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Neither Newton nor Leibnitz:
The Pre-History of Calculus in Medieval Kerala

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The Circle and Sin

•Sin and other related functions are these days thought of in terms of triangles. In India they were more closely associated with the circle. In fact the Indian version of the sin was a quantity with the dimensions of length.

•We will follow the discussion in *The Golden Age of Indian Mathematics* by S. Parameswaran pub by Swadeshi Science in Kochi, India.

•Imagine a circle of center O, andOS a line to a point on the circumference of the circle. Draw a perpendicular to OC intersecting the circle at A and B. The chord AB intersects OS at M.

•The sanskrit word *jya* means chord. AB is called *samasta-jya* the full-chord while AM is the ardha-jya, or half-chord. In practice the *ardha-jya* appears much more often and by default jya came to mean this half-chord. It is associated to the arc of the circle SA. Thus the jya as well as the arc are lengths proportional to the radius of the circle.

•The radius is chosen for convenience.

•The line MS is called the *saram* or 'arrow'. The picture of a bow and arrow with a chord stretching across is clear.

•The hypotenuse of a right triangle was called *karna*-literally, 'ear'. The other two sides are called *bhuja*-arm- and *kodi*.

•The length OM is also called *koti-jya*. It corresponds to $\cos$.

•The chord is also often called *jiva* (Sanskrit names have cases and have to adapted according to the context). For example, the formula for the *jya* of the sum of two arcs is called *jiveparasparanayaya*.

•Aryabhatta's work was famous in the Arab world. (Al-Biruni refers to him as 'Aryabhajos'.) In translations *jiva* became confused with the Arab word *jaib* which means 'fold' or 'bay'. Arabic is written without vowels and you have to know how to read the word from the context.

•Translations to Latin rendered *jaib* as sinus which mean fold. Then sinus got abbreviated to $\sin$. Along the way the radius was set to unity so that $\sin$ became a dimensionless quantity; also the arc came to measured in radians rather than degrees.

•Something similar happened with π. The Indian texts speak of the circumference of a circle of a given radius, it being understood that the circumference is proportional to the radius, which is then chosen for convenience. It was Euler who introduced the now standard notation of π for the ratio of the circumference to the diameter.

•Bhaskara gives a rational approximation to sin. It is equivalent to the formula

$$\sin x = \frac{4x(180 - x)}{40500 - x(180 - x)}. \tag{1}$$

when x is in the first quadrant. The remaining cases can be brought to this by the symmetries of the sin. The error is never worse than 0.01.

•To prove this, make the approximation that sin (with the angle expressed in degrees) is the ratio of two quadratic polynomials in the range $0 < x < 180$ degrees. Use the symmetry $x \to 180 - x$ of the sine to see that it must have the form

$$f(x) = \frac{a + bx(180 - x)}{p + qx(180 - x)}. \tag{2}$$

Then use the known values at $x = 0, 30, 90$ to determine the three constants. (An overall constant cancels out.)

The Circumference of the Circle

•It has always been of great interest to geometers and astronomers to relate the circumference of a circle to the its diameter.

•The basic method has been to inscribe or circumscribe a regular polygon. The problem then is to find the side of the polygon as a multiple of the diameter.

•Approximate formulae good enough for practical purposes had been known for a long time-$\pi \approx \frac{22}{7}$ is enough for most engineers.

•Archimedes of Syracuse (287-212 BCE) obtained the value $3\frac{10}{71} < \pi < 3\frac{10}{70}$ considering by considering a regular polygon of 91 sides.

•The *aaryabhatiiya* (499 CE) gives a value accurate to four decimal places: "The circumference of a circle of diameter 20,000 is 62832": or $\pi \approx 3.1416$.

•Bhaskara (1114-1185(?) CE) says that the circumference of a circle of diameter 1250 is 3927 by considering an inscribed regular polygon of 384 sides-correspond to $\pi \approx 3.14155$. Getting close!

•These days the value of π to a thousand decimals is just two clicks away if you have Mathematica! $\pi \approx =$

3.14159265358979323846264338327950288419716939937510582097494459230781640628620 8998628034825342117067982148086513282306647093844609550582231725359408128481117450284102701938521105559644622948954930381964428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273724587006606315588174881520920962829254091715364367892590360011330530548820466521384146951941511609433057270365759591953092186117381932611793105118548074462379962749567351885752724891227938183011949129833673362440656643086021394946395224737190702179860943702770539217176293176752384674818467669405132000568127145263560827785771342757789609173637178721468440901224953430146549585371050792279689258923542019956112129021960864034418159813629774771309960518707211349999998372978049951059731732816096318595024459455346908302642522308253344685035261931188171010003137838752886587533208381420617177669147303598253490428755468731159562863882353787593751957781857780532171226806613001927876611195909216420199

Which is displayed on the page as:

3.1415926535897932384626433832795028841971693993751058209749445923078164062862 0899862803482534211706798214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196442881097566593344612847564823378678316527120190914564856692346034861045432664821339360726024914127372458700660631558817488152092096282925409171536436789259036001133053054882046652138414695194151160943305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912983367336244065664308602139494639522473719070217986094370277053921717629317675238467481846766940513200056812714526356082778577134275778960917363717872146844090122495343014654958537105079227968925892354201995611212902196086403441815981362977477130996051870721134999999837297804995105973173281609631859502445945534690830264252230825334468503526193118817101000313783875288658753320838142061717766914730359825349042875546873115956286388235378759375195778185778053217122680661300192787661119590921642019

Chapter Six of the Yuktibhasha

•What is new with the Kerala school is a convergent infinite process that can give the value of π to *arbitrary* accuracy. There were several such processes known to this school, we will study in detail two of them, explained in detail in the sixth chapter of the *Yuktibhasha*.

