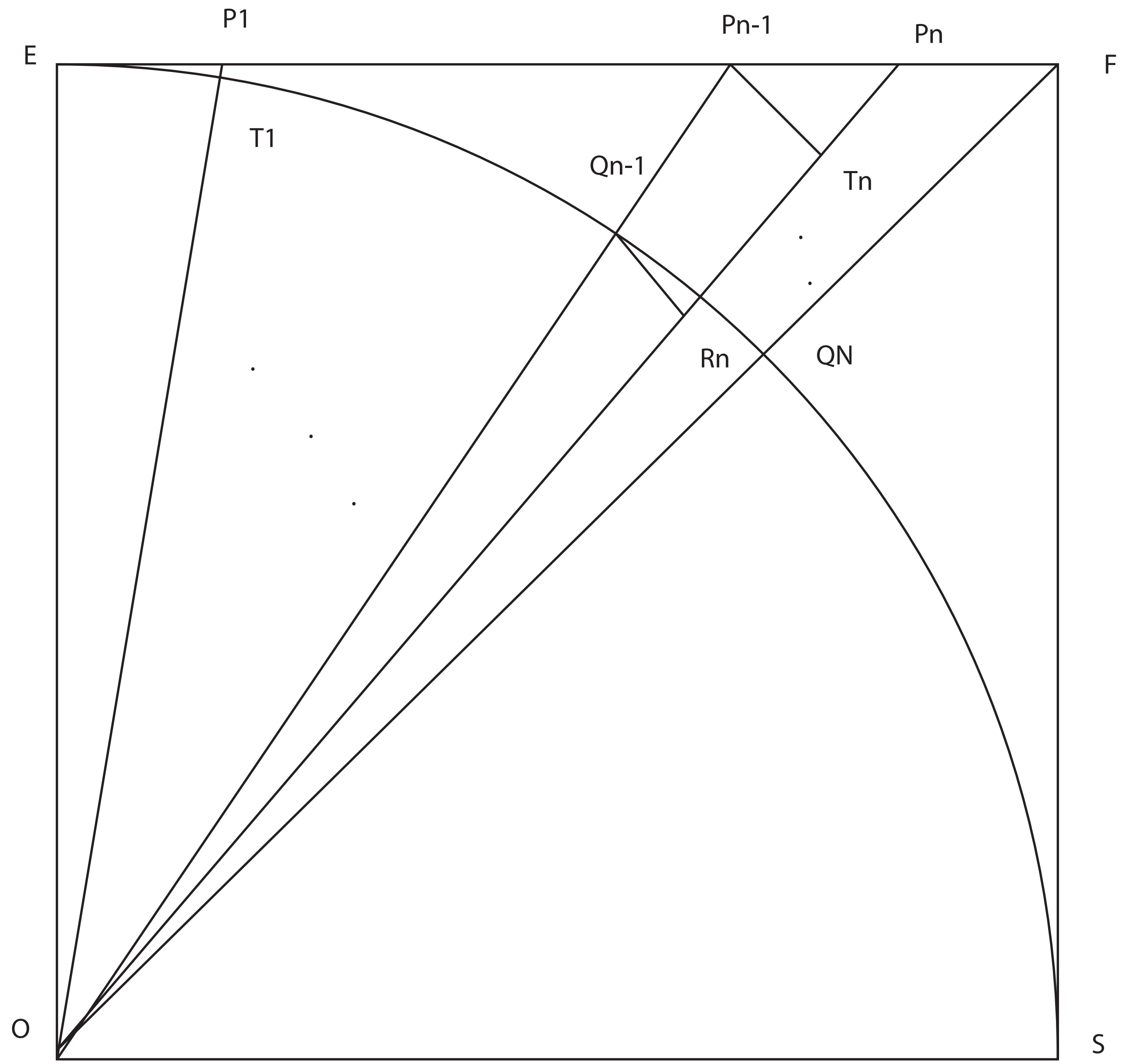
P1
Pn-1
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E
F
T1
Qn-1
Tn
Rn
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S

Prepared using the Latex package *pdfslide* written by C. V. Radhakrishnan of Trivandrum,India

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Neither Newton nor Leibnitz:
The Pre-History of Calculus in Medieval Kerala

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Circumference without Square Root

We might prefer a rational formula that avoids the calculation of square roots. The price for this will be much slower convergence. Still it leads to an entirely different approach to the calculation of the circumference, based on infinite series and integrals. The series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

is derived by an ingenious limiting procedure.

