

# MULTIPLICATIVE PARTIAL ISOMETRIES AND $C^*$ -ALGEBRAIC QUANTUM GROUPOIDS

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ABSTRACT. Generalizing the notion of a multiplicative unitary (in the sense of Baaĵ–Skandalis), which plays a fundamental role in the theory of locally compact quantum groups, we develop in this paper the notion of a *multiplicative partial isometry*. The axioms include the pentagon equation, but more is needed. Under suitable conditions (such as the “manageability”), it is possible to construct from it a pair of  $C^*$ -algebras having the structure of a  $C^*$ -algebraic quantum groupoid of *separable type*.

## 0. INTRODUCTION

In the theory of locally compact quantum groups, the *multiplicative unitary operators* (see [1], [28]) play a fundamental role. They give rise to the left/right regular representations of the associated quantum groups, while encoding their duality picture. Refer to the general theory on locally compact quantum groups [13], [14], [16].

In addition, the multiplicative unitaries have been useful in the construction of quantum groups, for instance as providing a way to describe their comultiplications [1], [24], [18], [8].

Meanwhile, Enock and Vallin introduced the notion of *pseudomultiplicative unitaries* [6], [7], [25]. They are defined on relative tensor products of Hilbert spaces and are rather technical, but they play a fundamental role in the theory of *measured quantum groupoids* by Lesieur and Enock [15], [5]. Measured quantum groupoids provide a general framework for studying quantum groupoids in the von Neumann algebra setting. In the finite-dimensional case, they become *weak Hopf algebras* [2], [3] or *finite quantum groupoids* [26], [17].

In the  $C^*$ -algebra setting, the status is not as satisfactory. Timmermann developed the notion of  $C^*$ -pseudomultiplicative unitaries and Hopf  $C^*$ -bimodules [21], [22], but the most general theory of  $C^*$ -algebraic quantum groupoid seems elusive at present. The reason is partly because the theory of pseudomultiplicative unitaries and that of measured quantum groupoids use some primarily von Neumann algebraic tools such as the *fiber product*, whose  $C^*$ -algebraic counterpart is not clearly established. A separate approach needs to be developed for the  $C^*$ -algebraic framework, which is on-going (see works by Timmermann [22], [23]).

At a reduced scale, the author, together with Van Daele, recently developed a  $C^*$ -algebraic framework for a subclass of quantum groupoids, namely the *locally compact quantum groupoids of separable type* [10], [11]. In this theory, we naturally obtain certain “multiplicative partial isometries”. As in the case of multiplicative unitaries for quantum groups, such partial isometries give rise to the left/right regular representations, encode the duality picture, and play important roles in the construction of the antipode map.

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Unlike the case of the multiplicative unitaries or that of the pseudomultiplicative unitaries, however, an axiomatic approach to multiplicative partial isometries has not been developed yet. The aim of this paper is to address this situation.

It has been known since Enock and Vallin's work [6] that in the finite-dimensional case, pseudomultiplicative unitaries become partial isometries, where the relative tensor product spaces associated with a pseudomultiplicative unitary become the initial and the terminal spaces of the corresponding partial isometry. The associated measured quantum groupoids would become finite quantum groupoids or weak Hopf algebras. Also Böhm and Szlachnyi provided a systematic treatment of finite-dimensional multiplicative partial isometries in [4], taking advantage of the results from the weak Hopf algebra theory.

Non-finite case is not as simple. But loosely speaking, multiplicative partial isometries *should* be a special case of pseudomultiplicative unitaries. At the same time, the locally compact quantum groupoids of separable type *should* be a special case of measured quantum groupoids. However, as alluded to above, the situation is not as straightforward as one would hope. Despite some known results, the way from pseudomultiplicative unitaries to multiplicative partial isometries is not completely understood even in the finite dimensional setting (see comments given in [4]). One primary reason is because of the von Neumann algebraic tools not translating well into the  $C^*$ -algebraic setting, and it has also to do with the fact that the theory of  $C^*$ -algebraic quantum groupoids based on multiplicative partial isometries (such as [10], [11]) has not been developed until recently.

The aim of this paper, as well as the theory of the  $C^*$ -algebraic quantum groups of separable type [10], [11], is an attempt at bridging this gap. In particular, our modest goal is to establish and understand the relationship between the multiplicative partial isometries discussed below and the  $C^*$ -algebraic quantum groupoids of separable type. By reducing the scope, the technical difficulties become milder. On the other hand, while it is true that such quantum groupoids and the multiplicative partial isometries do not cover the full generality of the  $C^*$ -algebraic quantum groupoids, these intermediate steps have sufficiently rich structure to help us gain valuable insights toward the ultimate goal of developing a fully general  $C^*$ -algebraic theory of locally compact quantum groupoids. This is the underlying guideline.

This paper is organized as follows: In Section 1, we gather some basic results and establish notations concerning partial isometries, then give conditions for a *multiplicative partial isometry*. A couple of them are variations of the “pentagon equation”, and we also need two other conditions that would have been trivial in the unitary case. As a consequence, we can associate two subalgebras of  $\mathcal{B}(\mathcal{H})$ , as well as the comultiplication maps. We do not know if they are  $*$ -algebras at this stage.

In a previous version of this paper, we required the existence of approximate units on these subalgebras. While algebraically useful, a serious drawback is that such an attempt involves counits, and they turn out to cause some unboundedness issues. As such, in the current version of the paper, we introduce in §2.1 a “fullness” condition on  $W$ , which in turn would help us establish the subalgebras as represented non-degenerately on the Hilbert space. Note that this is a condition that is extra in our setting, which did not have to be required in the unitary case.

Under the fullness assumption on  $W$ , we then define in 2.2 the *manageability condition* for a multiplicative partial isometry operator  $W$ . This is our main assumption, which is motivated by Woronowicz's notion in the unitary case [28], with some modifications. As a consequence, we can now show that the pair of subalgebras obtained as a consequence of the multiplicativity property are in fact  $C^*$ -algebras.

In Section 3, we study the coalgebra structures on the pair of  $C^*$ -algebras associated with our partial isometry. The projection  $E = W^*W$  can be regarded as  $\Delta(1)$ , and we can gather several of its properties. It plays an important role as the canonical idempotent.

We next turn our focus to studying the four spaces associated to the projections  $W^*W$  and  $WW^*$ . They are also shown to be  $C^*$ -algebras. This is done in Section 4. They are essentially the source and the target algebras of the dual pair of quantum groupoids corresponding to our multiplicative partial isometry. We will consider their von Neumann algebra counterparts first, and introduce certain “distinguished weights” on them, before considering the base  $C^*$ -algebras. There will be certain densely-defined maps between these subalgebras.

By this stage, we will have constructed sufficient structure on the pair of  $C^*$ -algebras, so that they can be regarded more or less as quantum groupoids. Indeed, with the additional conditions on the existence of certain invariant weights, we would be able to say that the resulting structure gives rise to a pair of *locally compact quantum groupoids of separable type*, in the sense of [10], [11]. We will stop short of considering the invariant weights in this paper, but in Section 5, we will give some indications on how the antipode map would be incorporated, by working with a characterization that does not explicitly rely on the invariant weights.

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## 1. MULTIPLICATIVE PARTIAL ISOMETRIES

Let  $\mathcal{H}$  be a (separable) Hilbert space, not necessarily finite-dimensional. Let  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  be a partial isometry, satisfying  $WW^*W = W$ .

Write  $E = W^*W$  and  $G = WW^*$ . By the general theory on partial isometries, it is known that  $E$  is a projection onto  $\text{Ran}(W^*W) = \text{Ran}(W^*) = \text{Ker}(W)^\perp$ , while  $G$  is a projection onto  $\text{Ran}(WW^*) = \text{Ran}(W) = \text{Ker}(W^*)^\perp$ . These spaces are necessarily closed in  $\mathcal{H} \otimes \mathcal{H}$ . In addition,  $W$  is an isometry from  $\text{Ran}(W^*W)$  onto  $\text{Ran}(WW^*)$ , and similarly,  $W^*$  is an isometry from  $\text{Ran}(WW^*)$  onto  $\text{Ran}(W^*W)$ . All these are standard results.

Write  $\widehat{W} := \Sigma W^* \Sigma$ , where  $\Sigma$  denotes the flip on  $\mathcal{H} \otimes \mathcal{H}$ . It is evident that  $\widehat{W}$  is also a partial isometry, with the associated projections  $\widehat{E} = \widehat{W}^* \widehat{W} = \Sigma W W^* \Sigma = \Sigma G \Sigma$  and  $\widehat{G} = \widehat{W} \widehat{W}^* = \Sigma W^* W \Sigma = \Sigma E \Sigma$ .

For  $E = W^*W$ , consider the following spaces:

$$N := \overline{\text{span}\{(\text{id} \otimes \omega)(W^*W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}} \subseteq \mathcal{B}(\mathcal{H}),$$

$$L := \overline{\text{span}\{(\omega \otimes \text{id})(W^*W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}} \subseteq \mathcal{B}(\mathcal{H}).$$

They are closed subspaces under the weak operator topology in  $\mathcal{B}(\mathcal{H})$ , but at present we cannot expect them to be subalgebras. Still, they will play important roles down the road. Similarly for  $\widehat{E} = \Sigma WW^*\Sigma$ , we can consider

$$\begin{aligned}\widehat{N} &:= \overline{\text{span}\{(\text{id} \otimes \omega)(\widehat{E}) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}} = \overline{\text{span}\{(\omega \otimes \text{id})(WW^*) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}}, \\ \widehat{L} &:= \overline{\text{span}\{(\omega \otimes \text{id})(\widehat{E}) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}} = \overline{\text{span}\{(\text{id} \otimes \omega)(WW^*) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}},\end{aligned}$$

which are also WOT-closed subspaces in  $\mathcal{B}(\mathcal{H})$ .

While we cannot claim that  $N, L, \widehat{N}, \widehat{L}$  are subalgebras, they are all closed under taking adjoints. That will be useful. We will come back to study these spaces in later sections.

Let us begin our discussion on multiplicative partial isometries, first by giving the definition:

**Definition 1.1.** Let  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  be a partial isometry. We will call  $W$  a *multiplicative partial isometry*, if the following conditions hold on  $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ :

$$W_{23}W_{12}W_{23}^* = W_{12}W_{13} \quad (1.1)$$

$$W_{12}^*W_{23}W_{12} = W_{13}W_{23} \quad (1.2)$$

$$W_{23}^*W_{23}W_{12} = W_{12}W_{23}^*W_{23} \quad (1.3)$$

$$W_{12}W_{12}^*W_{23} = W_{23}W_{12}W_{12}^* \quad (1.4)$$

*Remark.* Here, we are using the standard three-leg notation. Equations (1.1) and (1.2) resemble the ‘‘pentagon equation’’, as in the case of multiplicative unitaries [1], [28]. However, with  $W$  not being a unitary, these two conditions are not necessarily equivalent. Equations (1.3), (1.4) become trivial in the unitary case, but they are needed for our purposes. Equation (1.3) would imply that the elements of the spaces  $N$  and  $L$  commute, while Equation (1.4) gives the commutativity between the elements of  $\widehat{N}$  and  $\widehat{L}$ . See Proposition 1.2 below. Meanwhile, Equations (1.1) and (1.2) allow the construction of two subalgebras  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  of  $\mathcal{B}(\mathcal{H})$ , on which we can later build the quantum groupoid structure. See Proposition 1.3 below.

Here are some immediate consequences of Definition 1.1.

**Proposition 1.2.** *Let  $W$  be a multiplicative partial isometry, and consider the spaces  $N, L, \widehat{N}, \widehat{L}$  as above. We have:*

- (1) *For any  $b \in N$  and  $c \in L$ , we have  $bc = cb$ .*
- (2) *For any  $\widehat{b} \in \widehat{N}$  and  $\widehat{c} \in \widehat{L}$ , we have  $\widehat{b}\widehat{c} = \widehat{c}\widehat{b}$ .*

*Proof.* (1). Consider  $b = (\text{id} \otimes \omega)(W^*W) \in N$  and  $c = (\omega' \otimes \text{id})(W^*W) \in L$ , for any  $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$ . We can see that

$$\begin{aligned}cb &= (\omega' \otimes \text{id} \otimes \omega)(W_{12}^*W_{12}W_{23}^*W_{23}) = (\omega' \otimes \text{id} \otimes \omega)(W_{12}^*W_{23}^*W_{23}W_{12}) \\ &= (\omega' \otimes \text{id} \otimes \omega)(W_{23}^*W_{23}W_{12}^*W_{12}) = bc,\end{aligned}$$

by applying Equation (1.3) twice.

(2). Proof of (2) is similar, now using Equation (1.4).  $\square$

Given a multiplicative partial isometry  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ , we can associate to it the following two subspaces of  $\mathcal{B}(\mathcal{H})$ :

$$\mathcal{A} := \text{span}\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}, \quad \text{and} \quad \widehat{\mathcal{A}} := \text{span}\{(\omega \otimes \text{id})(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}.$$

It can be shown that  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  are subalgebras of  $\mathcal{B}(\mathcal{H})$ . The proof given below is essentially the same as in [1].

**Proposition 1.3.** *Let  $W$  be a multiplicative partial isometry, then the spaces  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  defined above are subalgebras of  $\mathcal{B}(\mathcal{H})$ .*

*Proof.* (1). Consider  $x = (\text{id} \otimes \omega)(W)$ ,  $x' = (\text{id} \otimes \omega')(W) \in \mathcal{A}$ , where  $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$  are arbitrary. By Equation (1.1), we have

$$xx' = (\text{id} \otimes \omega \otimes \omega')(W_{12}W_{13}) = (\text{id} \otimes \omega \otimes \omega')(W_{23}W_{12}W_{23}^*) = (\text{id} \otimes \theta)(W) \in \mathcal{A},$$

where  $\theta \in \mathcal{B}(\mathcal{H})_*$  is such that  $\theta(T) = (\omega \otimes \omega')(W(T \otimes 1)W^*)$ , for  $T \in \mathcal{B}(\mathcal{H})$ .

(2). Proof for  $\widehat{\mathcal{A}}$  being also a subalgebra is similar. For  $y = (\omega \otimes \text{id})(W^*)$ ,  $y' = (\omega' \otimes \text{id})(W^*) \in \widehat{\mathcal{A}}$ , by using Equation (1.2), we can show that

$$yy' = (\omega \otimes \omega' \otimes \text{id})(W_{13}W_{23}) = (\omega' \otimes \omega \otimes \text{id})(W_{12}^*W_{23}W_{12}) = (\theta \otimes \text{id})(W) \in \widehat{\mathcal{A}},$$

where  $\theta \in \mathcal{B}(\mathcal{H})_*$  is such that  $\theta(T) = (\omega \otimes \omega')(W^*(1 \otimes T)W)$ ,  $\forall T \in \mathcal{B}(\mathcal{H})$ .  $\square$

*Remark.* It is not difficult to see that if  $W$  is a multiplicative partial isometry, then  $\widehat{W} = \Sigma W^* \Sigma$  is also multiplicative. Note also that in terms of  $\widehat{W}$ , we have:

$$\widehat{\mathcal{A}}^* = \text{span}\{(\omega \otimes \text{id})(W^*) : \omega \in \mathcal{B}(\mathcal{H})_*\} = \text{span}\{(\text{id} \otimes \omega)(\widehat{W}) : \omega \in \mathcal{B}(\mathcal{H})_*\}.$$

At present, we do not know if  $\widehat{\mathcal{A}}^* = \widehat{\mathcal{A}}$ , however.

