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Neither Newton nor Leibnitz: The Pre-History of Calculus in Medieval Kerala

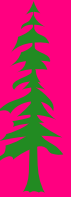
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Kerala

- Kerala is a small coastal state of India: thirty two million people on 15,000 square miles most of whom speak Malayalam.
- It is far away from the heartland of India, a blessing in the times of invasions, famine and pestilence of the late middle ages 1300-1700 CE during which our story takes place.
- The current capital is Thiru-Ananta-Puram (Trivandrum). Ancient ports are at Kollam(Quilon), Kozhikode(Calicut) , Kodungalloor(Muziris) and Kochi (Cochin) and Varkala.
- Kerala was one of the wealthier parts of the ancient world; wealth from trade in luxury goods such as spices (black pepper, cardamom, nutmeg), silk, metal mirrors, cotton, dyes, sugar (*sarkara*).
- Rice was the staple food and the mainstay of the domestic economy. The English word rice comes from the Tamil and Malayalam word *Arissu*.





Sources of Kerala History

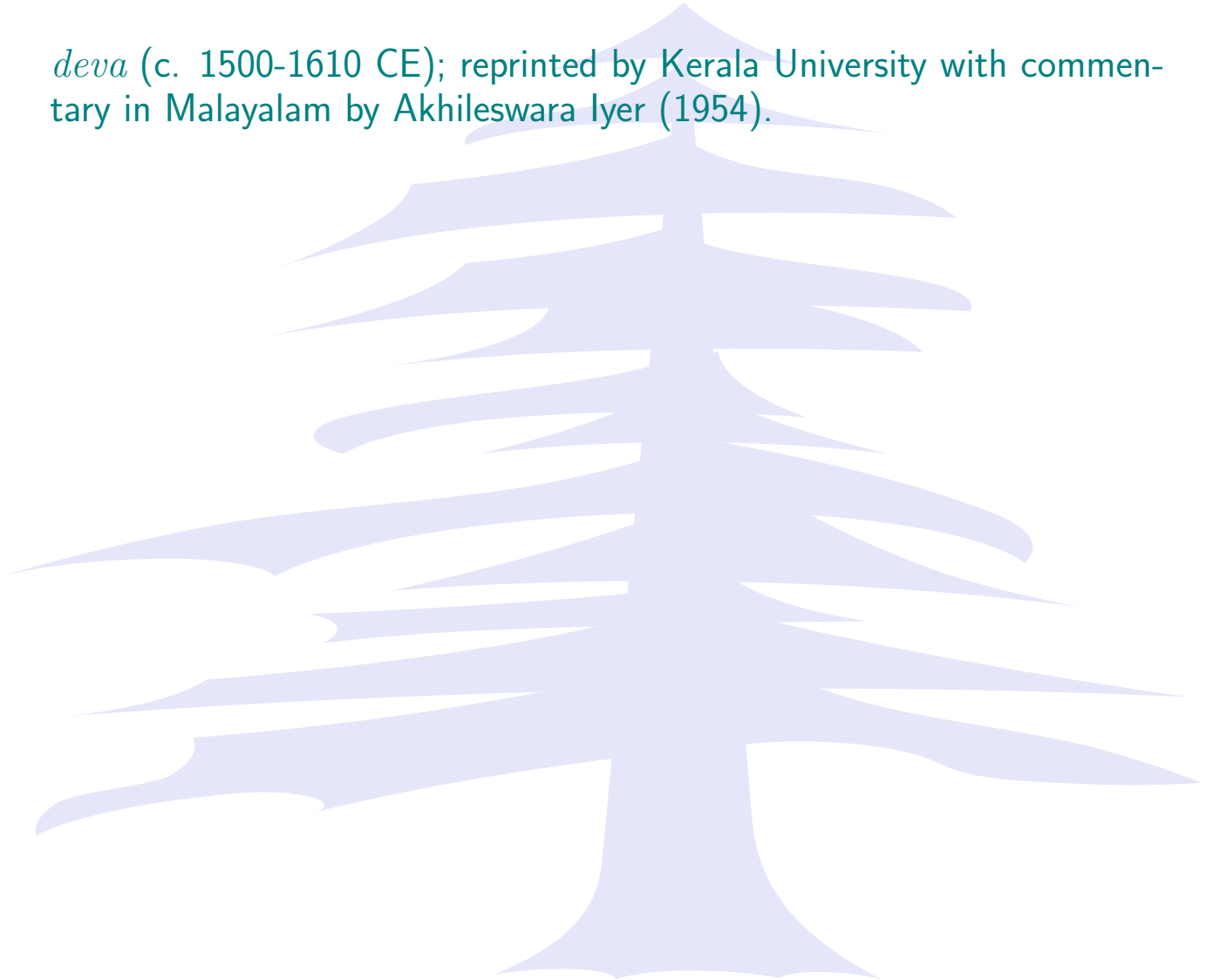
- The text "Circumnavigation of the Eritrean Sea" (written in Greek around 1st century CE) describes this trade; many of the ports still exist.
- Marco Polo passed through Kerala on his way back from China in the early 1300's
- Ibn Batuta also passed through during his raucous adventure in India a generation later.
- Al-Biruni's book *India* (*Al-hind*, c. 1000 CE) is the source of much information on North India. He was a great scholar in astronomy, mathematics, literature and philosophy.
- We also can know of the history of this period early Malayalam literature :*History of Malayalam Literature* (in Malayalam,1940s) by Ulloor Paramesware Iyer, a great modern poet.
- Extensive studies by K. V. Sharma *History of the Kerala School of Hindu Astronomy* Vishveswara-Anand Institute Hoshiarpur (1972)
- Yukti-bhasha* a calculus textbook in Malayalam written by *Jyeshtha-*