The first steps are similar to those above.

Now we will see how to find the circumference of a circle without using square-roots.Imagine a square whose four sides are equal to the diameter of the circle; and inside it imagine a circle. The arc of the circle should touch the sides of the square. Then, through the center of the circle imagine the East-West and North-South diameters (literally *suutra*, thread) connecting the arc of the circle which is also the midpoint of the sides of the square. Then from the East radius tip to the top-right hand corner will be equal to the half-diameter. There, mark some points, close to each other, the division calculated so that they are more or less equally separated. The more such points there are the more accurate will be the perimeter. Then from the circle-center,ending at these points, construct hypotenuses. The East radius will be the common height (of triangles). The line from the East radius to the tip of the hypotenuses will be the arms (of these triangles). There, the closest hypotenuse to the South of the East radius will have arm equal to one step ; the second hypotenuse will have arm which is two steps long. This way each successive hypotenuse will have arm one step more. Thus the corner of the square will have the longest arm of all. The height of all these hypotenuses is just the East-radius which is the half-diameter. Therefore the square root of the sum of the squares of the half-diameter and the respective arm is equal to the respective hypotenuse.

We are to imagine here a series of right angled triangles, with the

East radius as a common side and the other side increasing length by one interval as we move to the square-corner.

Then the distance from the East-radius-tip to the tip of the nearest hypotenuse multiplied by the half-diameter and divided by the first hypotenuse is the distance from the East-radius-tip to the first hypotenuse, orthogonal to it. There is a triangle with this line as the height . The arm of this triangle is from the intersection of this edge with the first hypotenuse to the tip of the first hypotenuse. The hypotenuse of this (new) triangle will the distance from the East-radius-tip to the original hypotenuse-tip on the side of the square. This is the desired desired (i.e., to be determined) part of a pair similar triangles. The reference triangle this is given next: the East radius is the height , the original hypotenuse-line is the hypotenuse, the distance between the hypotenuse-tips is the arm. This reference triangle is similar in shape to the desired triangle. The reason: the arm of the reference triangle is parallel to the hypotenuse of the desired triangle,the arm of the desired triangle in parallel to the reference hypotenuse. Then, the East radius which is the height of the reference triangle is orthogonal to the hypotenuse of the desired triangle which is in turn the common interval on the square-side. The long arm of the desired triangle is orthogonal to the hypotenuse of the reference triangle as well. This is the reason why the two triangles are similar shaped. Thus here for the two triangles the arm and the height are mutually parallel, the hypotenuse and the long arm are orthogonal, which makes them similar in shape. If the three sides are either parallel or perpendicular they will be similar shaped.

(We skip a short passage using analogies with carpentry.)

It is easy to see that two triangles are similar when the corresponding sides are parallel. It is argued here that the same is true if the corresponding sides are perpendicular as well. The author then gives an analogy with the beams of a roof.

Then there is a third triangle here: For this, the hypotenuse becomes the East-radius. The perpendicular from the East-radius to the first hypotenuse is its arm, which is also the long-side of the previous desired-triangle.The segment from the meeting point of this arm and the first hypotenuse to the circle-center is the long-side. So that is it.

The arm is the line connecting the East radius to the first diagonal. The segment of the original hypotenuse from the center of the circle to the meeting point of this arm and the original hypotenuse is the height .