Here are some more consequences of Definition 1.1:

**Lemma 1.4.** *Let  $W$  be a multiplicative partial isometry. Then the following results hold:*

$$W_{12}W_{13}W_{23} = W_{23}W_{12} \tag{1.5}$$

$$W_{12}^*W_{12}W_{13} = W_{13}W_{23}W_{23}^* \tag{1.6}$$

*Proof.* From Equation (1.2), we have  $W_{12}^*W_{23}W_{12} = W_{13}W_{23}$ . Multiply  $W_{12}$  to both sides, to obtain  $W_{12}W_{12}^*W_{23}W_{12} = W_{12}W_{13}W_{23}$ . Apply Equation (1.4) to the left side, which becomes  $W_{23}W_{12}W_{12}^*W_{12} = W_{23}W_{12}$ , as  $W$  is a partial isometry. In this way, we prove that  $W_{23}W_{12} = W_{12}W_{13}W_{23}$ .

By Equation (1.1), we have  $W_{12}^*W_{12}W_{13} = W_{12}^*W_{23}W_{12}W_{23}^*$ . Apply to the right side Equation (1.2), obtaining  $W_{12}^*W_{12}W_{13} = W_{13}W_{23}W_{23}^*$ .  $\square$

*Remark.* Equation (1.5) is exactly the pentagon equation of Baaj–Skandalis [1]. Here, we obtain it as a consequence. Note that the Equations (1.5), (1.6), (1.3), (1.4) have been chosen as the axioms by Böhm and Szlachnyi in [4]. It is not difficult to show that these four imply the four conditions (1.1), (1.2), (1.3), (1.4) chosen in our Definition 1.1, and vice versa.

Let us construct maps  $\Delta$  and  $\widehat{\Delta}$ , which would become comultiplications later, at first as maps from  $\mathcal{B}(\mathcal{H})$  into  $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ :

$$\Delta(X) = W^*(1 \otimes X)W \quad \text{and} \quad \widehat{\Delta}(X) = \Sigma W(X \otimes 1)W^*\Sigma, \quad \text{for } X \in \mathcal{B}(\mathcal{H}).$$

As a consequence of Lemma 1.4, we can show that  $\Delta$  and  $\widehat{\Delta}$  satisfy the ‘‘coassociativity’’ property, which will be useful later:

**Proposition 1.5.** *We have the following:*

- (1)  $\Delta : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  satisfies the property:  $(\Delta \otimes \text{id})\Delta(X) = (\text{id} \otimes \Delta)\Delta(X)$ ,  $\forall X \in \mathcal{B}(\mathcal{H})$ .
- (2)  $\widehat{\Delta} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  satisfies the property:  $(\widehat{\Delta} \otimes \text{id})\widehat{\Delta}(X) = (\text{id} \otimes \widehat{\Delta})\widehat{\Delta}(X)$ ,  $\forall X \in \mathcal{B}(\mathcal{H})$ .

*Proof.* For  $X \in \mathcal{B}(\mathcal{H})$ , we have:

$$\begin{aligned} (\Delta \otimes \text{id})\Delta(X) &= W_{12}^* W_{23}^* (1 \otimes 1 \otimes X) W_{23} W_{12} = W_{23}^* W_{13}^* W_{12}^* (1 \otimes 1 \otimes X) W_{12} W_{13} W_{23} \\ &= W_{23}^* W_{13}^* (1 \otimes 1 \otimes X) W_{12}^* W_{12} W_{13} W_{23} \\ &= W_{23}^* W_{13}^* (1 \otimes 1 \otimes X) W_{13} W_{23} W_{23}^* W_{23} \\ &= W_{23}^* W_{13}^* (1 \otimes 1 \otimes X) W_{13} W_{23} = (\text{id} \otimes \Delta)\Delta(X), \end{aligned}$$

where we used Equations (1.5) and (1.6), together with the fact that  $W$  is a partial isometry.

Proof for  $\widehat{\Delta}$  is similar. We may either give the proof directly, or use the multiplicativity property of  $\widehat{W}$  and use the result above, as we can write  $\widehat{\Delta}(X) = \widehat{W}^*(1 \otimes X)\widehat{W}$ .  $\square$

Consider the norm-closures of the algebras  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  in  $\mathcal{B}(\mathcal{H})$ . That is, define:

$$A := \overline{\text{span}\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}, \quad \text{and} \quad \widehat{A} := \overline{\text{span}\{(\omega \otimes \text{id})(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}.$$

Going forward, they will be our main objects of study. Eventually, they will be shown to be  $C^*$ -algebras, and the maps  $\Delta$  and  $\widehat{\Delta}$  will be restricted to  $A$  and  $\widehat{A}$ , on which we will construct the quantum groupoid structures.

However, some extra conditions need to be introduced for our program to work. For instance, unlike  $N, L, \widehat{N}, \widehat{L}$ , there is no reason to believe that  $A$  and  $\widehat{A}$  would be self-adjoint. This was already a problem even when  $W$  is a multiplicative unitary, so some extra conditions like the ‘‘regularity’’ (see section 3 of [1]) or the ‘‘manageability’’ (see [28]) had to be assumed to ensure that  $A$  and  $\widehat{A}$  are closed under the involution. We will discuss these matters in the ensuing sections.

Before wrapping up this section, here are some more consequences that follow from the operator  $W$  being a multiplicative partial isometry:

**Lemma 1.6.** *Let  $W$  be a multiplicative partial isometry. Then the following results hold:*

$$W_{12} W_{23}^* = W_{23}^* W_{12} W_{13} \tag{1.7}$$

$$W_{12}^* W_{23} = W_{13} W_{23} W_{12}^* \tag{1.8}$$

$$W_{13}^* W_{13} W_{23} = W_{23} W_{12}^* W_{12} \tag{1.9}$$

$$W_{12} W_{13} W_{13}^* = W_{23} W_{23}^* W_{12} \tag{1.10}$$

*Proof.* From Equation (1.1), we have  $W_{23}^* W_{23} W_{12} W_{23}^* = W_{23}^* W_{12} W_{13}$ . Then apply Equation (1.3) to the left side, which becomes  $W_{12} W_{23}^* W_{23} W_{23}^* = W_{12} W_{23}^*$ , because  $W^* W W^* = W^*$ . Combining, we prove Equation (1.7). Similarly, for Equation (1.8), use Equation (1.2) and Equation (1.4).

From Equation (1.2), we have:  $W_{13}^* W_{13} W_{23} = W_{13}^* W_{12}^* W_{23} W_{12}$ . Note that from Equation (1.7) we know  $W_{13}^* W_{12}^* W_{23} = W_{23} W_{12}^*$ . Combining, we obtain  $W_{13}^* W_{13} W_{23} = W_{23} W_{12}^* W_{12}$ , thereby proving Equation (1.9). Similarly, using Equation (1.1) and the adjoint of Equation (1.8), we can prove Equation (1.10).  $\square$

## 2. THE MANAGEABILITY CONDITION

**2.1. Fullness condition.** Let  $W$  be a multiplicative partial isometry, and consider the subalgebras  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$ . As we do not know if they are  $*$ -algebras, even in the finite-dimensional case we cannot be sure whether they are unital subalgebras. This is different from the case of multiplicative unitaries: For a multiplicative unitary, if the Hilbert space on which it is acting is finite-dimensional, then it is known that the norm-closures of  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  always become unital  $C^*$ -algebras (finite-dimensional Kac algebras). See Theorem 4.10 of [1].

To see what can happen in the general case, observe the example below. (This is essentially the example given by Böhm and Szlachnyi in [4], with only minor differences.)

**Example 2.1.** Let  $\mathcal{H} = \mathbb{C}^2$  and consider  $W = e_{21} \otimes e_{11} + e_{22} \otimes e_{22}$ , where the  $e_{ij} \in \mathcal{B}(\mathcal{H})$ ,  $1 \leq i, j \leq 2$ , are the matrix units. Then  $W$  is a multiplicative partial isometry. But the associated subalgebra  $\mathcal{A}$  is non-unital.

*Proof.* Suppose  $(\xi_k)$  denotes an orthonormal basis for  $\mathcal{H}$ . By definition,  $e_{ij} \in \mathcal{B}(\mathcal{H})$  is such that  $e_{ij}(\mathbf{v}) := \langle \mathbf{v}, \xi_j \rangle \xi_i$ . By standard Linear Algebra, it is easy to verify that  $e_{ij}e_{kl} = \delta_{jk}e_{il}$  and  $(e_{ij})^* = e_{ji}$ . As a consequence, it is easy to verify that

$$\begin{aligned} WW^*W &= (e_{21} \otimes e_{11} + e_{22} \otimes e_{22})(e_{12} \otimes e_{11} + e_{22} \otimes e_{22})W \\ &= (e_{22} \otimes e_{11} + e_{22} \otimes e_{22})(e_{21} \otimes e_{11} + e_{22} \otimes e_{22}) = e_{21} \otimes e_{11} + e_{22} \otimes e_{22} = W. \end{aligned}$$

So  $W$  is a partial isometry. We also have:

$$\begin{aligned} W_{12}W_{13} &= (e_{21} \otimes e_{11} \otimes 1 + e_{22} \otimes e_{22} \otimes 1)(e_{21} \otimes 1 \otimes e_{11} + e_{22} \otimes 1 \otimes e_{22}) \\ &= e_{21} \otimes e_{22} \otimes e_{11} + e_{22} \otimes e_{22} \otimes e_{22}, \\ W_{23}W_{12}W_{23}^* &= (1 \otimes e_{21} \otimes e_{11} + 1 \otimes e_{22} \otimes e_{22})(e_{21} \otimes e_{11} \otimes 1 + e_{22} \otimes e_{22} \otimes 1)W_{23}^* \\ &= (e_{21} \otimes e_{21} \otimes e_{11} + e_{22} \otimes e_{22} \otimes e_{22})(1 \otimes e_{12} \otimes e_{11} + 1 \otimes e_{22} \otimes e_{22}) \\ &= e_{21} \otimes e_{22} \otimes e_{11} + e_{22} \otimes e_{22} \otimes e_{22}, \end{aligned}$$

verifying Equation (1.1):  $W_{23}W_{12}W_{23}^* = W_{12}W_{13}$ . Equations (1.2), (1.3), (1.4) are also easily verified, so  $W$  is indeed a multiplicative partial isometry.

However, if we consider  $\mathcal{A} := \text{span}\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}$ , we can quickly observe that  $e_{11} + e_{22} \notin \mathcal{A}$ , and actually non-unital. The other subalgebra,  $\widehat{\mathcal{A}}$ , contains the unit.  $\square$

What this observation means is that unlike the case of multiplicative unitaries, an additional condition on  $W$  is required, to allow the associated subalgebras  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  become unital. See [4].

In our infinite-dimensional case, we cannot expect  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  to be unital. Nonetheless, it is apparent that some additional assumption on  $W$  is needed. That turns out to be related to non-degeneracy, so we will from now on require the following *fullness* condition on  $W$ :

**Definition 2.2.** Let  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  be a multiplicative partial isometry, and let  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  be the associated subalgebras of  $\mathcal{B}(\mathcal{H})$ . We will say that  $W$  is *full*, if  $[(\text{id} \otimes \omega)(W) = 0, \omega \in \mathcal{B}(\mathcal{H})_*] \implies [\omega = 0]$ , and  $[(\omega \otimes \text{id})(W) = 0, \omega \in \mathcal{B}(\mathcal{H})_*] \implies [\omega = 0]$ .

**2.2. Manageability condition.** From now on, we will assume that  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  is a multiplicative partial isometry satisfying the fullness condition (see §1 and §2.1). Motivated by Woronowicz's notion of the manageability for a multiplicative unitary [28], let us now introduce the *manageability* condition for a multiplicative partial isometry, then gather some resulting properties.

For our Hilbert space  $\mathcal{H}$ , denote by  $\overline{\mathcal{H}}$  its complex conjugation. For any  $\xi \in \mathcal{H}$ , the corresponding element will be denoted by  $\bar{\xi}$ . The map  $\mathcal{H} \ni \xi \mapsto \bar{\xi} \in \overline{\mathcal{H}}$  is a \*-anti-isomorphism. For  $\xi, \eta \in \mathcal{H}$ , we will have  $\langle \bar{\xi}, \bar{\eta} \rangle = \langle \eta, \xi \rangle$ .

If  $m$  is a closed operator on  $\mathcal{H}$ , then its transpose, written  $m^\top$ , is the operator on  $\overline{\mathcal{H}}$  such that  $\mathcal{D}(m^\top) = \overline{\mathcal{D}(m^*)}$  and  $m^\top \bar{\xi} = \overline{m^* \xi}$ , for  $\xi \in \mathcal{D}(m^*)$ . In particular, if  $m \in \mathcal{B}(\mathcal{H})$ , then  $m^\top \in \mathcal{B}(\overline{\mathcal{H}})$  such that  $\langle m^\top \bar{\eta}, \bar{\xi} \rangle = \langle \xi, m^* \eta \rangle = \langle m \xi, \eta \rangle$ , for  $\xi, \eta \in \mathcal{H}$ . It is clear that  $m \mapsto m^\top$  is a \*-anti-isomorphism. We may identify  $\overline{\overline{\mathcal{H}}} = \mathcal{H}$ , by  $\bar{\bar{\xi}} = \xi$ . Then we have  $(m^\top)^\top = m$ , for any  $m \in \mathcal{B}(\mathcal{H})$ .

With these notations set, we now give the definition for the manageability condition:

**Definition 2.3.** Let  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  be a multiplicative partial isometry. We say  $W$  is *manageable*, if there exist a densely-defined positive closed operator  $Q$  acting on  $\mathcal{H}$ ,  $\text{Ker}(Q) = \{0\}$ , and an operator  $\widetilde{W} \in \mathcal{B}(\widetilde{\mathcal{H}} \otimes \mathcal{H})$ , such that

- (1)  $W(Q \otimes Q) \subseteq (Q \otimes Q)W$ .
- (2)  $\langle W(\xi \otimes v), \eta \otimes u \rangle = \langle \widetilde{W}(\bar{\eta} \otimes Q^{-1}v), \bar{\xi} \otimes Qu \rangle$ , for any  $\xi, \eta \in \mathcal{H}$ ,  $v \in \mathcal{D}(Q^{-1})$ ,  $u \in \mathcal{D}(Q)$ .
- (3) We also require:  $\widetilde{W}_{13}\widetilde{W}_{23}\widetilde{W}_{23}^* = W_{12}^{\top \otimes \top} W_{12}^*{}^{\top \otimes \top} \widetilde{W}_{13}$ , and  $W_{23}W_{23}^*\widetilde{W}_{13} = \widetilde{W}_{13}\widetilde{W}_{12}\widetilde{W}_{12}^*$ .

*Remark.* This is a modification of Woronowicz's notion (see Definition 1.2 in [28]). In (1), we replaced his condition  $W^*(Q \otimes Q)W = Q \otimes Q$ , which is no longer true as  $W$  is not unitary, with the inclusion above. The characterizing equation in (2) is the same as in the unitary case. Meanwhile we included the two conditions in (3), which would have been trivial when  $W$  and  $\widetilde{W}$  are unitaries.

*Remark.* It eventually turns out that the operator  $\widetilde{W}$  is itself a partial isometry. However, the direct proof is not easy at present. We prove the result in Section 5.

In the below are some consequences of the inclusion,  $W(Q \otimes Q) \subseteq (Q \otimes Q)W$ .

**Proposition 2.4.** Write  $E = W^*W$  and  $G = WW^*$  as before. We have:

- (1)  $(Q \otimes Q)E = E(Q \otimes Q)E$  and  $(Q \otimes Q)G = G(Q \otimes Q)G$ .
- (2) It follows as a result that  $(Q \otimes Q)|_{\text{Ran}(E)}$ ,  $(Q \otimes Q)|_{\text{Ran}(G)}$ ,  $(Q \otimes Q)|_{\text{Ker}(W)}$ ,  $(Q \otimes Q)|_{\text{Ker}(W^*)}$  become valid operators on the respective subspaces  $\text{Ran}(E)$ ,  $\text{Ran}(G)$ ,  $\text{Ker}(W)$ ,  $\text{Ker}(W^*)$ .
- (3) For any  $z \in \mathbb{C}$ , we have:  $W(Q^z \otimes Q^z) \subseteq (Q^z \otimes Q^z)W$  and  $W^*(Q^z \otimes Q^z) \subseteq (Q^z \otimes Q^z)W^*$ .