deva (c. 1500-1610 CE); reprinted by Kerala University with commentary in Malayalam by Akhileswara Iyer (1954).



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Madhava of Sangama-grama

● Irinhalakkuta is today a small unremarkable town. There lived there in during the time 1340-1425 CE a mathematical genius named *Madhava* who discovered many of the basic ideas of calculus: the solution of transcendental equations by iteration, the infinite series for \sin , \cos and \arctan , integration, integration of series term by term, tests of convergence of infinite series, approximation of transcendental numbers by continued fractions etc.

● Madhava belonged to the *Aryabhata* school of astronomy (as opposed to the *Brahmagupta* school). There was astronomic work in Kerala as early as 4th century CE (*Vararuchi*), but *Madhava* made substantial breakthroughs.

● The *Aryabhateeyam* (composed in 499 CE) was an enormously influential text not only in India, but through translations, in the Arab world and indirectly through them in early Europe.





The Kerala School of Astronomy

- He also founded a lineage of astronomer-mathematicians which lasted till the early eighteenth century. His followers elaborated and refined his theories and composed hundreds of mathematical works.

- *Parameswara* (1360-1455) student of Madhava. Discovered *drk-ganita*, a mathematical model of astronomy based on observations. He made observations for a period of 55 years and had the most accurate observations of planetary motion before Tycho Brahe. Authored about 30 works in mathematics and astronomy

- *Damodara* (1410-1510) was *Paramswara's* son and student. His works survive only in quotations in others.

- *Neelakanta Somayaji* (1444-1545) Student of Damodara. Author of *Tantra-Sangraha* a comprehensive treatise on astronomy and related mathematics as well as many other extremely influential books. His masterpiece was his commentary on the *Aryabhatteyam* which contains many results on calculus. *Grahapareeksaakrama* is a manual on how



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to make observations in astronomy using instruments of that time.

- *Jyeshthadeva* (c. 1500-1610) also a student of Damodara is the author of the *yuktibhasha* as well as *Drk-karana* on observations.

- *Achyuta-Pisharati* (c. 1550-1621) was a student of *Jyeshthadeva*. Discovered the technique of 'reduction to the ecliptic' ; author of *sphuta-nirnaya* and *Rasi-gola-sphuta-neeti*.

- *Melpathur Narayana-Bhatta-thiri* student of Achyuta was a mathematical linguist (vyakarana). His masterpiece is *Prkriya-sarvawom* which sets forth an axiomatic system elaborating on the classical system of *Paanini*. But he is more famous for his devotional poem *Narayaneeyam* still sung at the temple where he worked, *Guruvayoor*.

- The lineage continues down to modern times, with new Sanskrit and Malayalam texts and commentaries written as late as the nineteen fifties. But the original research ends around the time of *Narayana*.

