Hurst exponent. His results can be found in tables 1-3. Whether looking at the returns of individual stocks, foreign markets, or currencies, they all appear to
have a correlation with past returns. Of the given markets, the Singapore dollar is the only market that does not exhibit non-random behavior. Peters included R/S analysis on the Singapore dollar as another test. The Singapore government has the banks control the valuation of their dollar so that it tracks the U.S. dollar. The exchange rate between the U.S. dollar and the Singapore dollar is controlled to stay at a constant rate. The only expected fluctuations in the exchange rate are random delays in the timing of adjusting the Singapore dollar to any changes in the U.S. dollar. Since its only expected fluctuations are random, the Hurst exponent of 0.5 supports the accuracy of R/S analysis (Peters, 1996).

<table>
<thead>
<tr>
<th></th>
<th>Hurst Exponent (H)</th>
<th>Cycle (Months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.78</td>
<td>48</td>
</tr>
<tr>
<td>IBM</td>
<td>0.72</td>
<td>18</td>
</tr>
<tr>
<td>Xerox</td>
<td>0.73</td>
<td>18</td>
</tr>
<tr>
<td>Apple Computer</td>
<td>0.75</td>
<td>18</td>
</tr>
<tr>
<td>Coca-Cola</td>
<td>0.70</td>
<td>42</td>
</tr>
<tr>
<td>Anheuser-Busch</td>
<td>0.64</td>
<td>48</td>
</tr>
<tr>
<td>McDonald's</td>
<td>0.65</td>
<td>42</td>
</tr>
<tr>
<td>Niagara Mohawk</td>
<td>0.69</td>
<td>72</td>
</tr>
<tr>
<td>Texas State Utilities</td>
<td>0.54</td>
<td>90</td>
</tr>
<tr>
<td>Consolidated Edison</td>
<td>0.68</td>
<td>90</td>
</tr>
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</table>

Table 1: Hurst Exponents and related cycles of individual stocks’ monthly returns from January 1963 to December 1989


<table>
<thead>
<tr>
<th></th>
<th>Hurst Exponent (H)</th>
<th>Cycle (Months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.78</td>
<td>48</td>
</tr>
<tr>
<td>MSCI Germany</td>
<td>0.72</td>
<td>60</td>
</tr>
<tr>
<td>MSCI Japan</td>
<td>0.68</td>
<td>48</td>
</tr>
<tr>
<td>MSCI U.K.</td>
<td>0.68</td>
<td>30</td>
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</table>

Table 2: Hurst Exponents and related cycles of international stock indices’ monthly returns from January 1959 to February 1990


<table>
<thead>
<tr>
<th></th>
<th>Hurst Exponent (H)</th>
<th>Cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Japanese yen</td>
<td>0.64</td>
<td>Unknown</td>
</tr>
<tr>
<td>German mark</td>
<td>0.64</td>
<td>Unknown</td>
</tr>
<tr>
<td>U.K. pound</td>
<td>0.61</td>
<td>Unknown</td>
</tr>
<tr>
<td>Singapore dollar</td>
<td>0.50</td>
<td>None</td>
</tr>
</tbody>
</table>

Table 3: Hurst Exponents and related cycles of U.S. dollar exchange rates’ daily changes from January 1973 to December 1989

From the various R/S analyses shown it is clear that market returns are correlated with past market returns. The market returns are shown to be fractal in nature by their fractal dimension, and follow fractal distributions belonging to the Lévy alpha-stable distribution family. The evidence that market returns are not independent of each other or normally distributed is overwhelming. The random walk theory and EMH have been shown to contain serious flaws.

V. Where do we now stand in the theories of investment?

The traditional theories of investing developed back in the mid 1900s had sound logic and an easy application to the markets. Over time evidence has continually surfaced bringing the assumptions of these theories into question. It is clear that they are wrong in assuming the rationality of investors, the normal distribution of market returns, and the independence between current market returns and those of the past. These investing theories are still prominently taught throughout business school across the country despite their underlying flaws. These investment theories need to be drastically revised in order to account for the complexity of the markets they have failed to address. The emerging field of behavioral finance shows a clear need to incorporate investor psychology into the investment formulas. The formulas also need to address the fact that the markets are much more volatile than originally assumed. There are black swans in the market that traditional investment theories fail to account for. The fractal nature of market returns is clear, but the tools to handle this compelling geometry have yet to be developed.

A modern day equivalent to the Capital Asset Pricing Model (CAPM) would forever change the field of investing. Models have been worked on and proposed, but none yet have been developed and withstood the test of time. Much more analysis on the markets will need to be done before we can expect such a model. The expanding fields of behavioral finance and neurofinance will hopefully provide deeper insights into how investors’ actions are driven. Every day more market data is produced, giving
us the opportunity to perform more robust and thorough analysis. Continual analysis of this data will further elucidate the fractal nature of the market. It is unrealistic to believe a perfect model for investing will ever be developed, as perfection is often not attained but instead endlessly pursued. With this new understanding of the markets, the pursuit continues on.
Works Cited


