J Peters' lecture notes

II The univariate Bernstein-Bézier form of a polynomial on the interval $[0, 1]$

The Bernstein basis functions of degree d are

$$B_{d-i,i} : u \mapsto \binom{d}{i}(1-u)^{d-i}u^i.$$

When the degree is understood, we drop the subscript $d - i$. A polynomial of degree d in Bernstein (Bézier) form has the representation

$$\sum_{i=0}^{d} c(i)B_i.$$

If $c(i) = f(i/d)$ for some continuous function f then the polynomial is called *Bernstein polynomial of* f. This special type of polynomial in Bernstein form popular in analysis will only be of marginal interest in the following. We are interested in the polynomial piece traced out when $u \in [0, 1]$ (otherwise see *reparametrization* below).

a Symmetry:

The equivalent representation

$$B_i = \frac{d!}{i!j!}u^i v^j, \qquad \text{where } u + v = 1, i + j = d,$$

shows the symmetry $B_i(v) = B_j(u)$.

b Partition of unity and positivity

$$\sum_{i=0}^{d} B_i = 1,$$

$$1 \geq B_i(u) \geq 0 \text{ for } u \in [0..1].$$

Hence the polynomial is in the *convex hull* of the coefficients for $u \in [0, 1]$ and the form is *affinely invariant* for all u. For $A_1, A_2 \in \mathbb{R}$

$$A_2 + A_1 \sum_{i=0}^{d} c(i)B_i(u) = \sum_{i=0}^{d} (A_2 + A_1 c(i))B_i(u).$$

B.II.a – Exercise [3]: Prove that $\sum_{i=0}^{d} B_i(u) = 1$, and $1 \geq B_i(u) \geq 0$.

B.II.b – Exercise [5]: Prove that $B_{d-i,i}$ has its maximum over $[0, 1]$ at the *Greville abscissa* $x_i := i/d$.

B.II.c – Exercise [5]: Draw the degree 3 basis functions B_i for $i = 0..3$ on the interval $[0, 1]$.

c Recurrence:

The recurrence

$$B_{d+1-i,i}(u) = (1 - u)B_{d-i,i}(u) + uB_{d-i+1,i-1}(u).$$

is the basis for evaluating polynomials in Bernstein form. The recurrence is consistent with the definition of the Bernstein form:

$$\begin{aligned}
B_{d-i+1,i}(u) &= \binom{d}{i} \frac{d+1}{d+1-i} v v^{d-i} u^{i-1} u \\
&= \binom{d}{i} \frac{d+1-i}{d+1-i} v v^{d-i} u^{i} + \binom{d}{i} \frac{i}{d+1-i} v^{d+1-i} u^{i-1} u \\
&= vB_{d-i,i}(u) + uB_{d+1-i,i-1}(u)
\end{aligned}$$

B.II.d – Exercise [2]: Give an alternative interpretation in terms of probabilities.

The corresponding nested multiplication for evaluating $p_d = \sum_{i=0}^{d} c(i)B_i$ at x is called

De Casteljau's algorithm:

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for $i = 0 : d, p(d-i, i) := c(i)$
for $l = 1 : d$,
    for $i = 0 : d - l$,
        $p(d-l-i, i) = (1-x)p(d-l+1-i, i)$
                        $+ xp(d-l-i, i+1)$
    end
end
$p_d(x) = p(0, 0)$.
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De Casteljau's algorithm actually does more than generate the value. The identity

$$B_{d-i,i}(xu) = \sum_{k \geq i}^{d} B_{k-i,i}(x) B_{d-k,k}(u)$$

implies that the restriction of p to the interval $[0, x]$ is given by

$$\begin{aligned} p_{[0,x]}(u) &:= \sum_{i=0}^{d} a(i) B_{d-i,i}(xu) \\ &= \sum_{i=0}^{d} \left(\sum_{k=i}^{d} a(i) B_{k-i,i}(x) \right) B_{d-k,k}(u) \\ &= \sum_{k=0}^{d} \left(\sum_{i=0}^{k} a(i) B_{k-i,i}(x) \right) B_{d-k,k}(u) \\ &= \sum_{k=0}^{d} p(k, 0) B_{d-k,k}(u). \end{aligned}$$

d Differentiation

Differentiation = Differencing coefficients

The derivative of a polynomial in Bernstein form must be writable as a polynomial of one degree less in Bernstein form:

$$D\Big(\sum_{i=0}^{d} c(i) B_{d-i,i}\Big) = \sum_{i=0}^{d-1} b(i) B_{d-1-i,i}$$

B.II.e – Exercise [5]: Show that the coefficients of the derivative are

$$b(i) := d(\ c(i+1) - c(i)).$$

B.II.f – Example: The second derivative of

$$p_2 = 3B_{0,2} + 5B_{1,2} + 8B_{2,2}.$$

is

$$\begin{aligned} p_2(u) &= 3(1-u)^2 + 5 * 2(1-u)u + 8u^2 \\ Dp_2(u) &= 2(5-3)B_{0,1} + 2(8-5)B_{1,1} \\ &= 4(1-u) + 6u \\ D^2p_2(u) &= 1 * (6-4)B_{0,0} \\ &= 2 \end{aligned}$$

Indeed, applying the generic rules of differentiation to $p_2(u)$, we get

$$D^2p_2(u) = 3 * 2 + 5 * (-4) + 8 * 2 = 2.$$

B.II.g – Exercise [5]: Compute coefficients $c(j)$ such that

$$D^2\Big(\sum_{i=0}^{d} c(i)B_{d-i,i}\Big) = \sum_{j=0}^{d-2} c(j)B_{d-2-j,j}.$$

e Integration

Integration = Summing coefficients

B.II.h – Exercise [10]: Show that

$$\int_0^1 \sum c(i)B_{d-i,i} = \sum c(i)/(d+1).$$

f Multiplication

Multiplication =
Collecting coefficients
with equal index sums

B.II.i – Exercise [10]: Show that

$$\sum_{i=0}^{d_1} c_1(i) B_{d_1-i,i} * \sum_{i=0}^{d_2} c_2(i) B_{d_2-i,i} = \sum_{i=0}^{d} c(i) B_{d-i,i}$$

where $d = d_1 + d_2$, and

$$c(i) = \sum_{i_1+i_2=i} \frac{\binom{d_1}{i_1}\binom{d_2}{i_2}}{\binom{d}{i}} c_1(i_1) c_2(i_2).$$

Hint: Show that for $v := 1 - u$, $j = d - i$

$$\bar{B}_{j,i} := v^j u^i, \quad \bar{B}_{i_1,j_1} \bar{B}_{i_2,j_2} = \bar{B}_{i_1+i_2,j_1+j_2}$$

Multiplication by $(1-u)+u$ is called *degree-raising*. Uses are data conversion and approximate evaluation.

B.II.j – Exercise [5]: Write out the coefficients of a quadratic polynomial that is degree-raised to a cubic polynomial.

B.II.k – Exercise [5]: Show that the coefficients of the degree-raised polynomial are convex combinations of pairs of coefficients of the original polynomial.

B.II.l – Exercise [2]: Do degree-raising and differentiation commute?

g Hermite Interpolation

Interpolation =
Matching differences
of coefficients

B.II.m – Exercise [1]: Show that

$$B_i(0) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{else} \end{cases}, \quad B_i(1) = \begin{cases} 1 & \text{if } i = d \\ 0 & \text{else} \end{cases}$$

Hence for $p_d = \sum c(i) B_i$,

$$\begin{aligned} p_d(0) = c(0), &\quad p_d(1) = c(d), \\ Dp_d(0) = d(c(1) - c(0)), &\quad Dp_d(1) = d(c(d) - c(d-1)) \\ &\quad \text{etc.} \end{aligned}$$

h Control polygon

The points

$$(x_k, c(k)) \text{ with Greville abscissae } x_k := \frac{k}{d}$$

are called *control points*. The *control polyline* ℓ of p is a broken line connecting the control points. Its kth segment $\ell_{[x_k,x_{k+1}]}$ on the interval $[x_k, x_{k+1}]$, is defined by

$$\ell_{[x_k,x_{k+1}]}(u) := c(k)\frac{x_{k+1}-u}{x_{k+1}-x_k} + c(k+1)\frac{u-x_k}{x_{k+1}-x_k}.$$

B.II.n – Exercise [5]: Show that x_k has to be the Greville abscissa if the control polyline of a linear polynomial is to agree with its graph.