- Manuscripts of the Kerala school can be found all over India. A measure of their influence is that many imitation texts were written with the word Kerala inserted into the title to give them an aura of authenticity.





Early Results

- The *Aryabhateeyam* (499 CE) is a text on astronomy with chapters on spherical trigonometry *gola* and mathematics *ganita*. It contained a table of sines (*jya*). The word \sin is derived from the latin *sinus* (meaning 'fold') which is a translation of *jaib* in Arabic, which in turn is a mis-transliteration of *jiva* (*jya*) the sanskrit word for chord. *Aryabhata*, the author of this work was famous throughout the ancient world. He is mentioned in Al-Biruni's *India* as the source of Arab trigonometry.

- Bhaskara's text *Leelavati* has a part called *Bijaganita* which is the source of algebra. Its Arab translations influenced *Al-Khwarizimi* who in turn is the source of the flowering of European algebra with Cardano and others.

- Bhaskara had an interesting rational approximation for \sin :

$$\sin x = \frac{4x(180 - x)}{40500 - x(180 - x)}. \quad (1)$$





The Circumference of the Circle

- It has always been of great interest to geometers and astronomers to relate the circumference of a circle to its diameter.

- The basic method has been to inscribe or circumscribe a regular polygon. The problem then is to find the side of the polygon as a multiple of the diameter.

- Approximate formulae good enough for practical purposes had been known for a long time- $\pi \approx \frac{22}{7}$ is enough for most engineers.

- Archimedes of Syracuse (287-212 BCE) obtained the value $3\frac{10}{71} < \pi < 3\frac{10}{70}$ considering by considering a regular polygon of 91 sides.

- The *aaryabhataiiya* (499 CE) gives a value accurate to four decimal places: "The circumference of a circle of diameter 20,000 is 62832": or $\pi \approx 3.1416$.

- Bhaskara (1114-1185(?) CE) says that the circumference of a circle of diameter 1250 is 3927 by considering an inscribed regular polygon of 384 sides-correspond to $\pi \approx 3.14155$. Getting close!



● These days the value of π to a thousand decimals is just two clicks away if you have Mathematica! $\pi \approx =$

3.1415926535897932384626433832795028841971693993751058209749445923078164062862 089986280348253421170679821480865132823066470938446095505822317253594081284811
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Chapter Six of the Yuktibhasha

•What is new with the Kerala school is a convergent infinite process that can give the value of π to *arbitrary* accuracy. There were several such processes known to this school, we will study in detail two of them, explained in detail in the sixth chapter of the *Yuktibhasha*.

- There are two different approaches to calculating the circumference.
- The first will give an algebraic recursion relation-involving a square root- that converges to the exact value. In modern notation,

$$x_0 = 1, \quad x_{n+1} = \frac{\sqrt{1 + x_n^2} - 1}{x_n}, \quad \pi = 4 \lim_{n \rightarrow \infty} 2^n x_n \quad (2)$$

•The second method-really a succession of improvements- goes much further. It starts as a way to avoid square roots in the calculation of the circumference.

•A finite series-whose terms depend on the number of terms in the series- is obtained which converges to the circumference as the number



of terms grows. Again in our notation,

$$\pi = 4 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left[\frac{1}{1 + \left(\frac{n}{N}\right)^2} \right]. \quad (3)$$

• We can recognize the sum as tending to $\int_0^1 \frac{dx}{1+x^2}$.

• Then this series is re-expressed in a way that the terms don't depend on the number of terms. Taking the limit this gives the fundamental infinite series

$$4D \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right] \quad (4)$$

for the circumference of a circle of diameter D .

• The integral was discovered in this context!

• Formulae such as

$$1 + 2 + 3 \dots N = \frac{N(N+1)}{2} \quad (5)$$

$$1^2 + 2^2 + 3^2 \dots N^2 = \frac{N(N+1)(2N+1)}{6} \quad (6)$$



$$1^3 + 2^3 + 3^3 \dots N^3 = \frac{N^2(N+1)^2}{4} \quad (7)$$

for powers up to four were known.

•The key step was to realize that for large N (small steps in the rectification of the circle)

$$1^k + 2^k + \dots N^k \approx \frac{N^{k+1}}{k+1} \quad (8)$$

so that in the limit we can replace

$$\sum_{n=1}^N \left[\frac{n}{N} \right]^k \approx \frac{N}{k+1} \quad (9)$$





Quotation from the Tantra-Sangraha

- Of course this modern notation was not used.
- The language is tortured in the Yuktibhasha as the arguments gets harder and harder. The final result is quite simple and is expressed in an elegant poem quoted from the Tantra-Sangraha (by *Neelakanta Somayaji*, the result is attributed to Madhava though).

