1 An Algebraic Recursion formulae for π

The essence of the first argument is to circumscribe the circle in regular polygons: an idea going back to Archimedes,but improved here to give arbitrary accuracy. The end result will be a remarkable recursion relation involving square roots which if repeated, gives increasingly accurate approximations for the circumference.

We will give a more or less literal translation of the text of the *Yuktibhasha*, in san serif font, interspersed by a commentary (in roman font) which will give the argument in modern algebraic notation.

Now we will see how to construct a circle from a square. Imagine a square whose side you may choose. We want to know the circumference of the circle whose diameter is the side of the square. Through the center of this imagined square draw East-West and North-South lines. Now we have four squares. Then from the center of the large square draw a line to the corner. This will be the hypotenuse[1] (*karna*). Imagine it to be in the 'fire corner' (top right corner, while facing East).Draw another hypotenuse from the South radius to the East radius This contains the circle centered at the center of the square.

The circle is being thought of as drawn by a thread stretched from the center; the word for radius *suutraagra* literally means 'thread-end'. This radius always has a specified direction, a vector in modern language. The radius of the circle in the sense of a magnitude or scalar is called *vyaasaardha* or half-diameter. We will keep this distinction in the translation, when needed.

Now, for any triangle, imagine (like a carpenter making a beam for a roof) the largest of the three sides to touch the ground and the two other sides meeting directly overhead. Hang a weight from the meeting point by a thread. This thread is called a *plumbline*. The side on the ground is called the *base* (literally, *bhoomi*, the Earth). The two pieces of the base from the point where the plumbline touches the base are called the *segments* (*Aaabaadha*).

Here, the hypotenuse from the center to the corner is imagined to be the base. The East radius and the South-half of the East side are the sides. Half of the hypotenuse from the East radius is the plumbline. The same way there is a triangle with the South-thread and the East-half of the South side as sides. The base being the same as before. This way, from a square, are two triangles made.

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[1] The sides of a triangle-*thryasha*-are the *karna*, the longest side which we will translate as the *hypotenuse*; the *kodi* which we will translate as the *long side*; and the *bhuja*, the *arm* or short side.

(Think of) the segment touching the corner as the denominator; the side from the corner to the East radius is the numerator. What remains of the hypotenuse after taking out the half-diameter from the base is the desired-multiple (of a similarity of triangles, see below). The result is measured from the corner in each direction to make two points and cut out the corner so formed. Then there is an octagon. Take out double this desired-multiple from the side of the original square. What remains is the side of the octagon.

The idea here is to construct a regular octagon circumscribing the circle and to find its side. A similarity is thought of as a standard-measure *pramaanam* P, its standard-multiple *pramanaphalam* p, the desired-measure *iccha* X, and the desired-multiple *icchaaphalam* x, related by $x = \frac{p}{P}X$. This is usually applied to similar triangles: the standard-measure and standard-multiple are two (usually known) sides of one of these triangles and the desired-measure and desired-result are the corresponding sides of the other.

If we work out the geometry (more detail below) we are being asked to measure out a distance equal to $\sqrt{2}(2-\sqrt{2})$ units from the corner in each direction; the side of the small square being $\sqrt{2}$ units and that of the original square being $2\sqrt{2}$ units. So the side of the octagon is $2\sqrt{2} - 2 \times \sqrt{2}(2-\sqrt{2}) = 2(2-\sqrt{2})$ units. This would be $(\sqrt{2}-1)$ times the diameter. The perimeter of the octagon is then $8(\sqrt{2}-1) \sim 3.3$ times the diameter. Next we will construct the sixteen-gon circumscribing the circle and find its side.

Then, the square root of the sum of the squares of the radius to the midpoint of an octagon-side and half the octagon-side is the hypotenuse from the center to the octagon-corner.(Using the Pythagoras theorem.) Imagine a plumbline to the triangle with this as base. This will fall from the midpoint of the octagon-side to the hypotenuse. There are segments into which the base is divided by this plumbline. The two sides of the triangle are the radius and half the octagon-side. The difference of the square of these segments is equal to the difference of these two sides.For, the square of the plumbline can be got from the sides and the segments in two different ways.(The Pythagoras theorem again.) Therefore, if we divide the difference of the square of the sides by the hypotenuse we get the difference of the segments, since the hypotenuse is the sum of the segments. The difference of squares divided by the sum is the difference.