Then there is a second reference triangle. Its long-side is the East-radius itself. From the tip of the radius two intervals on the square-side form the arm. The second hypotenuse from the circle-center is the hypotenuse (of this second triangle). This is the second reference triangle. Then its desired similar triangle: the height is the perpendicular from the first hypotenuse-tip to the second hypotenuse. The arm is from this intersection to the tip of the second hypotenuse. The second interval on the square-side is the hypotenuse. This is the desired triangle of the second similarity. (Omit some analogy to roofs.) Now multiply the second interval on the square-side by the radius which is the reference side divide by the second hypotenuse. The result is the height of the second desired triangle. Here too there is a third similarity: we can imagine this height to be the arm and line from its meeting point to the circle-center on the second hypotenuse to be the height ; the first hypotenuse as the hypotenuse.

This way, starting from the radius-tip to the the corner of the square, each piece of the square-side determines three similarities of triangles. There, starting from the radius-tip to the square-corner, multiply each interval by the radius and divide by the larger of the hypotenuses touching the boundary of the interval;the result is the perpendicular connector from the tip of previous hypotenuse to the current one.These are the height s of the desired triangle (of the similarity). These become then the arms (of the next similarity). The segment of the long hypotenuse starting from the intersection with this arm to the circle-center is the height . Then,the smaller of the two hypotenuses starting from the center and touching each square-side-segment is the hypotenuse (of the next similarity triangle).Thus exists certain similarities of triangles. For these there are certain reference triangles. The desired triangles of these reference triangles which are themselves imagined in the interior of the circle. Here the radius, which is almost the reference hypotenuse, is the desired one. The perpendicular from the tip of this radius to the large hypotenuse is the desired result. This way the perpendicular to each hypotenuse, the above result, becomes the Sine of the corresponding arc. Then, the intervals of the square-side starting from the East-radius multiplied twice by the corresponding half-diameter divided by the product of the two hypotenuses gives the Sine of the corresponding angle. Here as the square-side-intervals become small these Sines will become equal to the arc of the circle .

There,starting from the East-tip, divide by the square of the hypotenuses touching the North end of the intervals. First, dividing the radius by its square since the factor is itself, gives just the interval. The last hypotenuse is the diagonal. When divided by its square the result will be half the interval.For, twice the square of the radius is the last square of the last hypotenuse. Also, when a fraction, whose denominator twice the numerator, is multiplied by a quantity we just get half of it. Of all the intervals, only these two touch the first and last ends.

Back
Close

The sum over the reciprocals of the square of the first hypotenuses and the same of the second hypotenuses differ by the difference between the first term of the first sum and the last term of the second sum. These will be the half the interval. The middle terms are equal as they have the same denominator. The second term up to the penultimate term is the same (for the two series). The division by the first denominator is the interval itself and the division by the last denominator is half the interval. If we were to divide always by the square of the hypotenuses, the difference would be a quarter of the interval. When the intervals becomes small, its quarter can be dropped. Therefore we just have to use the square of the hypotenuse as the denominator.

Let us translate this into modern mathematical language. Draw a square of side equal to the radius of the circle having the center of the circle as one corner O, touching the circle at the points E (East) and S. We imagine E to the top of the page and S to the bottom right, following the Indian convention. The top right hand corner F is the 'fire corner' (a reference to the construction of Vedic sacrificial altars.)

Thus a quarter of the circle is contained in this square $OEFS$. Draw the diagonal OE.

Now choose points $P_1, P_2, \cdots P_N$ on the side EF, such that the intervals $P_{n-1}P_n$ are more or less equal and small. The larger number the of points and the closer they are, the more accurate will be our estimate of the circumference of the circle. Draw the lines $OP_1, OP_2, \cdots$ which form the hypotenuses of right triangles with the radius as a common (the longer) side OE. They will interest the circle at the points $Q_1, Q_2, \cdots$. Corresponding to the nth such interval on EF, we drop perpendiculars $P_{n-1}T_n$ and $Q_{n-1}R_n$ to the line OP_n.

