MAT 307: Noncommutative Functional Analysis

13 — 17 September 2004

Old Main 403 Canisius College: 5:00pm — 8:00pm Daily

Instructor: David P. Blecher
Associate Professor
Department of Mathematics
University of Houston
Houston, Texas

Dr Blecher has some experience in successfully navigating undergraduates through advanced topics. He co-ran a very successful Research Experience for Undergraduates (REU) in Houston in the Summer of 2001. The students who worked with him then have papers published or in the process of publication. Partly on the basis of this, one of these students (Kay Kirkpatrick) was a co-winner of the prestigious national award, the **Alice T. Schafer Prize for Excellence in Mathematics by an Undergraduate Woman** from the Association for Women in Mathematics, and has gone on to graduate school at UC Berkeley. The notes that Blecher produced for his REU students will form the basis of the first half of this Lecture Mini-Series, and is a manageable crash course in basic analysis for undergraduates, including the beginnings of operator theory (from a matrix perspective). The students will be expected to reread these notes between lectures (we have designed the lectures so they do not rely on the student doing this, although we hope that most students will). Blecher is also in the process of writing a book on the subject of noncommutative functional analysis, which is to appear in the Oxford University Press. Parts of Chapter 1 of this book will form the basis for the second half of this Lecture Mini-Series.

In the first half we also cover some ideas from "noncommutative mathematics" to acquaint and familiarize students with key concepts. The central idea being that of "quantization". We will show that studying some of the common spaces in mathematics (such as topological spaces, measure spaces, groups, and so on) is the same as studying "rings" of appropriate functions on these spaces. Typically, these rings are commutative in most classical settings. These commutative rings can then be replaced by noncommutative rings having the same formal properties. We will indicate ways in which the classical analysis of commutative rings may be generalized to various noncommutative settings. There are now important theories of noncommutative topology, noncommutative probability, noncommutative differential geometry, quantum groups, noncommutative functional analysis, and so on. We will only spend a few minutes briefly surveying each of these theories before narrowing our interest down to the last one: the theory of operator spaces.

A few selected features of operator space theory will be discussed in detail. We will discuss some connections with Banach space theory, and Choquet theory. The items we need from these theories will be stated of course, and examples given of their use. For example, we hope to treat something like the following sequence of topics:

1. The Hahn-Banach theorem, and a discussion of its operator space version.

2. A discussion of the noncommutative analogue of the Choquet and Shilov boundary, and some applications to Banach-Stone type theorems.

3. A discussion of injective operator spaces, culminating with some easy to understand recent results due to Blecher with Effros and Zarikian.

4. Application of the last item to the abstract characterization of operator algebras.

5. A presentation of some key points of a short proof due to Davidson and Paulsen of the recent solution by Pisier of the famous "Halmos similarity problem".

Prerequisites: Calculus I — III and Linear Algebra.

Biographical Information: Prof. Blecher is undoubtedly one of the great communicators of mathematics. A captivating speaker, Blecher is also a phenomenal researcher of mathematics. Blecher studied in Cambridge, England, and Edinburgh, Scotland, where he received his Ph.D. in 1998. Since then he has been at the University of Houston, with the exception of visiting posts at the University of California, Berkeley, and the University of Missouri.

Prof. Blecher's research interests include Operator Algebras and Functional Analysis. Blecher is a world-leading figure in the exciting new field of Operator Spaces. Recently his work has focused on Hilbert C^*-modules (which are a noncommutative generalization of a vector bundle), and on injectivity and extremal representations of Operator Spaces. Blecher has published numerous deep, field influencing, papers and monographs. With Muhly and Paulsen as co-authors, Blecher has published a Memoir of the American Mathematical Society *Categories of Operator Modules (Morita Equivalence and Projective Modules)*, **681** (2000), pp. 1-94.

MAT 308: Mathematics and Sex

27 September — 1 October 2004

Old Main 403 Canisius College: 5:00pm — 8:00pm Daily

Instructor: Clio Cresswell
Visiting Fellow
The School of Mathematics
The University of New South Wales
Sydney, Australia

For many people, just seeing the words *mathematics* and *sex* in the same sentence is odd enough, let alone discovering there is a deep relationship between the two. Well, now is the time to shatter stereotypes because the fact is sex is highly mathematical. And we're not just talking about numbers and probabilities. Mathematics is the study of patterns: their discovery, their interconnections and their implications. And in the context of sex, mathematics has uncovered a treasure-trove of sometimes unexpected but rich patterns and relationships. In this course, you will be presented with a sample of these. You'll get a taste of a number of mathematical areas; you'll see how mathematicians have considered sexual problems and you will learn how so called non-mathematicians have used mathematics to further their research. And, as if this wasn't enough already, you'll pick up some great relationship tips as well!

