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Neither Newton nor Leibnitz: The Pre-History of Calculus in Medieval Kerala

S. G. Rajeev

University of Rochester

email: rajeev@pas.rochester.edu

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The Recursion as a Dynamical System

Given a map $f : C \rightarrow C$ of the complex plane to itself and an initial point x_0 , it is possible to define a dynamical system See *John Milnor* by the recursion relation

$$x_{n+1} = f(x_n).$$

Our example is of course the case $f(x) = \frac{\sqrt{1+x^2}-1}{x}$, $x_0 = 1$ which converges to 0. what would have happened if we chose a different initial condition? Would it still converge to zero? Does the limit $\lim_{n \rightarrow \infty} 2^n x_n$ exist for any choice of x_0 and if so what is it as a function of x_0 ?

If we look through the derivation of the recursion earlier we will see that angle between the East-radius and the nearest corner of the regular polygon is being halved at each step of the iteration. Moreover half the side of the polygon is the tangent of this angle multiplied by the radius.



Hence

$$x_n = \tan \theta_n, \quad \theta_{n+1} = \frac{1}{2}\theta_n.$$

is just as good a way of thinking of the iteration. Moreover, at the beginning of the iteration when we have a square, $\theta_0 = \frac{\pi}{4}$ which is why $x_0 = 1$. Since

$$\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

we have

$$x_n = \frac{2x_{n+1}}{1 - x_{n+1}^2}.$$

Inverting this, which involves solving a quadratic equation,

$$x_{n+1}^2 x_n + 2x_{n+1} - x_n = 0$$

we get the recursion relation above:

$$x_{n+1} = \frac{-2 \pm \sqrt{4 + 4x^2}}{2x_n} \Rightarrow x_{n+1} = \frac{\pm \sqrt{1 + x_n^2} - 1}{x_n}.$$



We choose the root that gives a positive value for x_n , as it is supposed to be the half-length of a side of the regular polygon.

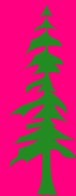
Thus the change of variables $x_n = \tan \theta_n$ has reduced our recursion relation to something very simple:

$$\theta_{n+1} = \frac{1}{2}\theta_n.$$

There is no doubt that this converges: for any choice of initial θ_0 , $\theta_n = 2^{-n}\theta_0$ which tends to zero. Moreover, $2^n x_n = 2^n \tan \theta_n \approx 2^n \theta_n$ since for large enough n , the angle will become small enough. Thus the sequence always converges to zero and the limit $\lim_{n \rightarrow \infty} 2^n x_n$ does always exist:

$$x_{n+1} = \frac{\sqrt{1 + x_n^2} - 1}{x_n} \Rightarrow \lim_{n \rightarrow \infty} 2^n x_n = \arctan x_0.$$

Thus this recursion relation is a way to calculate the value of arc-tangent.





The Circumference of an Ellipse

Considerably more complicated is the problem of determining the circumference of an ellipse. This problem was not solved until mid-nineteenth century, at the zenith of European mathematics and captured the attention of the best minds: Euler, Legendre, Gauss, Abel and Riemann made important contributions to the resulting theory of *elliptic functions*.

An *ellipse* is the curve determined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If $a = b$, this is just a circle of radius a . So we can assume that $a > b$ for a true ellipse. It is useful to define $k^2 = 1 - \frac{b^2}{a^2}$. It measures the departure of the ellipse from being a circle: if $k = 0$, we have $a = b$ which is a circle. The opposite extreme is $k = 1$ is the case where the



Back

Close



ellipse is so eccentric it has reduced to a straight-line.

Let us see how far the geometric ideas as of the *yuktibhasha* can be stretched to understand the ellipse. Of course there is no reason to believe that the medieval mathematicians went in this direction. Still, it is interesting to see if their ideas tell us something about the circumference of the ellipse.