The circumference will be approximated by the sum of the segments $ER_1 + Q_1R_2 + Q_2R_3 + \cdots$. To determine the distance $Q_{n-1}R_n$, we notice that the triangles $OQ_{n-1}R_n$ and $OP_{n-1}T_n$ are similar; for they are right triangles with one common angle. Similarly the big triangle OP_nE is similar to the little one $P_{n-1}P_nT_n$. Thus we get the similarities

$$Q_{n-1}R_n = OQ_{n-1}\frac{P_{n-1}T_n}{OP_{n-1}}, \quad P_{n-1}T_n = OE\frac{P_{n-1}P_n}{OP_n}.$$

Since $OE = OQ_{n-1} = r$, the radius of the circle,

$$Q_{n-1}R_n = P_{n-1}P_n\frac{r^2}{OP_nOP_{n-1}}$$

There if the square-side is equally divided, the first factors are equal and the other factor, square of the radius is anyway the same. The divisors, being the product of the two hypotenuses of each interval are different. In this situation, this product of hypotenuses can be thought of as (can be approximated by) half the sum of their squares, since the numbers are almost equal. Now, if we divide by each hypotenuse squared, add the results and halve it. This will be equal to the division by half the sum of the squares.

Now suppose that all the intervals $P_{n-1}P_n$ are equal to h. Then

$$Q_{n-1}R_n = hr^2\frac{1}{OP_nOP_{n-1}}$$

When these intervals are small, we can approximate

$$\frac{1}{OP_{n-1}OP_n} \approx \frac{1}{2}\left[\frac{1}{OP_{n-1}^2} + \frac{1}{OP_n^2}\right]$$

Thus the sum

$$\sum_{n=1}^{N} Q_{n-1}R_n \approx \sum_{n=1} hr^2 \frac{1}{2}\left[\frac{1}{OP_{n-1}^2} + \frac{1}{OP_n^2}\right]$$

is our approximation for the arc of the circle EQ_N, which is an eighth of the whole circumference.

Now let us rewrite the sum so that the square of the first hypotenuse is the denominator. Let us collect the term involving the first hypotenuse of each interval

$$hr^2 \frac{1}{2}\left[\frac{1}{OE^2} + \frac{1}{OP_1^2} + \cdots \frac{1}{OP_{N-1}^2}\right]$$

and the second hypotenuse:

$$hr^2\frac{1}{2}\left[\frac{1}{OP_1^2}+\cdots\frac{1}{OP_{N-1}^2}+\frac{1}{OF^2}\right]$$

since $P_0 = E$ and the last point is the corner $P_N = F$. Except for the first and last term, each such term appears twice in the sum. So we get for the eighth of the circumference

$$\frac{C}{8}\approx\frac{1}{2}h\frac{r^2}{OE^2}+hr^2\frac{1}{2}\left[\frac{1}{OP_1^2}+\cdots\frac{1}{OP_{N-1}^2}\right]+\frac{1}{2}h\frac{r^2}{OF^2}$$

But $OE = r$ and the first term is just $\frac{1}{2}h$. The square of the diagonal is twice that of the radius so the last term is $\frac{1}{4}h$.Together these make $\frac{3}{4}h$. Thus we can just sum over the hypotenuses, including the last one, if we subtract a quarter of the interval. In the limit of small interval, we can ignore this quarter of the interval. Thus

$$\frac{C}{8}\approx\sum_{n=1}^{N}h\frac{r^2}{OP_n^2}.$$

By Pythagoras theorem,

$$OP_n^2 = r^2 + n^2h^2.$$

so that

$$\frac{C}{8} \approx \sum_{n=1}^{N} h \frac{r^2}{r^2 + n^2h^2}.$$

If there are N divisions, $h = \frac{r}{N}$. Also nowadays we denote $\pi = \frac{C}{2r}$. Thus our result is

$$\pi \approx 4\frac{1}{N}\sum_{n=1}^{N}\left[\frac{1}{1+\left(\frac{n}{N}\right)^2}\right].$$

As the number of subdivisions grows this sum will tend to the exact answer:

$$\pi = 4\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left[\frac{1}{1+\left(\frac{n}{N}\right)^2}\right].$$

We thus get a sequence of rational numbers that tend to π as $N \to \infty$: already an interesting result. The sum tends to the integral

$$\pi = 4 \int_0^1 \frac{dx}{1+x^2},$$

of course.