*Proof.* These results are consequences of the fact that  $W$  and  $W^*$  are partial isometries. When restricted to subspaces, we may regard  $W|_{\text{Ran}(E)}$  and  $W^*|_{\text{Ran}(G)}$  as onto isometries between the subspaces  $\text{Ran}(E)$  and  $\text{Ran}(G)$ . As a consequence, a version of functional calculus can be applied.

See the results and proofs for Propositions 4.12 – 4.17 in [11], where a similar argument was carried out in more detail.  $\square$

We cannot do better than “ $\subseteq$ ” in general. However, if  $z \in \mathbb{C}$  is purely imaginary, or  $z = it$  for  $t \in \mathbb{R}$ , the operator  $Q^{it}$  is bounded. So the domain  $\mathcal{D}(Q^{it} \otimes Q^{it})$  becomes the whole space  $\mathcal{H} \otimes \mathcal{H}$ , and we obtain the following result:

**Proposition 2.5.** Let  $t \in \mathbb{R}$ . Then the following equality holds on the whole space  $\mathcal{H} \otimes \mathcal{H}$ :

$$(Q^{it} \otimes Q^{it})W(Q^{-it} \otimes Q^{-it}) = W.$$

*Proof.* Since  $Q^{it}$  is a bounded operator, there is no issue with the domains. As we already know  $W(Q^{it} \otimes Q^{it}) \subseteq (Q^{it} \otimes Q^{it})W$  from Proposition 2.4, we indeed have the equality:  $W(Q^{it} \otimes Q^{it}) = (Q^{it} \otimes Q^{it})W$ . This is equivalent to  $(Q^{it} \otimes Q^{it})W(Q^{-it} \otimes Q^{-it}) = W$ .  $\square$

Let us turn back our attention to the operator  $\widetilde{W}$ . As a consequence of Proposition 2.5 and the characterizing equation for the manageability, we obtain the following result:

**Proposition 2.6.** Let  $W$  be a manageable multiplicative partial isometry, and let  $Q$  and  $\widetilde{W}$  be the associated operators given in Definition 2.3. For any  $t \in \mathbb{R}$ , we have the following equality on the whole space  $\widetilde{\mathcal{H}} \otimes \mathcal{H}$ :

$$([Q^\top]^{-it} \otimes Q^{it})\widetilde{W}([Q^\top]^{it} \otimes Q^{-it}) = \widetilde{W}.$$

*Proof.* Suppose  $\xi, \eta \in \mathcal{H}$  and  $v \in \mathcal{D}(Q^{-1})$ ,  $u \in \mathcal{D}(Q)$ . Then for  $t \in \mathbb{R}$ , we can also say that  $Q^{-it}\xi, Q^{-it}\eta \in \mathcal{H}$  and  $Q^{-it}v \in \mathcal{D}(Q^{-1})$ ,  $Q^{-it}u \in \mathcal{D}(Q)$ , for instance by writing  $QQ^{-it}u = Q^{-it}Qu$ .



By (2) of Definition 2.3, we have:

$$\langle W(\xi \otimes v), \eta \otimes u \rangle = \langle \widetilde{W}(\bar{\eta} \otimes Q^{-1}v), \bar{\xi} \otimes Qu \rangle.$$

Noting that  $W = (Q^{it} \otimes Q^{it})W(Q^{-it} \otimes Q^{-it})$ , as in Proposition 2.5, the left side of the above equation can be expressed as follows:

$$\begin{aligned} (LHS) &= \langle (Q^{it} \otimes Q^{it})W(Q^{-it} \otimes Q^{-it}v), \eta \otimes u \rangle = \langle W(Q^{-it} \otimes Q^{-it}v), Q^{-it}\eta \otimes Q^{-it}u \rangle \\ &= \langle \widetilde{W}(\overline{Q^{-it}\eta} \otimes Q^{-1}Q^{-it}v), \overline{Q^{-it}\xi} \otimes QQ^{-it}u \rangle = \langle \widetilde{W}([Q^\top]^{it}\bar{\eta} \otimes Q^{-it}Q^{-1}v), ([Q^\top]^{it}\bar{\xi} \otimes Q^{-it}Qu) \rangle \\ &= \langle ([Q^\top]^{-it} \otimes Q^{it})\widetilde{W}([Q^\top]^{it} \otimes Q^{-it})(\bar{\eta} \otimes Q^{-1}v), \bar{\xi} \otimes Qu \rangle, \end{aligned}$$

where we used (2) of Definition 2.3 in the third equality. Comparing the right hand sides, as  $\xi, \eta, v, u$  are arbitrary, it follows that

$$\widetilde{W} = ([Q^\top]^{-it} \otimes Q^{it})\widetilde{W}([Q^\top]^{it} \otimes Q^{-it}) = (Q^\top \otimes Q^{-1})^{-it}\widetilde{W}(Q^\top \otimes Q^{-1})^{it},$$

which is true for all  $t \in \mathbb{R}$ .  $\square$

As a consequence of Proposition 2.6, which holds true for all  $t \in \mathbb{R}$ , we can see that  $\widetilde{W}$  and  $(Q^\top \otimes Q^{-1})^{-1}\widetilde{W}(Q^\top \otimes Q^{-1})$  will agree whenever they are valid. Considering the domains, we thus obtain the following result:

$$\widetilde{W}(Q^\top \otimes Q^{-1}) \subseteq (Q^\top \otimes Q^{-1})\widetilde{W} \quad \text{and} \quad \widetilde{W}(Q^\top \otimes Q^{-1})^{-1} \subseteq (Q^\top \otimes Q^{-1})^{-1}\widetilde{W}. \quad (2.1)$$

We formulate below an alternative characterizing equation that is equivalent to (2) of Definition 2.3. This will be useful throughout the paper.

**Proposition 2.7.** *Let  $W$  be a manageable multiplicative partial isometry, and let  $Q$  and  $\widetilde{W}$  be the associated operators given in Definition 2.3. Then for any  $\xi \in \mathcal{D}(Q)$ ,  $\eta \in \mathcal{D}(Q^{-1})$  and any  $v, u \in \mathcal{H}$ , we have:*

$$\langle W(\xi \otimes v), \eta \otimes u \rangle = \langle \widetilde{W}(Q^{-1\top}\bar{\eta} \otimes v), Q^\top\bar{\xi} \otimes u \rangle.$$

*Proof.* Let  $\xi \in \mathcal{D}(Q)$ ,  $\eta \in \mathcal{D}(Q^{-1})$ , and for the time being, let  $v \in \mathcal{D}(Q^{-1})$  and  $u \in \mathcal{D}(Q)$ . Then note that  $Q^{-1\top}\bar{\eta} \otimes v \in \mathcal{D}(Q^\top \otimes Q^{-1}) = \mathcal{D}(\widetilde{W}(Q^\top \otimes Q^{-1})) \subseteq \mathcal{D}((Q^\top \otimes Q^{-1})\widetilde{W})$ , by the inclusion (2.1), and we have:

$$(Q^\top \otimes Q^{-1})\widetilde{W}(Q^{-1\top}\bar{\eta} \otimes v) = \widetilde{W}(Q^\top \otimes Q^{-1})(Q^{-1\top}\bar{\eta} \otimes v) = \widetilde{W}(\bar{\eta} \otimes Q^{-1}v).$$

It follows that

$$\begin{aligned} \langle \widetilde{W}(Q^{-1\top}\bar{\eta} \otimes v), Q^\top\bar{\xi} \otimes u \rangle &= \langle (Q^\top \otimes Q^{-1})\widetilde{W}(Q^{-1\top}\bar{\eta} \otimes v), \bar{\xi} \otimes Qu \rangle \\ &= \langle \widetilde{W}(\bar{\eta} \otimes Q^{-1}v), \bar{\xi} \otimes Qu \rangle = \langle W(\xi \otimes v), \eta \otimes u \rangle. \end{aligned}$$

This is true for any  $v \in \mathcal{D}(Q^{-1})$  and  $u \in \mathcal{D}(Q)$ , but considering that  $W$  and  $\widetilde{W}$  are bounded operators, we may extend this result to all  $v, u \in \mathcal{H}$ .  $\square$

Recall that if  $W$  is a multiplicative partial isometry, then so is  $\widehat{W} = \Sigma W^* \Sigma$ . If  $W$  is further known to be a manageable multiplicative partial isometry, then it can be shown that  $\widehat{W}$  is also manageable (see a similar result in Proposition 1.4 of [28]).

**Proposition 2.8.** *Let  $W$  be a manageable multiplicative partial isometry, and let  $Q$  and  $\widetilde{W}$  be the associated operators given in Definition 2.3. Then the operator  $\widehat{W} = \Sigma W^* \Sigma$  is also a multiplicative partial isometry, with the same  $Q$  and  $\widehat{\widetilde{W}} = (\Sigma \widetilde{W}^* \Sigma)^\top \otimes \top$ .*

*Proof.* (1). From  $W(Q \otimes Q) \subseteq (Q \otimes Q)W$ , it is easy to see that  $\widehat{W}(Q \otimes Q) \subseteq (Q \otimes Q)\widehat{W}$ .

(2). Write  $\widetilde{W} = (\Sigma\widetilde{W}^*\Sigma)^{\top\top}$ . For any  $\xi, \eta \in \mathcal{H}$  and  $v \in \mathcal{D}(Q^{-1})$ ,  $u \in \mathcal{D}$ , observe that

$$\begin{aligned} \langle \widetilde{W}(\bar{\eta} \otimes Q^{-1}v), \bar{\xi} \otimes Qu \rangle &= \langle (\Sigma\widetilde{W}^*\Sigma)^{\top\top}(\bar{\eta} \otimes Q^{-1}v), \bar{\xi} \otimes Qu \rangle = \langle \Sigma\widetilde{W}^*\Sigma(\xi \otimes Q^{\top}\bar{u}), \eta \otimes Q^{-1\top}\bar{v} \rangle \\ &= \langle \widetilde{W}^*(Q^{\top}\bar{u} \otimes \xi), Q^{-1\top}\bar{v} \otimes \eta \rangle = \langle \widetilde{W}(Q^{-1\top}\bar{v} \otimes \eta), Q^{\top}\bar{u} \otimes \xi \rangle \\ &= \langle W(u \otimes \eta), v \otimes \xi \rangle = \langle W^*(v \otimes \xi), u \otimes \eta \rangle \\ &= \langle \Sigma W^*\Sigma(\xi \otimes v), \eta \otimes u \rangle = \langle \widehat{W}(\xi \otimes v), \eta \otimes u \rangle. \end{aligned}$$

In the fifth equality, we used the alternative characterizing equation given in Proposition 2.7.

(3). Finally, we need to verify the two conditions  $\widetilde{W}_{13}\widetilde{W}_{23}[\widetilde{W}_{23}]^* = \widehat{W}_{12}^{\top\top}[\widehat{W}_{12}^*]^{\top\top}\widetilde{W}_{13}$  and  $\widehat{W}_{23}\widehat{W}_{23}^*\widehat{W}_{13} = \widetilde{W}_{13}\widetilde{W}_{12}[\widetilde{W}_{12}]^*$ . Indeed we have:

$$\begin{aligned} \widetilde{W}_{13}\widetilde{W}_{23}[\widetilde{W}_{23}]^* &= \Sigma_{13}[\widetilde{W}_{13}^*]^{\top\top}\Sigma_{13}\Sigma_{23}[\widetilde{W}_{23}^*]^{\top\top}\Sigma_{23}\Sigma_{23}[\widetilde{W}_{23}]^{\top\top}\Sigma_{23} \\ &= \Sigma_{13}[\widetilde{W}_{13}^*]^{\top\top}[\widetilde{W}_{23}^*]^{\top\top}[\widetilde{W}_{12}]^{\top\top}\Sigma_{13} = \Sigma_{13}[(\widetilde{W}_{13}\widetilde{W}_{12}\widetilde{W}_{12}^*)]^{\top\top\top\top}\Sigma_{13} \\ &= \Sigma_{13}[(W_{23}W_{23}^*\widetilde{W}_{13})^*]^{\top\top\top\top}\Sigma_{13} = \Sigma_{13}[W_{23}^*]^{\top\top}W_{23}^{\top\top}[\widetilde{W}_{13}^*]^{\top\top}\Sigma_{13} \\ &= [W_{21}^*]^{\top\top}W_{21}^{\top\top}[\widetilde{W}_{31}^*]^{\top\top} = \widehat{W}_{12}^{\top\top}[\widehat{W}_{12}^*]^{\top\top}\widetilde{W}_{13}, \end{aligned}$$

where we used the fact that  $m \mapsto m^{\top}$  is a  $*$ -anti-isomorphism, and the second condition in (3) of Definition 2.3 (for the fourth equality).

Also we have:

$$\begin{aligned} \widetilde{W}_{13}\widetilde{W}_{12}[\widetilde{W}_{12}]^* &= \Sigma_{13}[\widetilde{W}_{13}^*]^{\top\top}\Sigma_{13}\Sigma_{12}[\widetilde{W}_{12}^*]^{\top\top}\Sigma_{12}\Sigma_{12}\widetilde{W}_{12}^{\top\top}\Sigma_{12} \\ &= \Sigma_{13}[\widetilde{W}_{13}^*]^{\top\top}[\widetilde{W}_{23}^*]^{\top\top}\widetilde{W}_{23}^{\top\top}\Sigma_{13} = \Sigma_{13}[(\widetilde{W}_{13}\widetilde{W}_{23}\widetilde{W}_{23}^*)]^{\top\top\top\top}\Sigma_{13} \\ &= \Sigma_{13}[(W_{12}^{\top\top}W_{12}^*{}^{\top\top}\widetilde{W}_{13})^*]^{\top\top\top\top}\Sigma_{13} = \Sigma_{13}W_{12}^*W_{12}[\widetilde{W}_{13}^*]^{\top\top}\Sigma_{13} \\ &= W_{32}^*W_{32}[\widetilde{W}_{31}^*]^{\top\top} = \widehat{W}_{23}\widehat{W}_{23}^*\widetilde{W}_{13}, \end{aligned}$$

where we used the first condition in (3) of Definition 2.3 (the fourth equality).

By (1), (2), (3), we conclude that  $\widehat{W} = \Sigma W^*\Sigma$  is also manageable, with the same  $Q$  and  $\widetilde{W} = (\Sigma\widetilde{W}^*\Sigma)^{\top\top}$ .  $\square$

In the lemma below, we obtain a result that relates the operators  $W$ ,  $\widetilde{W}$ ,  $Q$ , and the transpose map  $^{\top}$ . Here, the linear functional  $\omega_{a,b} \in \mathcal{B}(\mathcal{H})_*$ , for  $a, b \in \mathcal{H}$ , is defined by  $\omega_{a,b}(T) = \langle Ta, b \rangle$  for  $T \in \mathcal{B}(\mathcal{H})$ . This is a standard notation, and such functionals are dense in  $\mathcal{B}(\mathcal{H})_*$ .

