Neither Newton nor Leibnitz:
The Pre-History of Calculus in Medieval Kerala

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The Yuktibhasha

•It is also unusual in providing proofs rather than just statements of results. Moreover,it is written in prose rather than poetry. In many ways it is the last book in the old tradition of India and the first of the modern tradition.

•The first four chapters are quite elementary mathematics.

•Chapter five describes some calculations on calendars. Also, the solution of the equation $ax - by = c$ for given positive integers a, b, c; x and y are to be found as integers. This methods $Kuttakaram$ was known to Brahmagupta (tenth century CE).

• The beginning of the new ideas are in chapter six on the circumference of a circle.

Chapter one of the Yuktibhasha

•The book starts with the intrepretation of the product of two integers as the area of a rectangle. It is shown how to find the area of figures made of straghtlines intersecting at right angles by divinding them up into rectangles.

•This part of the book appears to be a prose commentary to the Sanskrit poetic text $tantra - sangraha$.

•Division of numbers is introduced next.

•Then the notion of a square $varga$ is introduced both as the product as two equal numbers and the area of a square.

•The formula $(a+b)^2 = a^2 + b^2 + 2ab$ is then proved by dividing up the square of side $a+b$ into rectangles of sides $a \times a, a \times b, b \times a$ and $b \times b$.

•After proving $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ by a similar method, the Pythagoras theorem is proved.

•This result is present in the ancient text $sulbasutra$ on the construction of sacrificial altars. It is proved here in a way that should be familiar to a modern student of mathematics, by constructing squares whose sides are each of the sides of a right triangle.

•The hypotenuse is called $karna$; the short side is called $bhuja$ (literally arm) and the long side is called $kodi$. All this sets the terminology for more interesting constructions later.

•The square root $vargamula$ is defined as the inverse of the square.

Chapters two, three and four

•Chapter two is about ten standard problems of elementary mathematics. These ten are:

•Given any two quantities among the sum, difference, product, sum of the squares or difference of squares, determine the underlying numbers.

•Of course the simplest of these problems is when the sum and difference are given.

•The formula $\sqrt{\frac{(a^2+b^2)-(a^2-b^2)}{2}} = a$ solves the most complicated of these.

• These elementary chapters are useful to understand the notation and terminology. Mathematical formulas are traditionally expressed as poems that can be more easily memorized. Here they are stated in prose which is obviously a translation of Sanskrit poems from some previous text such as *tantrasangraha*.

•It is an old Indian tradition to start mathematics discussions with the most elementary of processes; the mathematics chapter of the *aryabha-*

teeyam (from 495 CE) starts with a listing of the powers of ten. Explicit calculations are stressed at all levels.

•Usually the discussion proceeds very rapidly merely stating results with no proofs or explanations. These are to be provided by the teacher. There are also commentaries in which the results are explained and further refinements are made.

•Some of the deepest texts are in fact commentaries (*bhashya*) on classic texts: for example the commentary by *Neelakanta Somayaji* to the *aryabhatteyam* is a foundational reference for calculus. The *yuktibhasha* is much kinder to reader, to the point of being pedantic: even minor results are explained in detail.

•Chapter three is about fractions. The usual rules of additions of fractions are established using geometric arguments.

•Subdivision of rectangles into equal parts is the basic trick.

•Chapter four is very brief and is on proportions. Given three quantities in the statement $a : b :: c : d$, find the fourth.

•Proportions are viewed geometrically in terms of similar triangles, as well as in terms of problems in everyday life such as cost of rice.

Ch. Five: *Kuttakaram*

• *Kuttakaram* is a method of solving integer equations $Ax - By = C$. The method is from the *Leelavathi* of Bhaskara (11th century CE).

• The problem arises in the preparation of calendars, one of the main practical uses of astronomy. A solar month can be thought of as one twelfth of a solar year. This is close but not quite the same as a lunar month.

• Certain religious observances are tied to the moon (e.g.,fasts on the eleventh day of each half-period of the moon) while others are tied to the Sun (e.g., the start of a new year on the day of the vernal equinox). Converting from one system to the other leads to these kinds of problems.

• The practical problems are compounded by the fact that different regions of India used different calendars.