• *vyaase vaaridhi-nihate ruupahrte vyaasasaagaraabhihate
thri-saradi-vishamasamkhyaa-bhaktam r.n.am svam pr.that kra-
maal karyaat*

• K. V. Sharma's translation: "Multiply the diameter by four. Subtract from it and add to it alternately the quotients obtained by dividing four times the diameter to the odd numbers 3,5 etc. "

• This is not an absolutely convergent series; even when summed in the right order it is slowly converging. The commentator to the Yuktibhasha shows that summing 27 terms gives a value accurate to one (!) decimal place.





Estimates of Error

- One can add corrections to the truncated sum which estimate the terms omitted
- In the first direction there is

$$C \approx 4D \left[1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{n} \mp \frac{(n+1)/2}{(n+1)^2 + 1} \right] \quad (10)$$

- Here is an even better formula (also attributed to Madhava in the *Kriyakumari*) for the correction to the finite sum:

$$\frac{\left(\frac{n+1}{2}\right)^2 + 1}{\left(\frac{n+1}{2}\right) \left[4 \left(\frac{n+1}{2}\right)^2 + 1 \right]} \quad (11)$$





Convergent Series for the Circumference

- Or, we can look for new series that converge.
- A result of Madhava when translated to modern language is

$$\pi = \sqrt{12} \left[1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} \cdots \right] \quad (12)$$

- **Exercise** Prove this result by modern methods. Estimate the error if this series is stopped at the n th term.
- Madhava derived using this the result that the circumference of a circle of diameter 9^{11} is 2827433388233. He also derived a way to convert the radian to the degree.
- The Yuktibhasha also gives many rational approximations which have no parallel in modern mathematics. They are based on continued fractions and I have not been able to decipher them yet.





The Arctangent

•A poem of Madhava is quoted in the Yuktibhasha which gives the arc of the circle in terms of the ratio of jya (sin) and the koti (cos). (Remember that these quantities are proportional to the radius.)

•Based on a translation of K. V. Sharma: Multiply the jya by the trijya and divide the product by the koti. Multiply this by the square of the jya and divide by the square of the koti. We get a sequence of further results by repeatedly multiplying by the square of the jya and dividing by the square of the koti. Divide these in order by the odd numbers 1,3,5 and so on. Add the odd terms and subtract the even terms (preserving the order of the terms). This gives the dhanus (arc literally, bow) of these jya and koti. Here the smaller of the two sides should be taken as the jya as otherwise the result will be non-finite.

•If the jya is s and the koti is c and the trijya (radius) is R , we have

$$\frac{sR}{c} - \frac{1}{3} \frac{sR}{c} \left[\frac{s}{c} \right]^2 + \frac{1}{5} \frac{sR}{c} \left[\frac{s}{c} \right]^4 - \frac{1}{7} \frac{sR}{c} \left[\frac{s}{c} \right]^6 + \dots \quad (13)$$



●If we put $\frac{s}{c} = t$ as the tangent and measure the arc in units of the radius (as we would in modern notation) this is the infinite series for the arctangent:

$$t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \dots \quad (14)$$

Obtained a couple of centuries before Gregory after whom this series is named!

●Madhava also obtained the infinite series for \sin .



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Religion in Kerala

- The people of Kerala today belong to the three major religions: Christianity(20%) , Islam(20%) and Hinduism(60%).
- The proportions were different in the time we are speaking of: there was a small and ancient Christian church, founded by the Apostle St. Thomas himself if we are to believe in the legends.
- There was a tiny but vibrant jewish community. There were some converts into Islam along the coastal regions.
- But the vast majority of people followed the traditional religion of India known there simply as the 'Old Ways': the Sanaadhana Dharma. The Persians called the followers of this religion 'Hindus'- derived from their name for the Sindhu (Indus) river- which now is used even in India to describe them.



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The Hindu Religion-*Sanatana Dharma*

- Unlike modern religions (such as Buddhism, Christianity or Islam) Hinduism does not have a unique founder. Like Judaism it is a system of practices handed down from time immemorial.