**Lemma 2.9.** *For  $u \in \mathcal{D}(Q)$  and  $v \in \mathcal{D}(Q^{-1})$ , we have:*

$$(\text{id} \otimes \omega_{Q^{-1}v, Qu})(\widetilde{W}) = (\text{id} \otimes \omega_{v, u})(W)^{\top}.$$

*Proof.* Let  $\xi, \eta \in \mathcal{H}$  be arbitrary. We have:

$$\begin{aligned} \langle (\text{id} \otimes \omega_{v, u})(W)^{\top}\bar{\eta}, \bar{\xi} \rangle &= \langle (\text{id} \otimes \omega_{v, u})(W)\xi, \eta \rangle = \langle W(\xi \otimes v), \eta \otimes u \rangle \\ &= \langle \widetilde{W}(\bar{\eta} \otimes Q^{-1}v), \bar{\xi} \otimes Qu \rangle = \langle (\text{id} \otimes \omega_{Q^{-1}v, Qu})(\widetilde{W})\bar{\eta}, \bar{\xi} \rangle. \end{aligned}$$

We used the characterizing equation for  $\widetilde{W}$ , given in (2) of Definition 2.3.  $\square$

The next proposition provides some key observations:

**Proposition 2.10.** *Let  $W$  be a manageable multiplicative partial isometry, and let  $Q$  and  $\widetilde{W}$  be the associated operators given in Definition 2.3. Then we have:*

- (1)  $W_{12}^{\top\top} \widetilde{W}_{23} W_{12}^{*\top\top} = \widetilde{W}_{13} \widetilde{W}_{23}$
- (2)  $W_{12}^{*\top\top} W_{12}^{\top\top} \widetilde{W}_{23} = \widetilde{W}_{23} W_{12}^{*\top\top} W_{12}^{\top\top}$
- (3)  $\widetilde{W}_{23} W_{12}^{*\top\top} \widetilde{W}_{23}^* = W_{12}^{*\top\top} \widetilde{W}_{13}$

*Proof.* (1). Let  $\xi, \eta, r, s \in \mathcal{H}$ , and  $u \in \mathcal{D}(Q)$ ,  $v \in \mathcal{D}(Q^{-1})$ . Then:

$$\langle W_{12}^{\top\top} \widetilde{W}_{23} W_{12}^{*\top\top} (\bar{\eta} \otimes \bar{r} \otimes Q^{-1}v), \bar{\xi} \otimes \bar{s} \otimes Qu \rangle = \langle W^{\top\top} [1 \otimes (\text{id} \otimes \omega_{Q^{-1}v, Qu})(\widetilde{W})] W^{*\top\top} (\bar{\eta} \otimes \bar{r}), \bar{\xi} \otimes \bar{s} \rangle. \quad (2.2)$$

By Lemma 2.9, we have:

$$W^{\top\top} [1 \otimes (\text{id} \otimes \omega_{Q^{-1}v, Qu})(\widetilde{W})] W^{*\top\top} = W^{\top\top} [1 \otimes (\text{id} \otimes \omega_{v, u})(W)]^{\top} W^{*\top\top},$$

which is equal to  $(W^* [1 \otimes (\text{id} \otimes \omega_{v, u})(W)] W)^{\top\top}$  because  $\top\top$  is an anti-homomorphism. Moreover,

$$W^* [1 \otimes (\text{id} \otimes \omega_{v, u})(W)] W = (\text{id} \otimes \text{id} \otimes \omega_{v, u})(W_{12}^* W_{23} W_{12}) = (\text{id} \otimes \text{id} \otimes \omega_{v, u})(W_{13} W_{23}),$$

by Equation (1.2). Putting these observations together, we see that the right hand side of Equation (2.2) is equal to

$$(RHS) = \langle (\text{id} \otimes \text{id} \otimes \omega_{v, u})(W_{13} W_{23})^{\top\top} (\bar{\eta} \otimes \bar{r}), \bar{\xi} \otimes \bar{s} \rangle = \langle W_{13} W_{23} (\xi \otimes s \otimes v), \eta \otimes r \otimes u \rangle.$$

Apply here the alternative characterization of  $\widetilde{W}$  given in Proposition 2.7, and write  $v = QQ^{-1}v$  and  $u = Q^{-1}Qu$ . Then we have:

$$\begin{aligned} (RHS) &= \langle W_{13} \widetilde{W}_{23} (\xi \otimes Q^{-1\top} \bar{r} \otimes QQ^{-1}v), \eta \otimes Q^{\top} \bar{s} \otimes Q^{-1}Qu \rangle \\ &= \langle W_{13} (\xi \otimes [(Q^{-1\top} \otimes Q) \widetilde{W} (\bar{r} \otimes Q^{-1}v)]), \eta \otimes Q^{\top} \bar{s} \otimes Q^{-1}Qu \rangle \\ &= \langle \widetilde{W}_{13} (\bar{\eta} \otimes [(Q^{-1\top} \otimes \text{id}) \widetilde{W} (\bar{r} \otimes Q^{-1}v)]), \bar{\xi} \otimes Q^{\top} \bar{s} \otimes Qu \rangle \\ &= \langle \widetilde{W}_{13} \widetilde{W}_{23} (\bar{\eta} \otimes \bar{r} \otimes Q^{-1}v), \bar{\xi} \otimes \bar{s} \otimes Qu \rangle. \end{aligned}$$

In the second equality, we used the fact that  $\widetilde{W}(Q^{-1\top} \otimes Q) \subseteq (Q^{-1\top} \otimes Q) \widetilde{W}$ , from (2.1), because  $\bar{r} \otimes Q^{-1}v \in \mathcal{D}(Q^{-1\top} \otimes Q)$ . The third equality is using Definition 2.3, while the fourth equality using the fact that  $Q^* = Q$ .

Putting this result back into Equation (2.2) above, we obtain:

$$\langle W_{12}^{\top\top} \widetilde{W}_{23} W_{12}^{*\top\top} (\bar{\eta} \otimes \bar{r} \otimes Q^{-1}v), \bar{\xi} \otimes \bar{s} \otimes Qu \rangle = \langle \widetilde{W}_{13} \widetilde{W}_{23} (\bar{\eta} \otimes \bar{r} \otimes Q^{-1}v), \bar{\xi} \otimes \bar{s} \otimes Qu \rangle.$$

As  $\xi, \eta, r, s, u, v$  are arbitrary, this proves that  $W_{12}^{\top\top} \widetilde{W}_{23} W_{12}^{*\top\top} = \widetilde{W}_{13} \widetilde{W}_{23}$ .

(2). Let  $\xi, \eta, r, s \in \mathcal{H}$ , and  $u \in \mathcal{D}(Q)$ ,  $v \in \mathcal{D}(Q^{-1})$ . Then:

$$\langle W_{12}^{*\top\top} W_{12}^{\top\top} \widetilde{W}_{23} (\bar{\eta} \otimes \bar{r} \otimes Q^{-1}v), \bar{\xi} \otimes \bar{s} \otimes Qu \rangle = \langle W^{*\top\top} W^{\top\top} [1 \otimes (\text{id} \otimes \omega_{Q^{-1}v, Qu})(\widetilde{W})] (\bar{\eta} \otimes \bar{r}), \bar{\xi} \otimes \bar{s} \rangle.$$

By using Lemma 2.9 and the fact that  $\top\top$  is an anti-homomorphism, we have:

$$\begin{aligned} W^{*\top\top} W^{\top\top} [1 \otimes (\text{id} \otimes \omega_{Q^{-1}v, Qu})(\widetilde{W})] &= W^{*\top\top} W^{\top\top} [1 \otimes (\text{id} \otimes \omega_{v, u})(W)]^{\top\top} \\ &= (\text{id} \otimes \text{id} \otimes \omega_{v, u})(W_{23} W_{12} W_{12}^*)^{\top\top} = (\text{id} \otimes \text{id} \otimes \omega_{v, u})(W_{12} W_{12}^* W_{23})^{\top\top} \\ &= [1 \otimes (\text{id} \otimes \omega_{Q^{-1}v, Qu})(\widetilde{W})] W^{*\top\top} W^{\top\top}, \end{aligned}$$

where we also used Equation (1.4). From this it follows that

$$\langle W_{12}^{*\top\top} W_{12}^{\top\top} \widetilde{W}_{23} (\bar{\eta} \otimes \bar{r} \otimes Q^{-1}v), \bar{\xi} \otimes \bar{s} \otimes Qu \rangle = \langle \widetilde{W}_{23} W_{12}^{*\top\top} W_{12}^{\top\top} (\bar{\eta} \otimes \bar{r} \otimes Q^{-1}v), \bar{\xi} \otimes \bar{s} \otimes Qu \rangle.$$

As  $\xi, \eta, r, s, u, v$  are arbitrary, this proves that  $W_{12}^{*\top\top} W_{12}^{\top\top} \widetilde{W}_{23} = \widetilde{W}_{23} W_{12}^{*\top\top} W_{12}^{\top\top}$ .

(3). From (1), we have:  $W_{12}^{\top\top} \widetilde{W}_{23} W_{12}^{*\top\top} = \widetilde{W}_{13} \widetilde{W}_{23}$ . Multiply  $W_{12}^{*\top\top}$  from the left and multiply  $\widetilde{W}_{23}^*$  from the right. then it becomes:

$$W_{12}^{*\top\top} W_{12}^{\top\top} \widetilde{W}_{23} W_{12}^{*\top\top} \widetilde{W}_{23}^* = W_{12}^{*\top\top} \widetilde{W}_{13} \widetilde{W}_{23} \widetilde{W}_{23}^*.$$

In the (LHS), apply (2), while in the (RHS), apply the condition (3) given in Definition 2.3. Then it becomes:

$$\widetilde{W}_{23} W_{12}^{*\top\top} W_{12}^{\top\top} W_{12}^{*\top\top} \widetilde{W}_{23}^* = W_{12}^{*\top\top} W_{12}^{\top\top} W_{12}^{*\top\top} \widetilde{W}_{13}. \quad (2.3)$$

Recall that  $W$  is a partial isometry, so  $WW^*W = W$ . Apply here the involution and the transpose map, which are both anti-homomorphisms. We have:  $W^{*\top\top} W^{\top\top} W^{*\top\top} = W^{*\top\top}$ . From this observation, it follows from Equation (2.3) that

$$\widetilde{W}_{23} W_{12}^{*\top\top} \widetilde{W}_{23}^* = W_{12}^{*\top\top} \widetilde{W}_{13}.$$

□

*Remark.* In section 2 of [28], Woronowicz showed that for a manageable multiplicative unitary  $W$ , we have:  $V_{12}^{\top\top} \widetilde{W}_{23} V_{12}^{*\top\top} \widetilde{W}_{23}^* = \widetilde{V}_{13}$ , where  $V$  is a unitary operator “adapted” to  $W$ , meaning that  $W_{23}V_{12} = V_{12}V_{13}W_{23}$ . As  $W$  itself is adapted to  $W$  (by the pentagon equation), as a special case we have:  $W_{12}^{\top\top} \widetilde{W}_{23} W_{12}^{*\top\top} \widetilde{W}_{23}^* = \widetilde{W}_{13}$ . This was a key observation by Woronowicz that enabled him to prove several other results that followed in [28]. In an unpublished manuscript of his, Woronowicz makes this point more prominent, by introducing the notion of a  $\#$ -composability (The author is indebted to him for showing his manuscript, as well as for his valuable comments on this topic.). In particular, we would say  $(V^{*\top\top}, \widetilde{W})$  is  $\#$ -composable, written  $V^{*\top\top} \# \widetilde{W} = \widetilde{V}$ .

Our Proposition 2.10 above can be considered as a modification of Woronowicz’s observation that  $W^{*\top\top} \# \widetilde{W} = \widetilde{W}$ . Note here that unlike the case of a multiplicative unitary, the properties (1) and (3) of the proposition are not necessarily equivalent, so we needed separate proofs. It may be possible to further pursue the notion of the  $\#$ -composability in the multiplicative partial isometry setting, but for our current work purposes, we will choose not to do so.

As a consequence of Proposition 2.10, we are now ready to prove that the subalgebras  $A$  and  $\widehat{A}$  are closed under the involution, so  $C^*$ -algebras:

**Theorem 2.11.** *Let  $W$  be a manageable multiplicative partial isometry satisfying the fullness condition, and let  $Q$  and  $\widetilde{W}$  be the associated operators given in Definition 2.3. Also let  $A$  and  $\widehat{A}$  be as given in Section 1:*

$$A := \overline{\text{span}\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}, \quad \text{and} \quad \widehat{A} := \overline{\text{span}\{(\omega \otimes \text{id})(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}.$$

*Then  $A$  and  $\widehat{A}$  are separable  $C^*$ -algebras acting on  $\mathcal{H}$  in a non-degenerate way.*

*Proof.* Suppose  $\zeta \in \mathcal{H}$ ,  $\zeta \neq 0$ . we claim that we can then find  $\xi, \eta \in \mathcal{H}$  such that  $(\text{id} \otimes \omega_{\xi, \eta})(W)\zeta \neq 0$ . Otherwise, we would have  $\langle (\text{id} \otimes \omega_{\xi, \eta})(W)\zeta, \tilde{\zeta} \rangle = 0$ , for any  $\xi, \eta, \tilde{\zeta} \in \mathcal{H}$ . But then, that means we have  $\langle W(\zeta \otimes \xi), \tilde{\zeta} \otimes \eta \rangle = \langle (\omega_{\zeta, \tilde{\zeta}} \otimes \text{id})(W)\xi, \eta \rangle = 0$ . Since  $\xi, \eta$  is arbitrary, that would imply  $(\omega_{\zeta, \tilde{\zeta}} \otimes \text{id})(W) = 0$ , which, by the *fullness* assumption on  $W$  (see Definition 2.2) in turn means that  $\omega_{\zeta, \tilde{\zeta}} = 0$  in  $\mathcal{B}(\mathcal{H})_*$ . Since  $\tilde{\zeta}$  is arbitrary, it would have to mean that  $\zeta = 0$ , a contradiction.

Write  $a = (\text{id} \otimes \omega_{\xi, \eta})(W)$ . Such elements generate  $\mathcal{A}$ . The contradiction proof in the previous paragraph means that for any  $\zeta \in \mathcal{H}$ ,  $\zeta \neq 0$ , we can find an  $a \in \mathcal{A}$  such that  $a\zeta \neq 0$ . This shows that  $\mathcal{A}$  is represented in  $\mathcal{B}(\mathcal{H})$  in a non-degenerate way. So is  $\mathcal{A}^*$ .

Next, recall (3) of Proposition 2.10, or  $\widetilde{W}_{23}W_{12}^*{}^{\top\otimes\top}\widetilde{W}_{23}^* = W_{12}^*{}^{\top\otimes\top}\widetilde{W}_{13}$ . Apply here  $(\cdot)^*{}^{\top\otimes\top}$ . As the involution and the transpose map are both anti-homomorphisms, we obtain:

$$[\widetilde{W}_{23}^*]{}^{\top\otimes\top}W_{12}[\widetilde{W}_{23}]{}^{\top\otimes\top} = W_{12}[\widetilde{W}_{13}^*]{}^{\top\otimes\top}.$$

Then apply  $\text{id} \otimes \omega' \otimes \omega$ , where  $\omega' \in \mathcal{B}(\mathcal{H})_*$  and  $\omega \in \mathcal{B}(\overline{\mathcal{H}})_*$  are arbitrary. We have:

$$(\text{id} \otimes \omega' \otimes \omega)([\widetilde{W}_{23}^*]{}^{\top\otimes\top}W_{12}[\widetilde{W}_{23}]{}^{\top\otimes\top}) = (\text{id} \otimes \omega')(W)(\text{id} \otimes \omega)([\widetilde{W}^*]{}^{\top\otimes\top}).$$

This equation can be re-written as

$$(\text{id} \otimes \rho)(W) = (\text{id} \otimes \omega')(W)(\text{id} \otimes \omega)([\widetilde{W}^*]{}^{\top\otimes\top}), \quad (2.4)$$

where  $\rho \in \mathcal{B}(\mathcal{H})_*$  is such that  $\rho(T) = (\omega' \otimes \omega)([\widetilde{W}^*]{}^{\top\otimes\top}(T \otimes 1)\widetilde{W}{}^{\top\otimes\top})$ .