• In Kerala a calendar marking the rebuilding of the town of $Kollam$ was in wide use for civil purposes; the months are named as in the Greek calendar *Chingam* (Leo) being the first month.

•A completely different calendar was also in use mostly for religious observances: the *Kali* era starting on the day of the death of Krishna and the beginning of the 'modern' era of India according to mythology.

•The astronomers tried to avoid these problems by simply counting the number of revolutions of the Earth from a standard point (the first day of the *kali* era-mythically the day after Krishna's death) in time to denote each day.This day one is assumed to be a Friday.

•This would be a big but standard number (the *kali-dina-samkhya*, the kali-date-number) that could then be converted to whatever calendar you want: computing the day of the week, whatever system of months and years you use and what day of the phase of moon etc.

•Modern computer scientists use a similar tricks. In unix, the clock time is the number of milliseconds that have elapsed since 1 Jan 1970: the mythical beginning of the digital computer era. Is this the day that the last analogue computer was unplugged?

•This number is then converted to the date,month and year with corrections for leap years etc. built into the program. (The command 'time' in unix gives the number of milliseconds and 'date gives the latter.)

•For example, the year 1120 of the Kollam year corresponds to the year 5046 in the *kali* year, and 1949 in the Christian era.

This example is adapted from the commentary to the *Yuktibhasha* by *Akhileshara Iyer* and *Rama Varma* published by *Mangalodayam Press* of *Trissur* in the year 1952 CE.

So on 1/*Chingam*/1120 in the Kollam calendar, what is the day of the week?We need to find the kalidinasankhya of this date. Taken modulo seven it will give the day of the week; modulo twenty-eight it gives the lunar date and so on. We give just a fragment of this calcutaion to give a flavor.

The calculations proceeds in this way with several corrections added to produce the kali date number 1842853 for the required date. The mathematics is not interesting. It is interesting to check the accuracy of the astronomical observations for the ratio of the moon's revolution period to the Earth's rotation period; and to the period of revolution of the Earth around the Sun.

To find this number we start with number of solar months Earth since the beginning of the *kaliyuga* to the last day of the *kali* year 5045; this is $5045 \times 12 = 60540$.

To convert this into lunar months, a correction has to be applied since the lunar month is shorter than the solar month. This correction was known to be $1593320/51840000$ times the number of solar months. That is $60540 \times \frac{1593320}{51840000} = 1860$ in our case.

This ratio comes from measurements of the moon's period. It is expressed by the formula that that the length of a *yuga* (a mythical interval with no astronomical signifcance) is 4320000 solar years or 51840000 solar months; the moon makes 57753320 revolutions in this time. The difference between their number of revolutions is 5343320; the excess number of lunar months over solar months is 1593320.

On the first day of the *kali* year 5045, $60540 + 1860 = 62400$. To this we add four more months to take us to the first of $Chingam$ from the beginning of the kali year 5046.

Linear Diophantine Equations

•In calendar calculations, the following problem arises: Find all integer solutions to $ax - by = c$ given a, b, c.

•Clearly we can reduce a, b, c until they have no common factor.

•Given one solution (z, u) to the equation $ax - by = c$, all other solutions are of the form $x = z + mb, y = u + ma$. The idea is to apply this transformation repeatedly to reduce the equations to one with small coefficients, which is then solved 'by hand'.

•The method is called *kuttaka* after the iron rod that is used to pound grain into smaller chunks: the method 'pounds' on the numbers until they becomes small.

An example

•For example, solve $195x - 221y = 65$.

•$221 = 195 + 26$. So $195x_1 - 26y = 65$ where $x_1 = x - y$.

•$195 = 7 \times 26 + 13$. So $13x_1 - 26y_1 = 65$ where $y_1 = y - 7x_1$.

•Cancelling common factors $x_1 - 2y_1 = 5$ which has the solution $x_1 = 7, y_1 = 1$.

•Backtracking, $x = 57, y = 50$ is a solution. All others follow by the above transformation.

•**Exercise**: Write a computer program (in symbolic algebra language like Mathematica or in a numerical language like Fortran or C) to solve the equation $ax - by = c$ given a, b, c using the above idea; or determine that it does not have a solution.