- The basic spiritual texts are the four Veda, perhaps the oldest surviving texts of mankind. The word Veda simply means 'the knowledge'. These are supplemented in later times by the epics (puraana) (mainly the Ramaayana, MahaBhaaratha and the Bhaagavata) ; embedded in the epics are several important texts such as the Bhagavat Giita and the Yoga Vasishta. This classical literature is supplemented by the commentaries of saints the most important of whom is Sankara Achaarya.

- The Veda are a sort of encyclopedia of ancient knowledge. In addition to the hoary philosophy of the Upanishads, the Veda also contain the ancient rules of human behavior and of course, hymns and prayers.

- Although in its core Hinduism is not about Gods, but about a supreme existence of which we are all a part, much of the religious



practice has to do with a multitude of Gods: each of which represent an aspect of this supreme reality.

- In the Veda, many of the Gods are identified with natural phenomena: the Sun, the Moon and the planets are minor Gods the major ones being Indra, Vishnu, Siva and so on.

- It is difficult to convey that there is an essence to Hinduism that lies beyond the Gods to those from another cultural and religious background: it often looks like a bewildering array of colorful, even scary images connected together by fantastic legends, much like the pre-Christian religions of Europe. However, abstract notions of the impersonal infinite are still quite familiar to Hindus and the abstractions of mathematics were often derived from this common religious background.





A Glimpse of Infinity

●As an example, here is a verse from the Isaavasya Upanishad of the Yajur Veda that many of us repeat daily even today:

*purnamadah purnamidam purnaat purnamudachyate
purnasya purnaamadaya purnameva vashishyate*

That is the Universe, This is the Universe,
The Universe arises from itself, it is said
If you subtract the Universe from the Universe
There remains the Universe, indeed.



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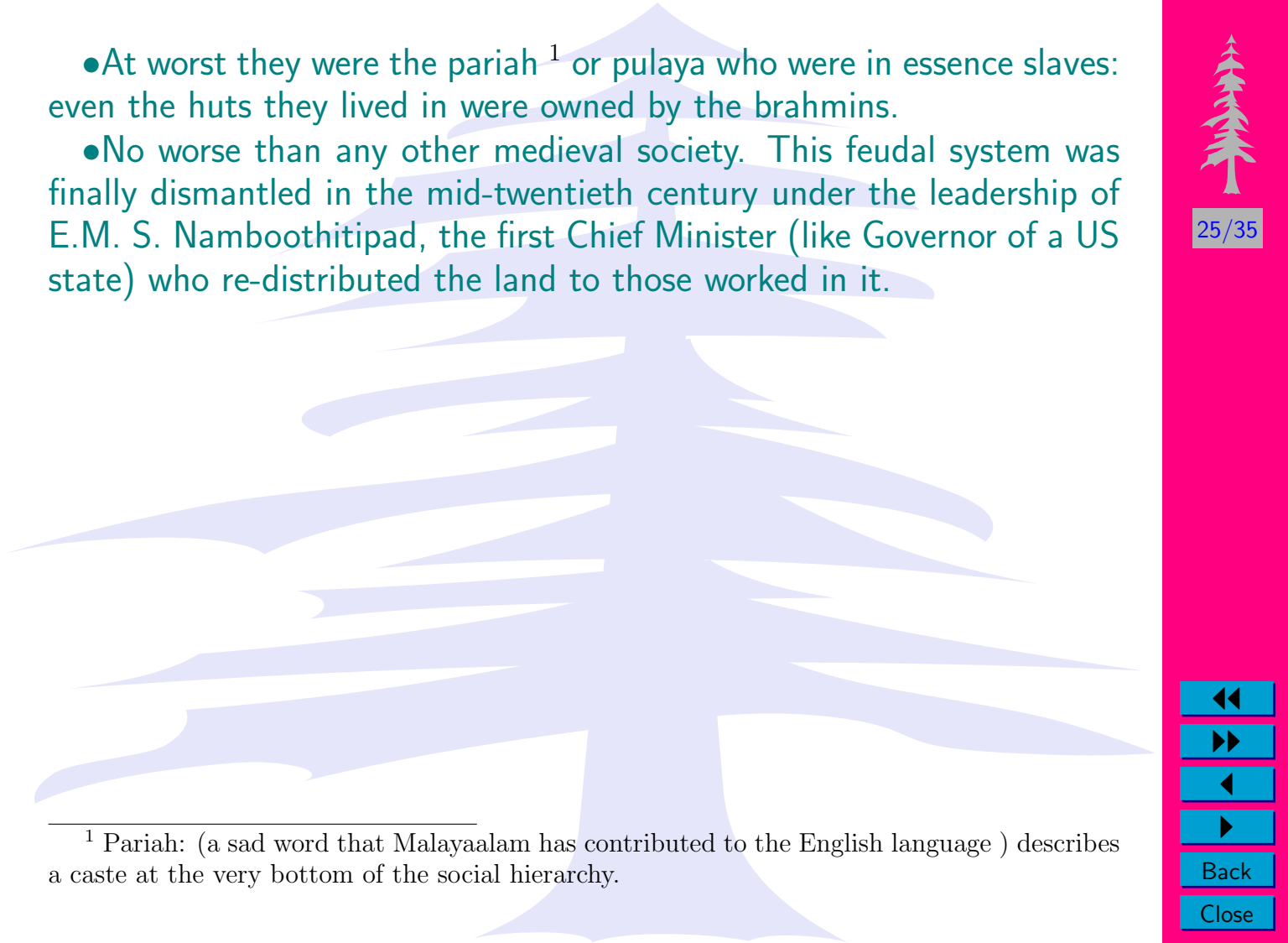
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Namboothiri: The High Priests