Then, subtracting from the hypotenuse the difference of segments and halving we get the smaller segment. Think of this segment as the new standard-measure. Half the octagon-side is the standard-multiple. Take from the hypotenuse the radius to get the desired-multiple. There is now a smaller pair of segments. The hypotenuse to these segments is half the octagon-side. Again, form the triangle and measure from the corners to find the two points that form the triangle. Cut out the triangle to get a sixteen-gon. Its side will be twice the above desired-multiple taken away from the half-octagon-side.

The argument that constructed the sixteen-gon can be used to make the thirtytwo-gon; and so on by doubling and doubling ; finding the perimeter as the number of corners becomes infinite we get close to a circle. Think of this as a circle. This circle will have as diameter the side of the original square.

Let us translate this into modern mathematical notation. As in the text above draw a square circumscribing a circle, so that its diameter is the side of the square. Draw the bisectors dividing the square into four smaller squares. It is enough to focus on the upper right hand quadrant. Let O be the center of the circle, E and S the point of contact of the circle with the square. ('East' and 'South'). Let the remaining vertex of the quadrant be called X_0. Draw the hypotenuse of the square OX_0. Let it meet the circle at Z_0. Let the tangent at Z_1 meet the line EX_0 at X_1.

Draw the perpendicular from E to the hypotenuse OX_0; let them meet at Y_0. Then the triangles EY_0X_0 and $X_1Z_0X_0$ are similar. Hence we have

$$X_1X_0 = \frac{EX_0}{Y_0X_0} Z_0X_0 = \sqrt{2}(OX_0 - OZ_0) = \sqrt{2}(\sqrt{2}-1)OE$$

so that

$$EX_1 = EX_0 - X_1X_0 = [\sqrt{2}-1]OE.$$

Now EX_1 and X_1Z_1 are two of the half-sides of a regular octagon circumscribing the circle. Its perimeter is therefore $8 \times 2EX_1 = 8(\sqrt{2}-1)(2OE)$. This gives the approximation $\pi \sim 8(\sqrt{2}-1)$.

We now repeat the construction by drawing the hypotenuse OX_1 of the octagon which meets the circle at Z_1; and the perpendicular EY_1 to this line. Also, the perpendicular to the hypotenuse at Z_1 meeting the side of the square at X_2. Again, EX_2 and X_2Z_1 are two of the half-sides of a sixteen-gon circumscribing the circle. The perimeter of the sixteen-gon, $16 \times 2EX_2$, gives a better approximation to the circumference of the circle. Let us find EX_2 in terms of EX_1. Clearly it is enough to find X_2X_1 since $EX_2 = EX_1 - X_2X_1$. Considering the similar triangles EY_1X_1 and $X_2Z_1X_1$ gives again

$$X_2X_1 = \frac{EX_1}{Y_1X_1} Z_1X_1.$$

Thinking of the perpendicular EY_1 in terms of the two right triangles to which it is a side gives

$$OE^2 - OY_1^2 = EX_1^2 - Y_1X_1^2$$

or

$$OE^2 - EX_1^2 = OY_1^2 - Y_1X_1^2.$$

Since $OX_1 = OY_1 + Y_1X_1$,

$$\frac{OE^2 - EX_1^2}{OX_1} = OY_1 - Y_1X_1.$$

Also,

$$\begin{aligned} Y_1X_1 &= \frac{1}{2}\left[OX_1 - (OY_1 - Y_1X_1)\right] = \frac{1}{2}\left[OX_1 - \frac{OE^2 - EX_1^2}{OX_1}\right] \\ &= \frac{1}{2OX_1}\left[OX_1^2 - OE^2 + EX_1^2\right] = \frac{EX_1^2}{OX_1}. \end{aligned}$$

In the last step we use the Pythagoras theorem on the triangle OEX_1. Putting this in,

$$\begin{aligned} EX_2 &= EX_1 - X_2X_1 = EX_1 - EX_1\frac{OX_1 - OZ_1}{Y_1X_1} \\ &= EX_1 - \frac{(OX_1 - OE)OX_1}{EX_1} = \frac{EX_1^2 - OX_1^2 + OE \times OX_1}{EX_1} \\ &= \frac{OE \times OX_1 - OE^2}{EX_1} \end{aligned}$$

So we have the recursion relation

$$EX_2 = \frac{OE\sqrt{OE^2 + EX_1^2} - OE^2}{EX_1}$$

This argument can be repeated by constructing a new hypotenuse OX_2, a new perpendicular EY_2 to it and so on. After n steps we will get a polygon with 4×2^n sides; a half side of the polygon will be given by the same recursion relation

$$EX_{n+1} = \frac{OE\sqrt{OE^2 + EX_n^2} - OE^2}{EX_n}.$$

The perimeter of this polygon will be $4 \times 2^n \times 2EX_n$ to be compared with the circumference of the circle $2\pi OE$. Thus the approximation to $\frac{\pi}{4}$ is $2^n\frac{EX_n}{OE}$. As the number of sides grow the perimeter will tend to the circumference of the circle from above: it will always be an overestimate as the polygon lies outside the circle.