We know that  $(\text{id} \otimes \omega')(W) \in \mathcal{A}$  and  $(\text{id} \otimes \rho)(W) \in \mathcal{A}$ . Meanwhile, to see where  $(\text{id} \otimes \omega)([\widetilde{W}^*]{}^{\top\otimes\top})$  belongs to, consider  $\omega = \omega_{\bar{v}, \bar{u}}$ , where  $v \in \mathcal{D}(Q^{-1})$ ,  $u \in \mathcal{D}(Q)$ . Note that we have:

$$(\text{id} \otimes \omega_{\bar{v}, \bar{u}})([\widetilde{W}^*]{}^{\top\otimes\top}) = ((\text{id} \otimes \omega_{u, v})(\widetilde{W})){}^{\top}{}^* = ((\text{id} \otimes \omega_{Qu, Q^{-1}v})(W)){}^* \in \mathcal{A}^*,$$

where we used the definition/property of the  $\top$  map, as well as Lemma 2.9. Functionals of the form  $\omega_{\bar{v}, \bar{u}}$  are dense in  $\mathcal{B}(\overline{\mathcal{H}})_*$ , so such elements are dense in  $\mathcal{A}^*$ .

As  $\omega'$ ,  $\omega_{\bar{v}, \bar{u}}$  are arbitrary, the observation made in Equation (2.4) means that  $\mathcal{A}\mathcal{A}^* \subseteq \mathcal{A}$ . Since  $\mathcal{A}^*$  is non-degenerately represented in  $\mathcal{B}(\mathcal{H})$ , we can see that  $\mathcal{A}\mathcal{A}^*$  is linearly dense in  $\mathcal{A}$ . With  $A = \overline{\mathcal{A}}^{\|\cdot\|}$ , it thus follows that  $AA^*$  is dense in  $A$ . Then since  $AA^*$  is  $*$ -closed, so should  $A$ , or  $A^* = A$ . Therefore  $A$  is a norm-closed  $*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$ , meaning that  $A$  is a  $C^*$ -algebra. Meanwhile, since  $\mathcal{H}$  is separable, the  $C^*$ -algebra  $A$  is separable.

Also by the fullness assumption on  $W$ , we see that  $\widehat{A}$  acts on  $\mathcal{H}$  in a non-degenerate way. Replacing  $W$  by  $\widehat{W} = \Sigma W^* \Sigma$  throughout, we can show that  $\widehat{A}^*$  is dense in  $\widehat{A}^* \widehat{A}$ , from which it follows that  $\widehat{A}^* = \widehat{A}$ , so  $\widehat{A}$  is also a (separable)  $C^*$ -algebra.  $\square$

### 3. THE COALGEBRA STRUCTURES ON $A$ AND $\widehat{A}$

Rest of the way, we assume that  $W$  is a full manageable multiplicative partial isometry, with the associated  $C^*$ -algebras  $A$  and  $\widehat{A}$ . In this section, we wish to explore the restrictions of the maps  $\Delta$  and  $\widehat{\Delta}$  considered in Proposition 1.5 to the subalgebras  $A$  and  $\widehat{A}$ , respectively, and show that they determine comultiplications on these subalgebras.

Before we construct the comultiplication map on  $A$ , let us prove the following lemma:

**Lemma 3.1.** (1) For any  $x \in A$ , we have:  $(1 \otimes x)WW^* = WW^*(1 \otimes x)$ .

(2) For any  $y \in \widehat{A}$ , we have:  $(y \otimes 1)W^*W = W^*W(y \otimes 1)$ .

*Proof.* (1). Let  $x = (\text{id} \otimes \omega)(W)$ , for an arbitrary  $\omega \in \mathcal{B}(\mathcal{H})_*$ . Note that

$$(1 \otimes x)WW^* = (\text{id} \otimes \text{id} \otimes \omega)(W_{23}W_{12}W_{12}^*) = (\text{id} \otimes \text{id} \otimes \omega)(W_{12}W_{12}^*W_{23}) = WW^*(1 \otimes x),$$

by Equation (1.4).

(2). The proof that  $(y \otimes 1)W^*W = W^*W(y \otimes 1)$ , for  $y \in \widehat{A}$ , is similarly done, using Equation (1.3).  $\square$

**Corollary.** (1) For any  $m \in M(A)$ , we have:  $(1 \otimes m)WW^* = WW^*(1 \otimes m)$ .

(2) For any  $n \in M(\widehat{A})$ , we have:  $(n \otimes 1)W^*W = W^*W(n \otimes 1)$ .

*Proof.* (1). Let  $m \in M(A)$ . Then for any  $a \in A$ , we know that  $am \in A$ . Then by the above lemma, we have  $(1 \otimes am)WW^* = WW^*(1 \otimes am)$ , or  $(1 \otimes a)(1 \otimes m)WW^* = WW^*(1 \otimes a)(1 \otimes m)$ . By applying the lemma again, this becomes

$$(1 \otimes a)(1 \otimes m)WW^* = WW^*(1 \otimes a)(1 \otimes m) = (1 \otimes a)WW^*(1 \otimes m).$$

As this result is true for any  $a \in A$ , it follows that  $(1 \otimes m)WW^* = WW^*(1 \otimes m)$ .

Proof for (2) is similar.  $\square$

Consider the map  $\Delta : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  introduced earlier, and consider its restriction to the subalgebra  $A$ . The next proposition shows that it determines a  $*$ -homomorphism on  $A$ , which extends to a  $*$ -homomorphism on  $M(A)$ .

**Proposition 3.2.** *Consider the map  $\Delta : A \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ , given by*

$$\Delta(x) = W^*(1 \otimes x)W, \quad x \in A.$$

*This determines a  $*$ -homomorphism on  $A$ , which extends to a  $*$ -homomorphism  $\Delta : M(A) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ .*

*Proof.* It is evident that  $\Delta$  is a  $*$ -map. Meanwhile, let  $a, b \in A$ . Then we have:

$$\Delta(a)\Delta(b) = W^*(1 \otimes a)WW^*(1 \otimes b)W = W^*WW^*(1 \otimes a)(1 \otimes b)W = W^*(1 \otimes ab)W = \Delta(ab),$$

by Lemma 3.1 (1). By its Corollary, we can also see that  $\Delta$  extends to a  $*$ -homomorphism on  $M(A)$ .  $\square$

In fact, we can show later that  $\Delta(A) \subseteq M(A \otimes A)$ . But for the time being, let us turn our attention to exploring the properties of the projection  $E = W^*W$ .

**Proposition 3.3.** (1) *We have:  $E = W^*W \in M(A \otimes A)$ .*

(2) *We have:  $E = W^*W = \Delta(1_{M(A)}) = \Delta(1)$ .*

*Proof.* (1). Knowing  $A = A^*$ , consider  $x = (\text{id} \otimes \omega)(W) \in A$  and  $y = (\text{id} \otimes \omega')(W^*) \in A$ , for  $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$ . Such elements are dense in  $A$ . Then we have:

$$\begin{aligned} E(x \otimes y) &= (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{12}^* W_{12} W_{13} W_{24}^*) \\ &= (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{13} W_{23} W_{23}^* W_{24}^*) = (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{13} W_{24}^* W_{43} W_{43}^*), \end{aligned}$$

where we used Equation (1.6) for the second equality and the conjugate of Equation (1.10) for the third.

Without loss of generality, we may take  $\omega = \omega(\cdot k)$  and  $\omega' = \omega(\cdot k')$ , where  $k, k' \in \mathcal{B}_0(\mathcal{H})$  are arbitrary compact operators. As  $W \in M(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}_0(\mathcal{H}))$ , we know we can approximate  $W_{21} W_{21}^*(k \otimes k')$  by the elements of the form  $p \otimes p'$ , where  $p, p' \in \mathcal{B}_0(\mathcal{H})$ . This means that we can approximate  $(\omega \otimes \omega')(\cdot W_{21} W_{21}^*(k \otimes k'))$  by the functionals of the form  $\theta \otimes \theta'$ , where  $\theta = \omega(\cdot p)$  and  $\theta' = \omega(\cdot p')$ . It follows that  $E(x \otimes y)$  can be approximated by the elements of the form  $(\text{id} \otimes \text{id} \otimes \theta \otimes \theta')(W_{13} W_{24}^*) = a_1 \otimes a_2$ , where  $a_1 = (\text{id} \otimes \theta)(W) \in A$ ,  $a_2 = (\text{id} \otimes \theta')(W^*) \in A$ , so  $E(x \otimes y) \in A \otimes A$ . As  $x, y$  are arbitrary, this shows that  $E \in M(A \otimes A)$ .

(2). We know that  $A$  acts on  $\mathcal{H}$  in a non-degenerate way. Then  $1_{M(A)} = \text{Id}_{\mathcal{B}(\mathcal{H})} = 1$ . The result of Proposition 3.2 confirms that  $E = \Delta(1_{M(A)})$ .  $\square$

Here is another result that seems natural, from the observation  $E = \Delta(1)$ :

**Proposition 3.4.**  *$E \otimes 1$  and  $1 \otimes E$  commute. That is, we have:*

$$(E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1),$$

*which is also equal to  $W_{12}^* W_{23}^* W_{23} W_{12} = W_{23}^* W_{12}^* W_{12} W_{23}$ .*

*Proof.* By Equation (1.3), we have:  $(E \otimes 1)(1 \otimes E) = W_{12}^* W_{12} W_{23}^* W_{23} = W_{12}^* W_{23}^* W_{23} W_{12}$ , while  $(1 \otimes E)(E \otimes 1) = W_{23}^* W_{23} W_{12}^* W_{12} = W_{23}^* W_{12}^* W_{12} W_{23} = W_{12}^* W_{12} W_{23}^* W_{23}$ , showing that the four expressions are all same.  $\square$

*Remark.* Proposition 3.4 is essentially saying that  $(\Delta \otimes \text{id})\Delta(1_{M(A)}) = (\text{id} \otimes \Delta)\Delta(1_{M(A)})$ , which seems natural, considering the result of Proposition 3.3 and the earlier observation on the coassociativity, in Proposition 1.5.

The following result shows that  $\Delta(A) \subseteq M(A \otimes A)$ , but we actually prove a stronger result:

**Proposition 3.5.** *Let  $a, b \in A$  be arbitrary. We have:*

$$\begin{aligned} (a \otimes 1)(\Delta b) &\in A \otimes A, & (\Delta a)(1 \otimes b) &\in A \otimes A, \\ (\Delta a)(b \otimes 1) &\in A \otimes A, & (1 \otimes a)(\Delta b) &\in A \otimes A. \end{aligned}$$

*Proof.* Let  $a = (\text{id} \otimes \omega)(W)$ ,  $b = (\text{id} \otimes \omega')(W)$ , for arbitrary  $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$ . Such elements are dense in  $A$ . We have:

$$\begin{aligned} (a \otimes 1)(\Delta b) &= (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{13} W_{12}^* W_{24} W_{12}) = (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{13} W_{14} W_{24}) \\ &= (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{34} W_{13} W_{34}^* W_{24}) = (\text{id} \otimes \text{id} \otimes \rho)(W_{13} W_{24}), \end{aligned}$$

where we used Equations (1.2) and (1.1) in the second and the third equalities, respectively. In the last line, the functional  $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})_*$  is such that  $\rho(S \otimes T) := (\omega \otimes \omega')(W(S \otimes 1)W^*(1 \otimes T))$ . As we may approximate  $\rho$  by the functionals of the form  $\theta_1 \otimes \theta_2$ , this means that  $(a \otimes 1)(\Delta b)$  can be approximated by the elements of the form  $(\text{id} \otimes \text{id} \otimes \theta_1 \otimes \theta_2)(W_{13} W_{24}) = a_1 \otimes a_2$ , where  $a_1 = (\text{id} \otimes \theta_1)(W) \in A$ ,  $a_2 = (\text{id} \otimes \theta_2)(W) \in A$ . So  $(a \otimes 1)(\Delta b) \in A \otimes A$ , proving the first result.

Similarly, for  $a = (\text{id} \otimes \omega)(W)$ ,  $b = (\text{id} \otimes \omega')(W)$ , we also have:

$$\begin{aligned} (1 \otimes a)(\Delta b) &= (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{24} W_{12}^* W_{23} W_{12}) \\ &= (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{24} W_{13} W_{23}) = (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{13} W_{24} W_{23}) \\ &= (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{13} W_{43} W_{24} W_{43}^*) = (\text{id} \otimes \text{id} \otimes \tilde{\rho})(W_{13} W_{24}), \end{aligned}$$

Here  $\tilde{\rho} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})_*$  is such that  $\tilde{\rho}(S \otimes T) := (\omega \otimes \omega')((S \otimes 1)W_{21}(1 \otimes T)W_{21}^*)$ , or equivalently  $\tilde{\rho}(S \otimes T) = (\omega' \otimes \omega)((1 \otimes S)W(T \otimes 1)W^*)$ . As before, we can approximate  $\tilde{\rho}$  by the functionals of the form  $\theta_1 \otimes \theta_2$ , so we can approximate  $(1 \otimes a)(\Delta b)$  by the elements of the form  $a_1 \otimes a_2$ , where  $a_1, a_2 \in A$ . It follows that  $(1 \otimes a)(\Delta b) \in A \otimes A$ , proving the fourth result.

The second and the third results can be obtained by taking the adjoints of the first and the fourth, respectively.  $\square$

**Corollary.** *For any  $a \in A$ , we have:  $\Delta a \in M(A \otimes A)$ .*

*Proof.* Let  $a \in A$  and consider  $\Delta a = W^*(1 \otimes a)W$ . Let  $b, c \in A$  be arbitrary. Then by Proposition 3.5, we have  $(\Delta a)(b \otimes 1) \in A \otimes A$ , so  $(\Delta a)(b \otimes c) \in A \otimes A$ . Similarly, we have  $(b \otimes c)(\Delta a) \in A \otimes A$ . It follows that  $\Delta a \in M(A \otimes A)$ .  $\square$

We thus have the \*-homomorphism  $\Delta : A \rightarrow M(A \otimes A)$ . It satisfies the following density results (so  $\Delta$  is “full”).

**Proposition 3.6.** *Let  $\Delta : A \rightarrow M(A \otimes A)$  be as defined above. Then the following subspaces are norm-dense in  $A$ :*

$$\begin{aligned} \text{span}\{(\theta \otimes \text{id})((a \otimes 1)(\Delta b)) : \theta \in A^*, a, b \in A\}, & \quad \text{span}\{(\text{id} \otimes \theta)((\Delta a)(1 \otimes b)) : \theta \in A^*, a, b \in A\}, \\ \text{span}\{(\theta \otimes \text{id})((\Delta b)(a \otimes 1)) : \theta \in A^*, a, b \in A\}, & \quad \text{span}\{(\text{id} \otimes \theta)((1 \otimes b)(\Delta a)) : \theta \in A^*, a, b \in A\}. \end{aligned}$$

*Proof.* Let  $a = (\text{id} \otimes \omega)(W)$ ,  $b = (\text{id} \otimes \omega')(W)$ , for  $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$ . Such elements are dense in  $A$ . We saw from the proof of Proposition 3.5 that  $(a \otimes 1)(\Delta b) \in A \otimes A$ , and that  $(a \otimes 1)(\Delta b) = (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{13}W_{12}^*W_{24}^*W_{12}) = (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_{13}W_{14}W_{24})$ . Apply here  $(\theta \otimes \text{id})$ , for any  $\theta \in A^*$ . As we know that  $A$  acts non-degenerately on  $\mathcal{H}$ , it is all right to take an arbitrary  $\theta \in \mathcal{B}(\mathcal{H})_*$ . Furthermore, without loss of generality, we may take  $\theta = \theta(k_0 \cdot)$ ,  $\omega = \omega(k_1 \cdot)$ ,  $\omega' = \omega'(k_2 \cdot)$ , where  $k_0, k_1, k_2 \in \mathcal{B}_0(\mathcal{H})$  are arbitrary compact operators. Then

$$(\theta \otimes \text{id})((a \otimes 1)(\Delta b)) = (\theta \otimes \text{id} \otimes \omega \otimes \omega')((k_0 \otimes 1 \otimes k_1 \otimes k_2)(W_{13}W_{14}W_{24})).$$

As  $W \in M(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}_0(\mathcal{H}))$ , we can approximate  $(k_0 \otimes k_1)W$  by the elements of the form  $p_0 \otimes p_1$ , where  $p_0, p_1 \in \mathcal{B}_0(\mathcal{H})$ . This means that the elements  $(\theta \otimes \text{id})((a \otimes 1)(\Delta b))$  can be approximated by the following elements:

$$(\theta \otimes \text{id} \otimes \omega \otimes \omega')((p_0 \otimes 1 \otimes p_1 \otimes k_2)(W_{14}W_{24})) = \omega(p_1)(\theta \otimes \text{id} \otimes \omega')((p_0 \otimes 1 \otimes k_2)W_{13}W_{23}).$$

Similarly, we can approximate  $(p_0 \otimes k_2)W$  by  $p \otimes p'$ , for  $p, p' \in \mathcal{B}_0(\mathcal{H})$ . In other words, the elements  $(\theta \otimes \text{id})((a \otimes 1)(\Delta b))$  can be approximated by the elements of the form  $\omega(p_1)\theta(p)(\text{id} \otimes \omega'(p' \cdot))(W)$ . As the functionals  $\theta, \omega, \omega'$  are arbitrary (so also  $p, p_1, p'$ ), such elements span a dense space in  $A$ . This shows that

$$\overline{\text{span}\{(\theta \otimes \text{id})((a \otimes 1)(\Delta b)) : \theta \in A^*, a, b \in A\}}^{\|\cdot\|} = A.$$

The proofs for the other three density results can be done similarly.  $\square$

The following theorem clarifies the comultiplication map  $\Delta : A \rightarrow M(A \otimes A)$ .