- Most of the astronomer-mathematicians of Kerala were *Namboothiris*, the highest ranking priests of the Hindu religion.
- They had the highest social status in the society, higher than the King.
- Although less than a half percent of the population, they controlled most of the wealth through land ownership.
- What they did not own out-right they controlled through the temple-trusts which were managed by them.
- The people who toiled in the fields (rice farming is very labor intensive) had no ownership of the land or its produce.
- Yet the enormous wealth that these rice plantations produced were entirely the product of their labor: without constant toil the land would have decayed out of over-cultivation in just a few years.
- At best, they were tenants who paid the landlords regularly for the privilege of cultivating the land.





- At worst they were the pariah¹ or pulaya who were in essence slaves: even the huts they lived in were owned by the brahmins.

- No worse than any other medieval society. This feudal system was finally dismantled in the mid-twentieth century under the leadership of E.M. S. Namboothiripad, the first Chief Minister (like Governor of a US state) who re-distributed the land to those worked in it.

¹ Pariah: (a sad word that Malayaalam has contributed to the English language) describes a caste at the very bottom of the social hierarchy.





How did the *namboothiris* Dominate ?

- The namboothiris are believed to have emigrated down the West coast of India starting with the fourth century CE. In any case by the ninth century every aspect of Kerala society came under their firm control.

- There are some sub-castes within the namboothiris who were later immigrants from *Thulu* region just to the North of Kerala. Madhava belonged to one of them: he was an *embranchiri*.

- The key to the dominance of the *namboothiris* was that they were the keepers of the sacred Hindu scriptures, the *Veda*.

- It was their sacred duty to perform the vedic sacrifices to maintain the harmony of the universe.

- The title Somayaji denoted someone who had performed one of the most difficult and ancient of these rituals, the *Soma-yaga*.

- Being the only allowed priests, they controlled the temples which were also the only centers of learning, art and culture.



- Each village had a small standing army of *Nairs* who protected the temple.

- The temples and could not be taxed; indeed the King was dependent on them even to raise an army.





Simplicity, Dedication, Discipline

- The life of a *Namboothiri* was comfortable but by no means luxurious. There are no castles or opulent palaces in Kerala.
- They led a life of scholarship and spirituality.
- Food was strictly vegetarian. No alcohol or other intoxicants were allowed. Even strong tastes like onions were forbidden as it could inflame sexual appetite.
- Fasts on the eleventh day of each half of the lunar cycle .
- Servants were not allowed to cook food; it was the domain of the *nambothiri* women.
- Exercise was built into the spiritual practices; e.g., the prayer to the Sun God is an excellent aerobic workout. You can learn it in yoga classes in the US. They walked everywhere.
- They lived in large joint families (many unmarried adult brothers and sisters, the children of the oldest brother, the grandparents and sundry dependents) in a simple single story home.