This time we draw a rectangle circumscribing the ellipse, so that its sides are $2a$ and $2b$. Draw the again the bisectors dividing the rectangle into four smaller ones, each of sides a and b . Again think of the upper right hand quadrant. Let O be the center of the ellipse, E and S its point of contact with the rectangle. Let the remaining vertex of the quadrant be called F . The perimeter of the rectangle is our first approximation for the circumference of the ellipse:

$$C_0 = 4(a + b) = 4a[1 + \sqrt{1 - k^2}).$$

It will be convenient to call $F = X_0, S = Z_0$ as we will try to get a sequence of points by a recursive procedure. Note that the tangent to





the ellipse at Z_0 meets the tangent through E at the point X_0 . Draw the line OX_0 . Call its meeting point with the ellipse Z_1 . Now draw the tangent through Z_1 , let it meet the tangent through E at X_1 .

In the n th step of the iteration we draw the tangent at Z_n , and determine the point X_n where it meets the tangent through E . Then we draw the line OX_n and define Z_{n+1} to be where it meets the ellipse. Thus we get a sequence of points on the tangent through E , X_0, X_1, \dots that start at F and tend to E .

We will have to find a formula that gives the co-ordinates of Z_{n+1} in terms of those of Z_n . It will be convenient to use the polar and cartesian co-ordinates although the argument can be phrased in the language of pure geometry easily. Let θ_n be the polar co-ordinate of Z_n measured from the axis OS in a counterclockwise direction. Then the cartesian co-ordinates of Z_n (centered at O and having OS as the ordinate) are

$$Z_n = (a \cos \theta, b \sin \theta_n)$$



The points on the tangent at Z_n are parametrized by a real number s :

$$(a \cos \theta_n, b \sin \theta_n) + s (-a \sin \theta_n, b \cos \theta_n).$$

The point X_n where it meets the tangent at E is determined by

$$b = b \sin \theta_n + sb \cos \theta_n, \Rightarrow s = \frac{1 - \sin \theta_n}{\cos \theta_n}.$$

Then EX_n , which is the ordinate of X_n is

$$a \cos \theta_n - sa \sin \theta_n = a \frac{1 - \sin \theta_n}{\cos \theta_n}$$

after a short calculation. Thus the polar co-ordinate of X_n , which is also that of Z_{n+1} is

$$\tan \theta_{n+1} = \frac{b \cos \theta_n}{a(1 - \sin \theta_n)}.$$

This is the recursion relation determining Z_{n+1} from Z_n . Equivalently, this is the algebraic recursion relation

$$\cot \theta_{n+1} = \frac{a \sqrt{1 + \cot^2 \theta_n} - 1}{b \cot \theta_n}.$$



Since $EX_n = b \cot \theta_{n+1}$ we can also write this relation as

$$EX_{n+1} = a \frac{\sqrt{b^2 + EX_n^2} - b}{EX_n}.$$

The recursion starts with $Z_0 = S$ which has $\theta_0 = 0$ which gives $\tan \theta_1 = \frac{b}{a}$ and $EX_0 = a$.

Exercise Derive this recursion relation without using co-ordinate geometry.

Notice that this is a natural generalization of the formula for the case of the circle. Unlike in the case of the circle, the perimeter is not just a multiple of this length, as the polygon circumscribing the ellipse is not regular. We need to determine its other sides.

Let us examine the first step of the iteration. Here we drew the tangent at Z_1 meeting EF at X_1 . If we extend it in the other direction, this tangent will meet the previous tangent through Z_0 at some point Y_1 . If we cut out the triangle $X_1X_0Y_1$ we will get a quarter of an



octagon that circumscribes the circle. Its perimeter is

$$C_1 = 4(EX_1 + X_1Z_1 + Z_1Y_1 + Y_1Z_0)$$

We already know $EX_1 = a \frac{\sqrt{a^2+b^2}-b}{b}$. We can find ¹ $X_nZ_n = s \sqrt{a^2 \sin^2 \theta_n + b^2 \cos^2 \theta_n}$ so that