**Theorem 3.7.** *Consider the restriction of the map  $\Delta : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ , to the subalgebra  $A$ . Namely,*

$$\Delta(x) = W^*(1 \otimes x)W, \quad \text{for } x \in A.$$

- (1) *This determines a \*-homomorphism  $\Delta : A \rightarrow M(A \otimes A)$ .*
- (2) *The comultiplication is “full”, in the sense that the density results of Proposition 3.6 hold.*
- (3) *We have:  $\overline{\Delta(A)(A \otimes A)}^{\|\cdot\|} = E(A \otimes A)$ , and  $\overline{(A \otimes A)\Delta(A)}^{\|\cdot\|} = (A \otimes A)E$ .*
- (4) *We do not expect  $\Delta$  to be non-degenerate, but it nonetheless extends to a \*-homomorphism  $\Delta : M(A) \rightarrow M(A \otimes A)$ .*
- (5) *The coassociativity property holds:*

$$(\Delta \otimes \text{id})\Delta(x) = (\text{id} \otimes \Delta)\Delta(x), \quad \text{for any } x \in A.$$

*As such, we will refer to the map  $\Delta$  as the comultiplication on  $A$ .*

*Proof.* (1). Proposition 3.2 showed that  $\Delta$  is a \*-homomorphism, and Proposition 3.5 and its Corollary showed that  $\Delta(A) \subseteq M(A \otimes A)$ .

(2). This is Proposition 3.6.

(3). For any  $a \in A$ , we saw that  $\Delta(a) \in M(A \otimes A)$ . So for  $b, c \in A$ , we can approximate  $\Delta(a)(b \otimes c)$  by the elements of the form  $a_1 \otimes a_2$ , where  $a_1, a_2 \in A$ . At the same time, note that  $\Delta a = W^*(1 \otimes a)W = W^*WW^*(1 \otimes a)W = E(\Delta a)$ . This means that any  $(\Delta a)(b \otimes c) = E(\Delta a)(b \otimes c)$  can be approximated by the elements of the form  $E(a_1 \otimes a_2)$ . Therefore, we have:  $\overline{\Delta(A)(A \otimes A)}^{\|\cdot\|} \subseteq E(A \otimes A)$ . As  $E$  is a projection, we have that  $E(A \otimes A)$  is already norm-closed.

Let  $(a_i)$  be an approximate unit in the  $C^*$ -algebra  $A$ . Then  $a_i \xrightarrow{\text{strictly}} 1$  in  $M(A)$ . We saw that  $\Delta$  is a \*-homomorphism on  $M(A)$ , so continuous. We thus have:  $\Delta(a_i) \xrightarrow{\text{strictly}} \Delta(1) = E$ . As a consequence, for any  $b, c \in A$ , we have the norm convergence,  $\Delta(a_i)(b \otimes c) \xrightarrow{\text{norm}} E(b \otimes c)$ .



In this way, we show that  $E(A \otimes A) \subseteq \overline{\Delta(A)(A \otimes A)}^{\|\cdot\|}$ . The two inclusions mean that we have  $\overline{\Delta(A)(A \otimes A)}^{\|\cdot\|} = E(A \otimes A)$ .

The proof for  $\overline{(A \otimes A)\Delta(A)}^{\|\cdot\|} = (A \otimes A)E$  can be done in a similar way, as we have  $\Delta a = (\Delta a)E$ , for any  $a \in A$ .

(4). As we noted in Proposition 3.3, we have:  $\Delta(1_{M(A)}) = E$ . Since  $E$  is only a projection, we do not have the non-degeneracy for  $\Delta$ . This can be also observed in (3) above, as  $\overline{\Delta(A)(A \otimes A)}^{\|\cdot\|} = E(A \otimes A) \subsetneq A \otimes A$ .

Nevertheless, it is possible to naturally extend  $\Delta$  to the level of  $M(A)$ . See Proposition 3.3 of [10]. The results (1), (2), (3) above provide the necessary conditions for the proposition to apply. The resulting extension map,  $\Delta : M(A) \rightarrow M(A \otimes A)$ , is a  $*$ -homomorphism that coincides with the extended  $\Delta$  map observed in Proposition 3.2.

(5). The coassociativity of  $\Delta$  has been already shown in Proposition 1.5.  $\square$

Replace  $W$  with  $\widehat{W} = \Sigma W^* \Sigma$ , which is also a manageable multiplicative partial isometry. We noted earlier that  $\overline{\{(\text{id} \otimes \omega)(\widehat{W}) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|} = \widehat{A}^* = \widehat{A}$ . As such, the results obtained in the earlier part of this section for  $(A, \Delta)$  will all have corresponding results, with the role of the canonical idempotent being played by  $\widehat{E} = \Sigma W W^* \Sigma$ . The main results are given in the following Theorem 3.8, clarifying the coalgebra structure on  $\widehat{A}$ .

**Theorem 3.8.** (1) Write  $\widehat{E} = \widehat{W}^* \widehat{W} = \Sigma W W^* \Sigma$ . We have:  $\widehat{E} \in M(\widehat{A} \otimes \widehat{A})$ .

(2)  $\widehat{E} = \Sigma W W^* \Sigma = \widehat{\Delta}(1_{M(\widehat{A})}) = \widehat{\Delta}(1)$ .

(3)  $(\widehat{E} \otimes 1)(1 \otimes \widehat{E}) = (1 \otimes \widehat{E})(\widehat{E} \otimes 1)$ .

(4) The restriction of  $\widehat{\Delta}$  to  $\widehat{A}$  determines a  $*$ -homomorphism  $\widehat{\Delta} : \widehat{A} \rightarrow M(\widehat{A} \otimes \widehat{A})$ . Namely,

$$\widehat{\Delta}(y) = \Sigma W(y \otimes 1)W^* \Sigma, \quad y \in \widehat{A}.$$

It is “full”, in the sense that it satisfies the following subspaces are norm-dense in  $\widehat{A}$ :

$$\begin{aligned} \text{span}\{(\theta \otimes \text{id})((c \otimes 1)(\widehat{\Delta}d)) : \theta \in \widehat{A}^*, c, d \in \widehat{A}\}, & \quad \text{span}\{(\text{id} \otimes \theta)((\widehat{\Delta}c)(1 \otimes d)) : \theta \in \widehat{A}^*, c, d \in \widehat{A}\}, \\ \text{span}\{(\theta \otimes \text{id})((\widehat{\Delta}d)(c \otimes 1)) : \theta \in \widehat{A}^*, c, d \in \widehat{A}\}, & \quad \text{span}\{(\text{id} \otimes \theta)((1 \otimes d)(\widehat{\Delta}c)) : \theta \in \widehat{A}^*, c, d \in \widehat{A}\}. \end{aligned}$$

(5) We have:  $\overline{\widehat{\Delta}(\widehat{A})(\widehat{A} \otimes \widehat{A})}^{\|\cdot\|} = \widehat{E}(\widehat{A} \otimes \widehat{A})$ , and  $\overline{(\widehat{A} \otimes \widehat{A})\widehat{\Delta}(\widehat{A})}^{\|\cdot\|} = (\widehat{A} \otimes \widehat{A})\widehat{E}$ .

(6)  $\widehat{\Delta}$  extends to a  $*$ -homomorphism  $\widehat{\Delta} : M(\widehat{A}) \rightarrow M(\widehat{A} \otimes \widehat{A})$ , and the coassociativity property holds:

$$(\widehat{\Delta} \otimes \text{id})\widehat{\Delta}(y) = (\text{id} \otimes \widehat{\Delta})\widehat{\Delta}(y), \quad \text{for any } y \in \widehat{A}.$$

*Proof.* (1), (2). These results are analogous to Proposition 3.3.

(3). This is analogous to Proposition 3.4.

(4). As in Proposition 3.2, we use Lemma 3.1 to prove that  $\widehat{\Delta}$  is a  $*$ -homomorphism. The results similar to Proposition 3.5 and its Corollary would show that  $\widehat{\Delta}(\widehat{A}) \subseteq M(\widehat{A} \otimes \widehat{A})$ . The “fullness” of  $\widehat{\Delta}$  is analogous to Proposition 3.6.

(5). Analogous to (3) of Theorem 3.7.

(6). Analogous to (4), (5) of Theorem 3.7. The coassociativity of  $\widehat{\Delta}$  has been already observed in Proposition 1.5.  $\square$

We now have a pair of  $C^*$ -bialgebras  $(A, \Delta)$  and  $(\widehat{A}, \widehat{\Delta})$ . In the ensuing sections, we will construct more structures on them.

## 4. THE BASE ALGEBRAS

Recall from Section 1 the following subspaces in  $\mathcal{B}(\mathcal{H})$ :

$$\begin{aligned} N &:= \overline{\text{span}\{(\text{id} \otimes \omega)(W^*W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}}, & L &:= \overline{\text{span}\{(\omega \otimes \text{id})(W^*W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}}, \\ \widehat{N} &:= \overline{\text{span}\{(\omega \otimes \text{id})(WW^*) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}}, & \widehat{L} &:= \overline{\text{span}\{(\text{id} \otimes \omega)(WW^*) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}}. \end{aligned}$$

In this section, we will first show that  $N, L, \widehat{N}, \widehat{L}$  are in fact  $W^*$ -subalgebras of  $\mathcal{B}(\mathcal{H})$ . In turn, we will later find their  $C^*$ -algebra counterparts.

We begin with a lemma, showing that the generators of  $N, L, \widehat{N}, \widehat{L}$  behave like multipliers in  $M(A)$  or  $M(\widehat{A})$ . In the below, note that  $b \in N, \hat{b} \in \widehat{N}, c \in L, \hat{c} \in \widehat{L}$ .

- Lemma 4.1.** (1) *Let  $b = (\text{id} \otimes \omega)(W^*W)$ , where  $\omega \in \mathcal{B}(\mathcal{H})_*$  is arbitrary. Then for any  $x \in A$ , we have  $bx \in A$ .*
- (2) *Let  $\hat{b} = (\omega \otimes \text{id})(WW^*)$ , where  $\omega \in \mathcal{B}(\mathcal{H})_*$  is arbitrary. Then for any  $y \in \widehat{A}$ , we have  $y\hat{b} \in \widehat{A}$ .*
- (3) *Let  $c = (\omega \otimes \text{id})(W^*W)$ , where  $\omega \in \mathcal{B}(\mathcal{H})_*$  is arbitrary. Then for any  $x \in A$ , we have:  $xc \in A$ . Also for any  $y \in \widehat{A}$ , we have:  $cy \in \widehat{A}$ .*
- (4) *Let  $\hat{c} = (\text{id} \otimes \omega)(WW^*)$ , where  $\omega \in \mathcal{B}(\mathcal{H})_*$  is arbitrary. Then for any  $x \in A$ , we have:  $x\hat{c} \in A$ . Also for any  $y \in \widehat{A}$ , we have:  $\hat{c}y \in \widehat{A}$ .*

*Proof.* (1). Let  $x = (\text{id} \otimes \theta)(W) \in A$ , for  $\theta \in \mathcal{B}(\mathcal{H})_*$ . By Equation (1.6), we have:

$$\begin{aligned} bx &= (\text{id} \otimes \omega \otimes \theta)(W_{12}^*W_{12}W_{13}) = (\omega \otimes \text{id} \otimes \theta)(W_{13}W_{23}W_{23}^*) \\ &= (\text{id} \otimes \theta)(W(1 \otimes (\omega \otimes \text{id})(WW^*))) = (\text{id} \otimes \theta)(W(1 \otimes q)) = (\text{id} \otimes \rho)(W) \in A, \end{aligned}$$

where  $q = (\omega \otimes \text{id})(WW^*)$ , and  $\rho(\cdot) = \theta(\cdot q) \in \mathcal{B}(\mathcal{H})_*$ . As  $\theta$  is arbitrary, this shows that  $bx \in A$  for any  $x \in A$ .

(2). Let  $y = (\theta \otimes \text{id})(W) \in \widehat{A}$ . Again by using Equation (1.6), we can show that

$$y\hat{b} = \cdots = (\theta \otimes \text{id})((p \otimes 1)W) = (\theta(p \cdot) \text{id})(W) \in \widehat{A},$$

where  $p = (\text{id} \otimes \omega)(W^*W)$ . Since  $\theta \in \mathcal{B}(\mathcal{H})_*$  is arbitrary, this means  $y\hat{b} \in \widehat{A}, \forall y \in \widehat{A}$ .

(3). Consider  $x = (\text{id} \otimes \theta)(W) \in A, \theta \in \mathcal{B}(\mathcal{H})_*$ . By a similar approach as above, but now using Equation (1.9), we can show that

$$xc = \cdots = (\text{id} \otimes \theta)((1 \otimes c)W) = (\text{id} \otimes \theta(c \cdot))(W) \in A.$$

This shows that  $xc \in A$  for any  $x \in A$ . Moreover, if we consider  $y = (\theta \otimes \text{id})(W) \in \widehat{A}$ , by using Equation (1.10), we observe that

$$cy = \cdots = (\theta \otimes \text{id})(W(c \otimes 1)) = (\theta(\cdot c) \otimes \text{id})(W) \in \widehat{A}.$$

This shows that  $cy \in \widehat{A}$  for any  $y \in \widehat{A}$ .

(4). As in (3), for  $\hat{c} = (\text{id} \otimes \omega)(WW^*)$ , we can show that  $x\hat{c} \in A$  for any  $x \in A$ , and  $\hat{c}y \in \widehat{A}$  for any  $y \in \widehat{A}$ .  $\square$

In the lemma above, we observe that while similar, the elements in  $L$  and  $\widehat{L}$  behave slightly differently than those in  $N$  and  $\widehat{N}$ . There seems to be a little more of a symmetric behavior going on for the elements in  $L$  and  $\widehat{L}$ . This is no accident, as we can see from the following proposition:

**Proposition 4.2.** *We have:  $L = \widehat{L}$ .*

*Proof.* Let  $\omega, \theta \in \mathcal{B}(\mathcal{H})_*$  be arbitrary. Write:  $y = (\omega \otimes \text{id})(W) \in \widehat{A}$  and  $x = (\text{id} \otimes \theta)(W) \in A$ . Consider  $\tilde{\omega} := \omega(\cdot x)$  and  $\tilde{\theta} := \theta(y \cdot)$ . Observe that

$$\begin{aligned} (\text{id} \otimes \tilde{\theta})(WW^*) &= (\text{id} \otimes \theta)((1 \otimes y)WW^*) = (\omega \otimes \text{id} \otimes \theta)(W_{13}W_{23}W_{23}^*) \\ &= (\omega \otimes \text{id} \otimes \theta)(W_{12}^*W_{12}W_{13}) = (\omega \otimes \text{id})(W^*W(x \otimes 1)) = (\tilde{\omega} \otimes \text{id})(W^*W). \end{aligned} \quad (4.1)$$

by Equation (1.6).