- The women rarely travelled outside: the Malayalam word *antarjanam* for a Namboothiri woman means 'woman inside' (the house).

- The houses had one or two internal courtyards without a roof and was built either as a square or double square around them. (There is some resemblance to floor plan of the Roman Triclinium.)

- The roof was tiled, or in the older days a thatched with coconut leaves. This kind of roof had to be replaced annually. The floor was bare, just a mud: no form of cement or marble was used.

- There would be a shed some distance away for the cows and a sizeable stack of hay to feed them.

- Dress was also very simple: a piece of cotton cloth around the waist, a towel around the shoulders. A thread around the body indicating *brahmin* status. Even on the most formal occasions, men were bare chested. Marco Polo was aghast at that!

- Here is a picture of the High priest during the year 2003-2004 of the biggest temple in Kerala (Ayyappa kshetram): this was the typical dress of a *namboothiri* of that time.

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The Education of a Namboothiri

- The education of a Namboothiri boy was intense, deep and broad.
- He started at the latest by the age of eight and continued at least to age sixteen.
- The teacher was often a family member, an uncle or father or grandfather.
- Every morning started before sunrise (always 6:00 am in these tropical parts) with prayers to the Sun God, Suurya. After a short break for lunch it would continue till sundown.
- The center piece of the education was the learning of the Veda. No other caste had the right to learn the Veda.
- The Veda could not be written down, the entire corpus had to be memorized. Each family inherited a piece of the Veda assigned to it according to tribal succession laws and passed it on to the next generation.
- The smartest boys learned the hardest and most abstract kind of



knowledge: the Upanishads, about the nature of knowledge itself, that ultimate knowledge from which all else follows.

- Even the dumb ones at least had to memorize the Veda by rote without understanding its meaning. The chanting of the Veda is the ultimate duty of the Namboothiri. Each of them were in essence a walking library of ancient knowledge.





Error Correction by Redundancy

- To make sure that no error would creep into the oral transmission of the Veda, there were intricate error correction techniques built into this rote learning.

- You would learn the Veda not only as if read from left to right, but also in the reverse order. Which would make equal sense to someone who doesn't understand the ancient Sanskrit to begin with. Not only that you would memorize each verse by taking a syllable from the middle then one to the left then one to the right and so on.

- This redundancy as well as the redundancy in the large number of people who learned the Veda compensated for the volatility of human memory. Indeed the Veda are extremely remarkably well preserved: you can compare the Veda as recited by a Namboothiri brahmin to a Kashmir pundit. There would be no difference not only in literal content but also in the pronunciation and rhythm of the singing.

- There are greater disputes over Shakespeare's writing than over the



text (*samhita*) of the Veda: in spite of the former being printed and much more recent. As the older generation who received this classical education die out, there is now the danger of entire branches of the Veda dying out with them. There is an ongoing project to record the chanting of the Veda before this happens.

- There were annual competitions in the recitation of the veda. Such a competition (*anyonyam*) still continues but at a much smaller scale.
- The various city states and temples competed to attract well known scholars to stay in residence. In return the scholars were expected to compose some salutary verses honoring the local ruler (Prasasthi) which they completed with some grumbling.
- Secular subjects such as poetics, rhetoric, grammar, logic, astronomy (of which mathematics is a part), medicine occupied an important place in the education, but were considered distinctly inferior to the study of the Veda.
- Each person has a guru or teacher responsible for his overall education; although occasionally there might be more than one teacher when someone has expertise in several areas.
- In this respect advanced education today at the level graduate school



holds a remarkable similarity to this ancient system. But the guru was often a relative: an uncle or ones father.

- The guru was held in the highest regard, indeed as a form of divinity.
- The guru was responsible for spiritual and moral development as well as education.In return the student was to obey and protect the guru for life.
- The word guru means literally 'heavy' or ponderous. In astronomy, guru is also the name of the planet that we call Jupiter in English: because of its ponderous motion across the sky with a period of twelve years. In mythology this guru represents the teacher of the Gods, a play on the meaning of the word.
- The schools or Madhams survived down to the early twentieth century. At this time the Namboothiris started to suffer from the lack of an English education. There was a popular reform movement which allowed the Namboothiris to adapt to the modern world.
- Now they have melted into the emerging vast Indian middle class as professionals: teachers, doctors, scientists and immigrants to the United States..