•There are two different approaches to calculating the circumference.

•The first will give an algebraic recursion relation-involving a square root- that converges to the exact value. In modern notation,

$$x_0 = 1, \quad x_{n+1} = \frac{\sqrt{1+x_n^2} - 1}{x_n}, \quad \pi = 4 \lim_{n\to\infty} 2^n x_n \tag{3}$$

•The second method-really a succession of improvements- goes much further. It starts as a way to avoid square roots in the calculation of the circumference.

•A finite series-whose terms depend on the number of terms in the series- is obtained which converges to the circumference as the number

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of terms grows. Again in our notation,

$$\pi = 4 \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left[\frac{1}{1 + \left(\frac{n}{N}\right)^2} \right]. \tag{4}$$

•We can recognize the sum as tending to $\int_0^1 \frac{dx}{1+x^2}$.

•Then this series is re-expressed in a way that the terms don't depend on the number of terms. Taking the limit this gives the fundamental infinite series

$$4D[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots] \tag{5}$$

for the circumference of a circle of diameter D.

•The integral was discovered in this context!

•Formulae such as

$$1 + 2 + 3 \cdots N = \frac{N(N+1)}{2} \tag{6}$$

$$1^2 + 2^2 + 3^2 \cdots N^2 = \frac{N(N+1)(2N+1)}{6} \tag{7}$$

$$1^3 + 2^3 + 3^3 \cdots N^3 = \frac{N^2(N+1)^2}{4} \tag{8}$$

for powers up to four were known.

•The key step was to realize that for large N (small steps in the rectification of the circle)

$$1^k + 2^k + \cdots N^k \approx \frac{N^{k+1}}{k+1} \tag{9}$$

so that in the limit we can replace

$$\sum_{n=1}^{N} \left[\frac{n}{N}\right]^k \approx \frac{N}{k+1} \tag{10}$$

Quotation from the Tantra-Sangraha

•Of course this modern notation was not used.

•The language is tortured in the Yuktibhasha as the arguments gets harder and harder. The final result is quite simple and is expressed in an elegant poem quoted from the Tantra-Sangraha (by *Neelakanta Somayaji*, the result is attributed to Madhava though).

• *vyaase vaaridhi-nihate ruupahrte vyaasasaagaraabhihate thri-saradi-vishamasamkhyaa-bhaktam r.n.am svam pr.that kra-maal karyaat*

•K. V. Sharma's translation: "Multiply the diameter by four. Subtract from it and add to it alternately the quotients obtained by dividing four times the diameter to the odd numbers 3,5 etc. "

•This is not absolutely convergent series; even when summed in the right order it is slowly converging. The commentator to the Yuktibhasha shows that summing 27 terms gives a value accurate to one (!)decimal place.

Estimates of Error

•One can add corrections to the truncated sum which estimate the terms omitted

•In the first direction there is

$$C \approx 4D\left[1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{n}\mp\frac{(n+1)/2}{(n+1)^2+1}\right] \tag{11}$$

•Here is an even better formula (also attributed to Madhava in the *Kriyakumari*) for the correction to the finite sum:

$$\frac{\left(\frac{n+1}{2}\right)^2+1}{\left(\frac{n+1}{2}\right)\left[4\left(\frac{n+1}{2}\right)^2+1\right]} \tag{12}$$

Convergent Series for the Circumference

•Or, we can look for new series that converge.

•A result of Madhava when translated to modern language is

$$\pi = \sqrt{12}\left[1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} \cdots\right] \tag{13}$$

•**Exercise** Prove this result by modern methods. Estimate the error if this series is stopped at the nth term.

•Madhava derived using this the result that the circumference of a circle of diameter 9^{11} is 2827433388233. He also derived a way to convert the radian to the degree.

•The Yuktibhasha also gives many rational approximations which have no parallel in modern mathematics. They are based on continued fractions and I have not been able to decipher them yet.

The Arctangent

• A poem of Madhava is quoted in the Yuktibhasha which gives the arc of the circle in terms of the ratio of jya ($\sin$) and the koti ($\cos$). (Remember that these quantities are proportional to the radius.)

• Based on a translation of K. V. Sharma: Multiply the jya by the trijya and divide the product by the koti. Multiply this by the square of the jya and divide by the square of the koti. We get a sequence of further results by repeatedly multiplying by the square of the jya and dividing by the square of the koti. Divide these in order by the odd numbers 1,3,5 and so on. Add the odd terms and subtract the even terms (preserving the order of the terms). This gives the dhanus (arc literally, bow) of these jya and koti. Here the smaller of the two sides should be taken as the jya as otherwise the result will be non-finite.

• If the jya is s and the koti is c and the trijya (radius) is R, we have

$$\frac{sR}{c} - \frac{1}{3}\frac{sR}{c}\left[\frac{s}{c}\right]^2 + \frac{1}{5}\frac{sR}{c}\left[\frac{s}{c}\right]^4 - \frac{1}{7}\frac{sR}{c}\left[\frac{s}{c}\right]^6 + \cdots \tag{14}$$

•If we put $\frac{s}{c} = t$ as the tangent and measure the arc in units of the radius (as we would in modern notation) this is the infinite series for the arctangent:

$$t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \cdots \tag{15}$$

Obtained a couple of centuries before Gregory after whom this series is named!

•Madhava also obtained the infinite series for $\sin$.