As  $\omega, \theta$  are arbitrary and since  $A$  and  $\widehat{A}$  act on  $\mathcal{H}$  in a non-degenerate way, it is evident that the functionals of the form  $\tilde{\omega}$  and  $\tilde{\theta}$  generate  $\mathcal{B}(\mathcal{H})_*$ . Therefore, the elements  $(\text{id} \otimes \tilde{\theta})(WW^*)$  are dense in  $\widehat{N}$  and the elements  $(\tilde{\omega} \otimes \text{id})(W^*W)$  are dense in  $N$ . So Equation (4.1) indicates that  $\widehat{N} = N$ .  $\square$

*Remark.* There is no such result for  $N$  and  $\widehat{N}$ . While it can be shown that  $N \cong \widehat{N}$ , we have  $N \neq \widehat{N}$ , in general.

We next turn our attention to proving that  $L$  is a subalgebra of  $\mathcal{B}(\mathcal{H})$ .

**Proposition 4.3.**  $L = \overline{\text{span}\{(\omega \otimes \text{id})(W^*W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}}$ , is a subalgebra in  $\mathcal{B}(\mathcal{H})$ .

*Proof.* Consider arbitrary  $\rho, \theta, \omega \in \mathcal{B}(\mathcal{H})_*$  and let  $x = (\text{id} \otimes \theta)(W) \in A$  and  $y = (\omega \otimes \text{id})(W) \in \widehat{A}$ . The elements of the form  $c = (\rho \otimes \text{id})(W^*W)$  and also those of the form  $c' = (\omega \otimes \text{id})(W^*W(x \otimes 1))$  are dense in  $L$ , because  $A$  is non-degenerate.

Meanwhile, by Equation (4.1), we know that  $(\omega \otimes \text{id})(W^*W(x \otimes 1)) = (\text{id} \otimes \theta)((1 \otimes y)WW^*)$ . As a result, we have:

$$\begin{aligned} cc' &= (\rho \otimes \text{id})(W^*W)(\omega \otimes \text{id})(W^*W(x \otimes 1)) = (\omega \otimes \rho \otimes \text{id})(W_{23}^*W_{23}W_{13}^*W_{13}(x \otimes 1 \otimes 1)) \\ &= (\text{id} \otimes \rho \otimes \theta)(W_{23}^*W_{23}(1 \otimes 1 \otimes y)W_{13}W_{13}^*) \\ &= (\text{id} \otimes \theta)((1 \otimes cy)WW^*) = (\text{id} \otimes \tilde{\theta})(WW^*), \end{aligned}$$

where  $\tilde{\theta} = \tilde{\theta}(cy \cdot)$ . So  $cc' \in \widehat{L}$ , thus  $cc' \in L$ , as we know from Proposition 4.2 that  $L = \widehat{L}$ .  $\square$

**Corollary.**  $\widehat{L} = \overline{\text{span}\{(\text{id} \otimes \omega)(WW^*) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}}$  is a subalgebra in  $\mathcal{B}(\mathcal{H})$ .

*Proof.* Since we know  $L = \widehat{L}$ , this is immediate from the proposition.  $\square$

We also wish to see that  $N$  and  $\widehat{N}$  are subalgebras in  $\mathcal{B}(\mathcal{H})$ . But before jumping into proving these results, let us first consider the following result, which is a consequence of the condition (3) of the manageability of  $W$  (Definition 2.3).

**Proposition 4.4.** Let  $b = (\text{id} \otimes \omega)(W^*W)$ , where  $\omega = \omega_{r,s} \in \mathcal{B}(\mathcal{H})_*$ , for  $r \in \mathcal{D}(Q^{-1})$ ,  $s \in \mathcal{D}(Q)$ . Such elements are dense in  $N$ . Write:

$$\kappa(b) = Q(\omega^\top \otimes \text{id})(\widetilde{W}\widetilde{W}^*)Q^{-1}, \quad (4.2)$$

where  $\omega^\top \in \mathcal{B}(\overline{\mathcal{H}})_*$  is such that  $\omega^\top(m^T) = \omega(m)$ , for  $m \in \mathcal{B}(\mathcal{H})$ . Then we have:

$$E(b \otimes 1) = E(1 \otimes \kappa(b)). \quad (4.3)$$

*Proof.* From the condition (3) of Definition 2.3, we have:  $\widetilde{W}_{13}\widetilde{W}_{23}\widetilde{W}_{23}^* = W_{12}^{\top \otimes \top}W_{12}^{*\top \otimes \top}\widetilde{W}_{13}$ . Apply here  $\text{id} \otimes \omega^\top \otimes \text{id}$ . Then we have:  $\widetilde{W}(1 \otimes y) = (x \otimes 1)\widetilde{W}$ , where  $y = (\omega^\top \otimes \text{id})(\widetilde{W}\widetilde{W}^*)$  and  $x = (\text{id} \otimes \omega^\top)(W^{\top \otimes \top}W^{*\top \otimes \top})$ . Note that

$$x^\top = [(\text{id} \otimes \omega^\top)(W^{\top \otimes \top}W^{*\top \otimes \top})]^\top = (\text{id} \otimes \omega)(W^*W) = b,$$

because  $m \mapsto m^\top$  is an anti-homomorphism. This means  $x = b^\top$ , and we have:

$$(b^\top \otimes 1)\widetilde{W} = \widetilde{W}(1 \otimes y).$$

As a consequence, for any  $\eta, v \in \mathcal{D}(Q)$ ,  $\xi, u \in \mathcal{D}(Q^{-1})$ , we have:

$$\langle \widetilde{W}(\bar{\eta} \otimes v), \bar{b}\bar{\xi} \otimes u \rangle = \langle (b^\top \otimes 1)\widetilde{W}(\bar{\eta} \otimes v), \bar{\xi} \otimes u \rangle = \langle \widetilde{W}(\bar{\eta} \otimes yv), \bar{\xi} \otimes u \rangle,$$

because  $\bar{b}\bar{\xi} = \overline{(b^*)^*\xi} = [b^*]^\top \bar{\xi} = [b^\top]^* \bar{\xi}$ . Apply to both sides the alternative characterizing equation for  $\widetilde{W}$ , from Proposition 2.7. Then this (formally) becomes:

$$\langle W(Q^{-1}b\xi \otimes v), Q\eta \otimes u \rangle = \langle W(Q^{-1}\xi \otimes yv), Q\eta \otimes u \rangle. \quad (4.4)$$

For Equation (4.4) to be valid, we actually need to be sure whether  $Q^{-1}b\xi$  is valid. But, from our assumption that  $\omega = \omega_{r,s}$ , for  $r \in \mathcal{D}(Q^{-1})$ ,  $s \in \mathcal{D}(Q)$ , we know  $Q^{-1}bQ$  is valid, as follows:

$$Q^{-1}bQ = Q^{-1}(\text{id} \otimes \omega_{r,s})(W)Q = (\text{id} \otimes \omega_{r,s}(Q \cdot Q^{-1}))(W) = (\text{id} \otimes \omega_{Q^{-1}r, Qs})(W),$$

because  $W(Q \otimes Q) \subseteq (Q \otimes Q)W$ . As a consequence, we can write:  $Q^{-1}b\xi = Q^{-1}bQQ^{-1}\xi$ , all valid since  $\xi \in \mathcal{D}(Q^{-1})$ .

Re-writing Equation (4.4), we then have:

$$\langle (Q^{-1}bQ \otimes 1)(Q^{-1}\xi \otimes v), W^*(Q\eta \otimes u) \rangle = \langle ((1 \otimes y)(Q^{-1}\xi \otimes v), W^*(Q\eta \otimes u) \rangle.$$

Compare the two sides, while noting that the elements  $W^*(Q\eta \otimes u)$  generate  $\text{Ran}(W^*) = \text{Ran}(E)$ . Since  $\text{Ran}(E) \subsetneq \mathcal{H} \otimes \mathcal{H}$ , we cannot say  $Q^{-1}bQ \otimes 1 = 1 \otimes y$ . Nevertheless, knowing  $E = W^*W$ , we can at least say the following:

$$E(Q^{-1}bQ \otimes 1) = E(1 \otimes y). \quad (4.5)$$

Equivalently, as we know  $E(Q \otimes Q) \subseteq (Q \otimes Q)E$ , we also have:

$$E(b \otimes 1) = E(1 \otimes QyQ^{-1}) = E(1 \otimes \kappa(b)), \quad (4.6)$$

where  $\kappa(b) = QyQ^{-1} = Q(\omega^\top \otimes \text{id})(\widetilde{W}\widetilde{W}^*)Q^{-1}$ . By the same reason as above, our choice of  $\omega$  means that  $QyQ^{-1}$  is valid.  $\square$

Utilizing Proposition 4.4, we can show that  $N$  is an algebra:

**Proposition 4.5.** *We have:*

$$N = \overline{\text{span}\{(\text{id} \otimes \omega)(W^*W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}} \text{ is a subalgebra in } \mathcal{B}(\mathcal{H}).$$

$$\widehat{N} := \overline{\text{span}\{(\omega \otimes \text{id})(WW^*) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\text{WOT}} \text{ is a subalgebra in } \mathcal{B}(\mathcal{H}).$$

*Proof.* (1). Let  $b = (\text{id} \otimes \omega)(W^*W)$ , where  $\omega = \omega_{r,s} \in \mathcal{B}(\mathcal{H})_*$ , for  $r \in \mathcal{D}(Q^{-1})$ ,  $s \in \mathcal{D}(Q)$ . Consider also  $b' = (\text{id} \otimes \theta)(W^*W)$ , for  $\theta \in \mathcal{B}(\mathcal{H})_*$ . We know that the elements of the form  $b, b'$  above span a dense subset in  $N$ .

Observe:

$$\begin{aligned} b'b &= (\text{id} \otimes \theta \otimes \omega)(W_{12}^*W_{12}W_{13}^*W_{13}) = (\text{id} \otimes \theta)(W^*W(b \otimes 1)) = (\text{id} \otimes \theta)(E(b \otimes 1)) \\ &= (\text{id} \otimes \theta)(E(1 \otimes \kappa(b))) = (\text{id} \otimes \rho)(W^*W), \end{aligned}$$

where  $\rho \in \mathcal{B}(\mathcal{H})_*$  such that  $\rho = \theta(\cdot \kappa(b))$ . We used the result of Proposition 4.4. This shows that  $b'b \in N$ .

(2). Replacing the role of  $W$  with that of  $\widehat{W} = \Sigma W^* \Sigma$ , we can quickly see that  $\widehat{N}$  is also an algebra.  $\square$

We have thus shown that  $L, N, \widehat{L}, \widehat{N}$  are WOT-closed subalgebras in  $\mathcal{B}(\mathcal{H})$ , and they are already closed under the involution. This means they are von Neumann algebras. Moreover, our canonical idempotent element,  $E$ , is contained in  $N \otimes L$  (similar for  $\widehat{E}$ ):

**Proposition 4.6.** *We have:  $E = W^*W \in N \otimes L$ , where  $\otimes$  is the von Neumann algebra tensor product. Similarly, we have:  $\widehat{E} = \Sigma W W^* \Sigma \in \widehat{N} \otimes \widehat{L}$ .*

*Proof.* This is evident from the way the algebras are defined. It is in fact easy to show that for any  $x \otimes y \in N' \otimes L'$ , we have  $E(x \otimes y) = (x \otimes y)E$ . So  $E \in N'' \otimes L'' = N \otimes L$ . Similarly, we have  $\widehat{E} \in \widehat{N} \otimes \widehat{L}$ .  $\square$

By the fullness assumption on  $W$ , together with the fact that  $W$  is a partial isometry, it is apparent that the von Neumann algebra  $L$  acts on  $\mathcal{H}$  in a non-degenerate way. In particular, if  $\zeta \neq 0$  in  $\mathcal{H}$ , there exists  $\omega \in \mathcal{B}(\mathcal{H})_*$  such that  $(\omega \otimes \text{id})(W^*W)\zeta \neq 0$ . Similarly, the algebras  $N, \widehat{L}, \widehat{N}$  act non-degenerately on  $\mathcal{H}$ . Therefore, combined with the observation given in Proposition 4.6 above, we can give the following alternative characterizations for the subalgebras:

$$\begin{aligned} N &= \overline{\text{span}\{(\text{id} \otimes \omega)(E) : \omega \in L^*\}}^{\text{WOT}}, & L &= \overline{\text{span}\{(\omega \otimes \text{id})(E) : \omega \in N^*\}}^{\text{WOT}}, \\ \widehat{N} &= \overline{\text{span}\{(\text{id} \otimes \omega)(\widehat{E}) : \omega \in \widehat{L}^*\}}^{\text{WOT}}, & \widehat{L} &= \overline{\text{span}\{(\omega \otimes \text{id})(\widehat{E}) : \omega \in \widehat{N}^*\}}^{\text{WOT}}. \end{aligned}$$

If, in particular,  $W$  is a multiplicative unitary, we would have  $W^*W = \text{Id}_{\mathcal{B}(\mathcal{H})} = WW^*$ , so we will have  $N = L = \widehat{N} = \widehat{L} = \mathbb{C}$ . In our case, however, as  $W$  is a partial isometry, we have to work with these non-trivial base subalgebras. As such, going forward, we will need to introduce suitable weights on them.

On the other hand, we cannot just consider any weight or a functional on  $N$ . The general theory on  $C^*$ -algebraic quantum groupoids of separable type [10], [11] suggests that our  $E$  will have to be a *separability idempotent*. This restricts the choice of a suitable weight. At the purely algebraic level, a separability idempotent (see [27]) is automatically equipped with certain “distinguished linear functionals”. But in the operator algebraic framework, such functionals (weights) have to be assumed as a part of the definition: See [9]. Considering these known facts, we will require the existence of a certain “distinguished weight”,  $\nu$ , which is a normal semi-finite faithful (n.s.f.) weight on  $N$  defined as follows:

**Definition 4.7.** Let  $\nu$  be an n.s.f. weight on  $N$ , together with its associated modular automorphism group  $(\sigma_t')_{t \in \mathbb{R}}$ , satisfying

$$(\nu \otimes \text{id})(E) = 1.$$

Then we will refer to  $\nu$  as the *distinguished weight* on  $N$ .

*Remark.* The condition given in the definition means that for any  $\omega \in L_+^*$ , we require  $(\text{id} \otimes \omega)(E) \in \mathfrak{M}_\nu (\subseteq N)$  and that  $\nu((\text{id} \otimes \omega)(E)) = \omega(1)$ . From this, it will follow that  $(\text{id} \otimes \omega)(E) \in \mathfrak{M}_\nu$  for any  $\omega \in L^*$ , and that  $\nu((\text{id} \otimes \omega)(E)) = \omega(1), \forall \omega \in L^*$ .

**Definition 4.8.** For  $b \in \mathcal{T}_\nu$  (the Tomita algebra for  $\nu$ ), define

$$\gamma_N(b) := (\nu \otimes \text{id})(E(b \otimes 1)).$$

*Remark.* Refer to the standard textbooks on the modular theory [20], [19] for the precise definition of the Tomita algebra, which is a certain strongly  $*$ -dense subalgebra in  $N$ , consisting of elements that are analytic with respect to the modular automorphism group  $(\sigma_t')$ . For any  $\theta \in \mathcal{B}(\mathcal{H})_*$ , we noted in the above Remark that  $(\text{id} \otimes \theta)(E) \in \mathfrak{M}_\nu$ . So, with  $b \in \mathcal{T}_\nu$ , it can be shown that  $(\text{id} \otimes \theta)(E)b \in \mathfrak{M}_\nu$ . As such, the definition above for  $\gamma_N(b)$  makes sense.