Lecture 1: Love & Marriage [Dynamical Systems]

Here we will introduce the concept of a dynamical system and its analysis. A dynamical system consists of a set of differential equations that capture the essential properties of an entity as it changes with time. It was Newton, no less, who was the first to use them. He was studying the motion of planets and cannon balls. Today, however, they are used in all fields, from chemistry to psychology to the stock market. The key to their success is that they are the only reliable fortune telling method we have. The motion of the planets depends upon the forces between them. What sorts of forces apply to lovers? Can we use these methods to predict successful relationships?

Lecture 2: Searching for The One [Search Algorithms & Probability]

With our everyday use of the Internet, we have become very familiar with different search methods or strategies, such as Google or Yahoo. We see how different search strategies yield different results, that the validity of the results can be quantified statistically, and that some search strategies may be more suited to a specific task than others. But it's not just the Internet, our lives are full of searches and therefore strategies: Which peach will be the juiciest at the store? Where shall I park? Whom shall I marry? We are always searching! And with divorce rates skyrocketing, the question is how can we devise a good search strategy to find that special someone? We will study a number of search strategies to address this, in conjunction with some rules of probability to account for our lives little imperfections.

Lecture 3: Physical Attractiveness [Genetic Algorithms & Scaling]

We appear to know when someone has a big nose or is too skinny, yet there is no set of exact individual measurements to lead us to such classifications. Attractiveness is about how everything fits together in the final structure. It's about the relationships between the different body parts. Simple measurements are not sufficient. Here, we will delve into some of the mathematical techniques that have been used to investigate this complex issue. These include scaling relations, genetic algorithms, and fractal geometry. Will we be able to teach a computer what is attractive? Or will the computer be showing us what we find attractive?

Lecture 4: People Matching & Dating Services [The "Marriage Problem" & Distance Measures]

How do people pair up in society? If everyone wants Cameron Diaz or Keanu Reeves but only one can have either, where does that leave the others? And where does that leave the less popular? When we look at people matching on a community scale, where everyone is vying for someone else about town, patterns emerge in terms of end results and satisfaction levels. We will discuss the mathematically notorious *marriage problem*. Now one place where matching happens in, let's say a more controlled fashion, is among the users of dating services. How do people get matched up with these services? Tackling this question will mean an encounter with distance measures, a mathematical notion developed in the 19^{th} century but thought to be completely useless at the time. Little did anybody know! Many of the ideas discussed in this lecture will be tested out in the classroom. How will people be matched up in this sample space? Hopefully neither Cameron nor Keanu will be attending otherwise our calculations may be terribly skewed.

Lecture 5: Feeling Hormonal? [Self-Organization & Stochastic Calculus]

No course on *Mathematics and Sex* would be complete without some of the mathematics used to study hormones. Our bodies are teaming with testosterone, oestrogen, progesterone, oxytocin — the list carries on. How do all these hormones coordinate themselves to have an effect? What are men's hormonal patterns? How do ovaries count out the months to know when it is time to release an egg? Self-organization and stochastic calculus are both topics that have come in handy here. The mathematics is of a high level and so this final lecture will, in comparison to previous ones, be more of a survey of results. A perfect way to finish as it will give you a flavor of how far mathematics goes and where mathematics can take you.

Prerequisites: One, preferably two, semesters of college level mathematics. Or, a strong high school background in mathematics (for example, advance placement calculus).

Biographical Information: The source of Clio Cresswell's fame lies in its incongruity — she is a mathematical celebrity! Besides being a mathematics lecturer at The University of New South Wales in Sydney, Australia, she is the author of an iconoclastic new book *Mathematics and Sex*, writes a 'relationships and love' Q&A column for *New Woman Magazine*, and appears regularly on radio and television. Clio's unconventional and dynamic career stems from her active pursuit of diversity and a desire to experience all the many fascinating aspects of life.

Born in England, she spent part of her childhood on a Greek island, and was then schooled in the south of France. Clio was studying Visual Art in Cannes, before she simultaneously discovered the joys of Australia and mathematics at the age of 18. With the door to mathematics opened, Clio studied mathematics at The University of New South Wales, winning the exceptionally prestigious University Medal for her Honors degree, and completing a PhD in 1999, from which she published a number of papers.

However, always keen to embrace a challenge, Clio decided against a traditional career and instead combined her love of mathematics with freelance writing and presenting, work which ranges from television appearances, to corporate speaking, to publishing.

Clio has interviewed such personalities as Dennis Rodman on the top rating Australian television show *The Panel*; delivered serious book reviews for the reputable *Science Show* on the Australian Broadcasting Corporation's *Radio National*; and has been a host of breakfast radio for one of Australia's leading commercial stations, *Triple M*. She has given relationship advice on *Beauty and The Beast*, one of Australia's longest running television programs, and has launched one of the latest bras from the *Triumph International* collection. In 2001, Clio's incongruous celebrity was cemented when she was voted one of the *25 Most Beautiful People* by *Who Weekly*, the Australian incarnation of *People Magazine*.