The control polyline is central to reasoning about nonlinearity and curved geometry due to convex hull property, the Hermite Interpolation property and the following theorem.

Theorem [NPL 98] The distance from the univariate, scalar-valued, degree d polynomial p to its control polyline ℓ is bounded as

$$\|p(t) - \ell(t)\|_{\infty,[0,1]} \le \mathtt{N}(d)\|\Delta_2 c\|_\infty$$

where

$$\|\Delta_2 c\|_\infty := \max_{0\le i\le d-2} |c(i) - 2c(i+1) + c(i+2)|$$
$$\mathtt{N}(d) := \frac{\lfloor d/2\rfloor\lceil d/2\rceil}{2d}$$

i Subdivision and approximate evaluation

Subdivision=Reparametrization

Representing the polynomial piece $p(u), u \in [0,1]$ as two pieces $p_1(u), u \in [0,1/2]$ and $p_2(u), u \in [1/2,1]$ has the advantage that the control polygons of p_1 and p_2 (on the finer subdivision) approximate the graph of the function more closely than the control polygon of p.

In particular, the theorem of the previous subsection implies

$$\|p_{[0,x]}(t) - \ell_{[0,x]}(t)\|_{\infty,[0,x]} \le x^2\mathtt{N}(d)\|\Delta_2 c\|_\infty$$

and m-fold subdivision results in the bound

$$\mathtt{x}^{2m}\mathtt{N}(d)\|\Delta_2 a\|_\infty \text{ where } \mathtt{x} := \max\{x, 1-x\}.$$

That is, the control polyline converges to the graph of the function like $\mathtt{x}^{2m}$.

B.II.o – Exercise [15]:

(a) Show that the control polygon of a degree-raised polynomial is obtained from that of the original polynomial by cutting off corners (linear interpolation).
(b) Next show that the r-fold degree-raised representation of the polynomial $\sum_i c(i)B_i$ has coefficients $b_i = \sum_j c_j \binom{n}{j} K(r,i,j)$ for some K that converges for increasing r by Stirling's formula to $t^j(1-t)^{d-j}$ where $t = i/(d+r)$.
(c) Use this to show the *variation diminishing property*: a line intersects the graph of a Bernstein polynomial at most as often as it intersects the control polyline.

j Conversion to power form

B.II.p – Exercise [10]: Check that

$$\sum_i B_{i,d} = 1,$$

$$\sum_i (\frac{i}{d}) B_{i,d}(u) = u, \quad \text{linear precision}$$

$$\sum_i (\frac{i}{d})^2 B_{i,d}(u) = (1 - 1/d)u^2 + u/d.$$

These are special cases of

$$\sum_{i=k}^{d} \binom{i}{k} B_i^d(t) = \binom{d}{k} t^k.$$

k Weierstrass' approximation theorem

Define the modulus of continuity of f:

$$\omega(f,\delta) := \text{lub}_{|b-a|\le\delta} |f(b) - f(a)|$$

where lub means least upper bound. Choose $\delta = 1/\sqrt{d}$ and let $d \to \infty$ then *any continuous function f can be approximated arbitrarily closely by polynomials*:

$$
\begin{aligned}
&|f - \sum f(j/d) B_j|_{[0,1]} \\
&\quad \le |\sum_{|u-j/d|\le\delta} (f(u) - f(j/d))B_j| \\
&\qquad + |\sum_{|u-j/d|>\delta} (f(u) - f(j/d))B_j| \\
&\quad \le \omega(f,\delta) + \sum_{>} (1 + |u - j/d|/\delta) B_j \omega(f,\delta) \\
&\quad \le \omega(f,\delta)[2 + \sum_{>} (u - j/d)^2/\delta^2 B_j] \\
&\quad = \omega(f,\delta)[2 + \sum (u^2 - 2u(j/d) + (j/d)^2) B_j/\delta^2] \\
&\quad = \omega(f,\delta)[2 + (u^2 - 2u^2 + (1 - 1/d)u^2 + u/d)/\delta^2] \\
&\quad = \omega(f,\delta)[2 + u(1-u)/(d\delta^2)] \\
&\quad \le \omega(f,\delta)[2 + 1/4] \to 0 \qquad \text{as } d \to \infty
\end{aligned}
$$

B.II.q – Exercise [2]: Show that polynomials are dense in L^2. Hint: Use that the continuous functions are dense in L^2.