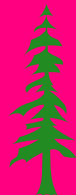
$$\begin{aligned} \frac{X_nZ_n}{EX_n} &= \frac{\sqrt{a^2 \sin^2 \theta_n + b^2 \cos^2 \theta_n}}{a} \\ &= \frac{b}{a} \sqrt{\frac{a^2 + EX_{n-1}^2}{b^2 + EX_{n-1}^2}} \end{aligned}$$

after some algebra. This gives

$$X_1Z_1 = \sqrt{2} \left(a - \frac{ab}{\sqrt{a^2 + b^2}} \right)$$

The lines Z_1Y_1 and Y_1S are obtained by reflection around the diagonal OF of Z_1X_1 and EX_1 . Thus their lengths are given by the same

¹ There is no extra work to do the general case, although for now only need $n = 1$.



formulae except for the interchange $a \leftrightarrow b$:

$$Y_1 S = b \frac{\sqrt{a^2 + b^2} - a}{a}, \quad Z_1 Y_1 = \sqrt{2} \left(b - \frac{ab}{\sqrt{a^2 + b^2}} \right)$$

Thus our octagonal approximation to the perimeter of the ellipse is

$$C_1 = 4 \left(a \frac{\sqrt{a^2 + b^2} - b}{b} + \sqrt{2} \left(a - \frac{ab}{\sqrt{a^2 + b^2}} \right) + a \leftrightarrow b \right).$$

That is

$$C_1 = 4 \left(\left[\frac{a}{b} + \frac{b}{a} \right] \sqrt{a^2 + b^2} + (\sqrt{2} - 1)(a + b) - \frac{2\sqrt{2}ab}{\sqrt{a^2 + b^2}} \right)$$

Exercise Derive this result yourself.





The Arithmetico-Geometric Mean

The idea of calculating transcendental quantity by iterating an algebraic formula is very much alive in modern mathematics. We will illustrate this by determining the circumference of an ellipse, a substantially more complicated problem than for the circle. The basic idea goes back to Gauss but we give a simplified description, looking at an elliptic integral of the first kind rather than the second kind that which gives the circumference of the ellipse².

The formula $\frac{\pi}{2} = \int_0^{\infty} \frac{dx}{1+x^2}$ is easily established by the substitution $x = \tan \theta$. A generalization of this which arises in the theory of elliptic

² determine the circumference of an ellipse using these ideas, we have to solve an inhomogeneous linear recursion relation. The function studied here also determines the period of oscillation of a pendulum



functions is

$$G(a, b) = \int_0^{\infty} \frac{dx}{\sqrt{(a^2 + x^2)(b^2 + x^2)}}.$$

We can always choose $a \geq b > 0$. Clearly $G(a, a) = \frac{\pi}{2a}$. The substitution $y = x + \sqrt{x^2 + ab}$ proves (after a long calculation) that

$$G(a, b) = G(a_1, b_1), \quad a_1 = \frac{a + b}{2}, \quad b_1 = \sqrt{ab}.$$

That is, $G(a, b)$ is unchanged if we replace a by the arithmetic mean and b by the geometric mean of a, b . Thus we can calculate the integral by repeated applications of this procedure: the two means will approach each other and tend to some common value $M(a, b)$ which is called the arithmetico-geometric mean. In that limit we know the value of the integral since it reduces to the circular case. Thus

$$G(a, b) = \frac{\pi}{2M(a, b)}.$$



The iteration

$$a_{n+1} = \frac{1}{2}[a_n + b_n], \quad b_n = \sqrt{a_n b_n}, \quad a_0 = a, \quad b_0 = b$$

determines the arithmetico-geometric mean:

$$M(a, b) = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$$

The convergence of this iteration is quite fast. The worst case is when a and b are far apart. But the extreme case $a = 1$ and $b = 0$ doesn't converge. But if $a = 1$ and $b = 0.01$,

$$\begin{aligned} a_1 &= .505, & b_1 &= .1, \\ a_2 &= .3025, & b_2 &= .31622, \\ a_3 &= 0.30936, & b_3 &= 0.309287; \end{aligned}$$

only after three iterations, they already agree to four decimal places!

Exercise Find $G(1.0, 0.001)$ to an accuracy of four decimal places.