It was also in 2001 that Clio was selected as one of 200 Australia Day (January 26) Ambassadors to promote Australian excellence during the Australia Day celebrations. She has been a Visiting Fellow in The School of Mathematics at The University of New South Wales since 2000, and now also holds the post of Development and Liaison Manager — a role focusing on promoting the school itself, as well as mathematics in general.

The North American release of Dr Cresswell's new book *Mathematics and Sex* will coincide with her visit to Canisius College and will be marked with various celebratory events. *Mathematics and Sex* is currently being translated into German, French and Japanese.

MAT 309: The Atiyah-Singer Index Theorem and Noncommutative Geometry

25 — 29 October 2004

Old Main 403 Canisius College: 5:00pm — 8:00pm Daily

Instructor: Toshikazu Natsume
Professor of Mathematics
Nagoya Institute of Technology
Japan

The focus for this short course will be the Atiyah-Singer Index Theorem for elliptic differential operators. This is one of the deepest mathematical results of the twentieth century. To give a perspective on the course, we briefly discuss this theorem now.

Since the time of Isaac Newton, differential operators have been an important tool for science, in particular, playing an important role in mathematical physics. A differential operator is a polynomial in partial derivatives, with coefficients being functions. Three most important classes of differential operators are elliptic, parabolic, and hyperbolic. In mathematical physics, elliptic operators appear as potential equations, parabolic ones appear as heat equations, and hyperbolic ones appear as wave equations.

Among these three classes, elliptic operators have particularly nice properties. Most of the results for elliptic operators on bounded domains in Euclidean spaces also hold for elliptic operators on more general curved spaces, such as the surface of a doughnut, or of a pretzel. Elliptic operators on such closed surfaces were observed to have an especially nice property: to each such operator, an integer called the "index" of the operator can be assigned. Curiously, when you continuously perturb an elliptic operator, the index stays fixed, whereas other important numerical invariants for the operator, such as the number of independent solutions, often vary irregularly. This fact that the index doesn't change under continuous perturbations led Izrail M. Gel'fand of Moscow to conjecture that this index has a deeper meaning relating to geometry and topology. In 1963 Michael F. Atiyah and Isadore M. Singer gave an affirmative answer to Gel'fand's conjecture. They showed that the index of any elliptic operator is equal to another invariant associated to the operator: a topological invariant. This result, the Atiyah-Singer Index Theorem, is at the heart of the interaction between three important branches of mathematics: analysis, topology and geometry. It gave birth to new fields of mathematical research, such as K-theory, and led to the noncommutative geometry of Alain Connes. Both Atiyah and Connes received the Fields Medal (the mathematics equivalent of the Nobel Prize) for this and related work.

In this short course historical background will be given, and the meaning and basic properties of the index will be explained. In fact the Atiyah-Singer Index Theorem holds on all sorts of very general surfaces (manifolds), but for this course we will focus on elliptic operators on Euclidean space, and prove the index theorem here (still a highly nontrivial result). The course will require background in multivariable calculus and linear algebra. An introductory survey of the additional tools (from functional analysis and topology) needed to prove the theorem will be given. Examples will be developed in two and three dimensions.

The Atiyah-Singer Index Theorem opened the door to a new world of interaction between different areas of mathematics, where analytic machinery such as operator algebras can play a significant role in topology and geometry. A crystallization of this idea is noncommutative geometry, currently important particularly in today's mathematical physics. The course will be concluded by explaining some fundamental ideas and remarkable results in noncommutative geometry.

Prerequisites: Calculus I — III and Linear Algebra.

Biographical Information: Prof. Natsume received a Doctor of Science from Kyoto University in 1986. During the period 1988-96 Natsume was an assistant and then associate professor of mathematics at SUNY Buffalo and he has been a full professor at the Nagoya Institute of Technology (Japan) since 1997. Natsume has held visiting positions at the Universities of Ottawa (1982) and Kyushu (1999 - present), and was a member of the Mathematical Sciences Research Institute (MSRI) at UC Berkeley in 1984 (which is an exceptional honor as MSRI is by invitation only).

Prof. Natsume has published 29 far-reaching papers, in mathematics journals of exceptional quality and prestige, in the areas of K-theory, C^*-algebras, Lie Algebras, and general Index Theory. Moreover, Natsume has given 14 invited talks at an array of stellar institutions (Copenhagen, Muenster, MSRI Berkeley, and so on) in the past four years, all attesting to his prominence as a researcher on the world stage.