It is convenient to set $x_n = \frac{EX_n}{OE}$. This leads to the recursion formula

$$x_0 = 1, \quad x_{n+1} = \frac{\sqrt{1 + x_n^2} - 1}{x_n}, \quad z_n = 2^n x_n.$$

where z_n is the nth approximation to $\frac{\pi}{4}$. Or, put another way

$$z_0 = 1, \quad z_{n+1} = 2\frac{4^n}{z_n}\left\{\sqrt{1 + \frac{z_n^2}{4^n}} - 1\right\}.$$

This gives the sequence of values (easily calculated with a standard calculator that can extract square roots)

$$z_1 = 0.82842712474619009760337744841940,$$

$$z_2 = 0.78793122685731402461758017033062,$$

$$z_3 = 0.78602959631147606568549314034102,$$

$$z_4 = 0.78555590748561421134655212674908,$$

$$z_5 = 0.78543759229224161477680340699333,$$

to be compared with the ‘exact’ value

$$\frac{\pi}{4} = 0.78539816339744830961566084581988 \cdots .$$

Assuming that you can calculate the square roots accurately, the error in this formula can be shown to be:

$$\left| \frac{z_{n+1} - z_n}{z_n} \right| \sim 4^{-n} \frac{\pi^2}{4^3}.$$

Thus this gives an algebraic recursion relation for π that converges exponentially fast. The difficulty of course, lies in extracting the square roots accurately. Calculating each square root itself involves an infinite recursion so the promised exponential convergence is hard to realize in practice.

2 The Recursion as a Dynamical System

Given a map $f : C \to C$ of the complex plane to itself and an initial point x_0, it is possible to define a dynamical system See John Milnor by the recursion relation

$$x_{n+1} = f(x_n).$$

Our example is of course the case $f(x) = \frac{\sqrt{1+x^2}-1}{x}, x_0 = 1$ which converges to 0. what would have happened if we chose a different initial condition? Would it still converge to zero? Does the limit $\lim_{n\to\infty} 2^n x_n$ exist for any choice of x_0 and if so what is it as a function of x_0?

If we look through the derivation of the recursion earlier we will see that angle between the East-radius and the nearest corner of the regular polygon is being halved at each step of the iteration. Moreover half the side of the polygon is the tangent of this angle multiplied by the radius. Hence

$$x_n = \tan\theta_n, \quad \theta_{n+1} = \frac{1}{2}\theta_n.$$

is just as good a way of thinking of the iteration. Moreover, at the beginning of the iteration when we have a square, $\theta_0 = \frac{\pi}{4}$ which is why $x_0 = 1$. Since

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$$

we have

$$x_n = \frac{2x_{n+1}}{1 - x_{n+1}^2}.$$

Inverting this, which involves solving a quadratic equation,

$$x_{n+1}^2 x_n + 2x_{n+1} - x_n = 0$$

we get the recursion relation above:

$$x_{n+1} = \frac{-2 \pm \sqrt{4+4x^2}}{2x_n} \Rightarrow x_{n+1} = \frac{\pm\sqrt{1+x_n^2}-1}{x_n}.$$

We choose the root that gives a positive value for x_n, as it is supposed to be the half-length of a side of the regular polygon.

Thus the change of variables $x_n = \tan\theta_n$ has reduced our recursion relation to something very simple:

$$\theta_{n+1} = \frac{1}{2}\theta_n.$$

There is no doubt that this converges: for any choice of initial θ_0, $\theta_n = 2^{-n}\theta_0$ which tends to zero. Moreover, $2^n x_n = 2^n \tan\theta_n \approx 2^n\theta_n$ since for large enough n, the angle will become small enough. Thus the sequence always converges to zero and the limit $\lim_{n\to\infty} 2^n x_n$ does always exist:

$$x_{n+1} = \frac{\sqrt{1+x_n^2}-1}{x_n} \Rightarrow \lim_{n\to\infty} 2^n x_n = \arctan x_0.$$

Thus this recursion relation is a way to calculate the value of arc-tangent.