**Proposition 4.9.**  $\gamma_N$  is densely-defined, and has a dense range in  $L$ .

*Proof.* As  $\mathcal{T}_\nu$  is dense in  $N$ , we know  $\gamma_N$  is densely-defined. Meanwhile, since  $\nu(\cdot b) \in N^*$ , we can see that  $\gamma_N(b) \in L$ . The functionals  $\nu(\cdot b)$  are dense in  $N^*$ , so  $\gamma_N$  has a dense range in  $L$ .  $\square$

More can be said about the  $\gamma_N$  map, but we will return to the discussion later. Instead, let us revisit the map  $\kappa$  that we saw earlier. The definition of  $\kappa$  is given in Equation (4.2) in Proposition 4.4, with Equation (4.3) obtained as a consequence. However, it actually turns out that Equation (4.3), namely  $E(b \otimes 1) = E(1 \otimes \kappa(b))$ , completely determines the map  $\kappa : b \mapsto \kappa(b)$ . To see this, assume  $E(b \otimes 1) = E(1 \otimes \kappa(b))$  and let  $b = 0$ . Then for any  $\omega \in \mathcal{B}(\mathcal{H})_*$ , we have:

$$(\omega \otimes \text{id})(W^*W)\kappa(b) = (\omega \otimes \text{id})(E(1 \otimes \kappa(b))) = (\omega \otimes \text{id})(E(b \otimes 1)) = 0.$$

The elements of the form  $(\omega \otimes \text{id})(W^*W)$  generate  $L$ , which acts non-degenerately on  $\mathcal{H}$ . So this observation means that  $b = 0$  implies  $\kappa(b) = 0$ . Phrased another way, the element  $\kappa(b)$  is uniquely determined by  $b$ . The main point of Proposition 4.4 is that there actually exists a (unique) map  $\kappa$  satisfying the characterizing equation:  $E(b \otimes 1) = E(1 \otimes \kappa(b))$ .

Using this new perspective, we can prove the following result:

**Proposition 4.10.** For  $b = (\text{id} \otimes \omega)(W^*W)$ , where  $\omega = \omega_{r,s}$ , for  $r \in \mathcal{D}(Q^{-1})$ ,  $s \in \mathcal{D}(Q)$ , the equation  $E(b \otimes 1) = E(1 \otimes \kappa(b))$  uniquely determines a densely-defined linear map  $\kappa$  from  $N$  into  $\mathcal{B}(\mathcal{H})$ . It is injective and anti-multiplicative.

*Proof.* We saw above that  $b \mapsto \kappa(b)$  is a valid function. It is a densely-defined function on  $N$ , such that  $\text{span}\{(\text{id} \otimes \omega_{r,s})(W^*W) : r \in \mathcal{D}(Q^{-1}), s \in \mathcal{D}(Q)\}$  forms a core.

A similar argument may be used to show that  $\kappa$  is injective: If  $\kappa(b) = 0$ , for any  $\omega \in \mathcal{B}(\mathcal{H})_*$ , we have:

$$(\text{id} \otimes \omega)(W^*W)b = (\text{id} \otimes \omega)(E(b \otimes 1)) = E(1 \otimes \kappa(b)) = 0.$$

The elements  $(\text{id} \otimes \omega)(W^*W)$  generate  $N$ , which is non-degenerate. So  $b = 0$ . In other words, we see that  $\kappa(b) = 0$  implies  $b = 0$ . As  $\text{Ker}(\kappa) = \{0\}$ , we see that  $\kappa$  is injective.

To prove the anti-multiplicativity, consider  $b_1, b_2 \in \mathcal{D}(\kappa)$ . Then

$$E(b_1 b_2 \otimes 1) = E(b_1 \otimes 1)(b_2 \otimes 1) = E(b_2 \otimes \kappa(b_1)) = E(1 \otimes \kappa(b_2)\kappa(b_1)).$$

By the characterization of  $\kappa$  give by Equation (4.3), this means that  $b_1 b_2 \in \mathcal{D}(\kappa)$  and that  $\kappa(b_1 b_2) = \kappa(b_2)\kappa(b_1)$ .  $\square$

In the proof of Proposition 4.4, in Equation (4.5), we also saw that

$$E(Q^{-1}bQ \otimes 1) = E(1 \otimes y),$$

where  $b = (\text{id} \otimes \omega)(W^*W)$ ,  $\omega = \omega_{r,s}$  with  $r \in \mathcal{D}(Q^{-1})$ ,  $s \in \mathcal{D}(Q)$ , and  $y = (\omega^\top \otimes \text{id})(\widetilde{W}\widetilde{W}^*)$ . In view of the knowledge that Equation (4.3) characterizes the  $\kappa$  map, we can now see from this observation that  $Q^{-1}bQ \in \mathcal{D}(\kappa)$  and that  $\kappa(Q^{-1}bQ) = y$ .

Next, consider the map  $R_\kappa : b \mapsto Q^{-1}\kappa(b)Q$ , for  $b \in \mathcal{D}(\kappa)$ . In particular, if  $b = (\text{id} \otimes \omega)(W^*W)$  as above, we know  $\kappa(b) = QyQ^{-1}$ , from an alternative definition of  $\kappa$  given in Equation (4.2). So we would have:

$$R_\kappa(b) = Q^{-1}\kappa((\text{id} \otimes \omega)(W^*W))Q = Q^{-1}QyQ^{-1}Q = y = (\omega^\top \otimes \text{id})(\widetilde{W}\widetilde{W}^*).$$

We show below that this map extends to a bounded map on  $N$ .

**Proposition 4.11.** (1) Consider the map  $R_\kappa : b \mapsto Q^{-1}\kappa(b)Q$ , for  $b \in \mathcal{D}(\kappa)$ . The map  $R_\kappa$  extends to a bounded map  $R_\kappa : N \rightarrow \mathcal{B}(\mathcal{H})$ . It can be also characterized by

$$R_\kappa(b) = \kappa(Q^{-1}(\text{id} \otimes \omega)(W^*W)Q).$$

(2) Write  $T := Q(\cdot)Q^{-1}$ . We have:

$$\kappa = T \circ R_\kappa = R_\kappa \circ T.$$

*Proof.* (1). As  $Q^{-1}(\cdot)Q$  naturally preserves multiplication, and obviously injective, we can see quickly from Proposition 4.10 that  $R_\kappa$  is an anti-multiplicative injective map. Meanwhile, for  $b = (\text{id} \otimes \omega)(W^*W) \in \mathcal{D}(\kappa)$ , we have:

$$R_\kappa(b^*) = R_\kappa((\text{id} \otimes \bar{\omega})(W^*W)) = (\bar{\omega}^\top \otimes \text{id})(\widetilde{W}\widetilde{W}^*) = [(\omega^\top \otimes \text{id})(\widetilde{W}\widetilde{W}^*)]^* = y^* = [R_\kappa(b)]^*,$$

because  $\bar{\omega}^\top = \overline{\omega^\top}$ . This means that  $R_\kappa$  is also a  $*$ -map. So  $R_\kappa$  is a  $*$ -anti-homomorphism, so bounded. Therefore, it extends to all of  $N$ .

Meanwhile, from  $E(Q^{-1}bQ \otimes 1) = E(1 \otimes y)$ , we obtain a different characterization:  $R_\kappa(b) = \kappa(Q^{-1}(\text{id} \otimes \omega)(W^*W)Q)$ . As  $R_\kappa$  is shown to be bounded, there is no reason to worry about its domain.

(2). From (1), we observe that:  $\kappa(b) = QR_\kappa(b)Q^{-1} = R_\kappa(QbQ^{-1}), \forall b$ . It follows that  $\kappa = T \circ R_\kappa = R_\kappa \circ T$ .  $\square$

Let us return to our  $\gamma_N$  map (Definition 4.8), and compare it with  $\kappa$ . Both are densely-defined maps on  $N$ . It turns out that when valid, we actually have:  $\kappa = \gamma_N$ . See below:

**Proposition 4.12.** (1) On  $\mathcal{D}(\kappa)$ , we have:  $\kappa = \gamma_N$ .

(2)  $\gamma_N$  is closed, injective, anti-multiplicative, and satisfies Equation (4.3):

$$E(b \otimes 1) = E(1 \otimes \gamma_N(b)).$$

*Proof.* (1). Let  $b \in \mathcal{D}(\kappa)$ , so that we have  $E(b \otimes 1) = E(1 \otimes \kappa(b))$ , by Equation (4.3). Then we have:

$$(\nu \otimes \text{id})(E(b \otimes 1)) = (\nu \otimes \text{id})(E(1 \otimes \kappa(b))) = (\nu \otimes \text{id})(E)\kappa(b) = \kappa(b).$$

Comparing this with the definition of  $\gamma_N$  given in Definition 4.8, we can say from this observation that  $\mathcal{D}(\kappa) \subseteq \mathcal{D}(\gamma_N)$  and that  $\kappa(b) = \gamma_N(b)$  for all  $b \in \mathcal{D}(\kappa)$ .

(2). As  $\mathcal{D}(\kappa)$  is already dense in  $N$  and since  $\kappa$  is a closed map (since  $\kappa = R_\kappa \circ T$ , where  $R_\kappa$  is bounded and  $T = Q^{-1}(\cdot)Q$  is a closed map), this result means that  $\gamma_N = \kappa$ . In particular,  $\gamma_N$  is closed, injective, anti-multiplicative, and satisfies Equation (4.3).  $\square$

We will from now on primarily work with the  $\gamma_N$  map, knowing that  $\kappa$  gives an alternative characterization. In the below, we prove a nice polar-decomposition result for the  $\gamma_N$  map:

**Proposition 4.13.** Let  $\gamma_N : N \rightarrow L$  be the injective, densely-defined map defined in Definition 4.8. We have:

(1) Consider the map,  $\widetilde{R} : N \rightarrow L$ , defined below:

$$\widetilde{R} := \gamma_N \circ \sigma_{-i/2}^\nu, \tag{4.7}$$

where  $\sigma_{-i/2}^\nu$  is the analytic generator of the modular automorphism group  $(\sigma_t^\nu)$ , at  $z = -\frac{i}{2}$ .

Then  $\widetilde{R}$  extends to a  $*$ -anti-isomorphism from  $N$  to  $L$ .

(2) We thus obtain the following polar decomposition of the map  $\gamma_N$ :

$$\gamma_N = \widetilde{R} \circ \sigma_{i/2}^\nu.$$

*Proof.* (1), (2). As  $\sigma_{-i/2}^\nu$  is an automorphism, while  $\gamma_N$  is an injective, densely-defined map having a dense range, so too is  $\widetilde{R}$ . We know from  $\gamma_N = \kappa$  that  $\gamma_N$  is anti-multiplicative. Since  $\sigma_{-i/2}^\nu$

is an automorphism, we can see quickly that  $\tilde{R}$  is anti-multiplicative:  $\tilde{R}(b_1 b_2) = \tilde{R}(b_2) \tilde{R}(b_1)$ . Furthermore, for  $b \in \mathcal{D}(\sigma_{-i/2}^\nu)$ , we have:

$$\begin{aligned} \tilde{R}(b)^* &= \gamma_N(\sigma_{-i/2}^\nu(b))^* = [(\nu \otimes \text{id})(E(\sigma_{-i/2}^\nu(b) \otimes 1))]^* \\ &= (\nu \otimes \text{id})((\sigma_{i/2}^\nu(b^*) \otimes 1)E) = (\nu \otimes \text{id})((\sigma_i^\nu(\sigma_{-i/2}^\nu(b^*)) \otimes 1)E) \\ &= (\nu \otimes \text{id})(E(\sigma_{-i/2}^\nu(b^*) \otimes 1)) = \gamma_N(\sigma_{-i/2}^\nu(b^*)) = \tilde{R}(b^*), \end{aligned}$$

showing that  $\tilde{R}$  is a \*-anti-homomorphism. So  $\tilde{R}$  is bounded.

As  $\tilde{R}$  is a bounded map from  $N$  to  $L$ , injective, densely-defined, having a dense range, it extends to a \*-anti-isomorphism  $\tilde{R} : N \rightarrow L$ . From the definition of  $\tilde{R}$ , we can see that  $\gamma_N = \tilde{R} \circ \sigma_{i/2}^\nu$ . This gives a polar decomposition for  $\gamma_N$ .  $\square$

Since  $\tilde{R} : N \rightarrow L$  is a \*-anti-isomorphism, we can consider  $\tilde{R}^{-1}$ , which would be a \*-anti-isomorphism from  $L$  to  $N$ . Using this, we can define the following n.s.f. weight,  $\mu$ , on  $L$ :

**Definition 4.14.** Let  $\mu := \nu \circ \tilde{R}^{-1}$ . It is an n.s.f. weight on  $L$ , together with its associated modular automorphism group  $(\sigma_t^\mu)_{t \in \mathbb{R}}$ , satisfying  $\sigma_t^\mu = \tilde{R} \circ \sigma_{-t}^\nu \circ \tilde{R}^{-1}$ . We will refer to  $\mu$  as the *distinguished weight* on  $L$ .

As one can imagine, the pair  $(L, \mu)$  behaves a lot like  $(N, \nu)$ . See below. Analogous results can be found in [9], though in the  $C^*$ -algebra framework.

**Proposition 4.15.** *Let  $\mu$  be as above. Then we have:*

- (1)  $(\text{id} \otimes \mu)(E) = 1$ .
- (2) For  $c \in \mathcal{D}(\sigma_{-i/2}^\mu)$ , write:  $\gamma_L(c) := (\tilde{R}^{-1} \circ \sigma_{-i/2}^\mu)(c)$ . This defines a closed, densely-defined map from  $L$  to  $N$ , having a dense range. It is also injective and anti-multiplicative.
- (3) For  $c \in \mathcal{D}(\gamma_L) = \mathcal{D}(\sigma_{-i/2}^\mu)$ , we have:  $(1 \otimes c)E = (\gamma_L(c) \otimes 1)E$ .
- (4) For  $c \in \mathcal{D}(\gamma_L)$ , we have:  $(\text{id} \otimes \mu)((1 \otimes c)E) = \gamma_L(c)$ .

*Proof.* (1). As before, the equation means that  $(\theta \otimes \text{id})(E) \in \mathfrak{M}_\mu$  for all  $\theta \in N^*$ , and that  $\mu((\theta \otimes \text{id})(E)) = \theta(1)$ . We can verify this for  $\theta = \nu(\cdot b)$ , for  $b \in \mathcal{D}(\gamma_N)$ . Such functionals are dense in  $N^*$ . Using the fact that  $\mu = \nu \circ \tilde{R}^{-1}$  and that  $\nu$  is  $\sigma^\nu$ -invariant, we have:

$$\begin{aligned} \mu((\theta \otimes \text{id})(E)) &= \mu((\nu \otimes \text{id})(E(b \otimes 1))) = \mu(\gamma_N(b)) \\ &= \mu((\tilde{R} \circ \sigma_{i/2}^\nu)(b)) = \nu(b) = \theta(1). \end{aligned}$$

(2). From  $(\nu \otimes \text{id})(E(b \otimes 1)) = \tilde{R}(\sigma_{i/2}^\nu(b))$ ,  $b \in \mathcal{D}(\gamma_N) = \mathcal{D}(\sigma_{i/2}^\nu)$ , take the adjoint. Since  $\tilde{R}$  is a \*-anti-isomorphism, we have:

$$(\nu \otimes \text{id})((b^* \otimes 1)E) = [\tilde{R}(\sigma_{i/2}^\nu(b))]^* = \tilde{R}([\sigma_{i/2}^\nu(b)]^*) = \tilde{R}(\sigma_{-i/2}^\nu(b^*)).$$

In other words, for  $x \in \mathcal{D}(\gamma_N)^* = \mathcal{D}(\sigma_{-i/2}^\nu)$ , which is dense in  $N$ , the expression  $(\nu \otimes \text{id})((x \otimes 1)E)$  is valid and  $(\nu \otimes \text{id})((x \otimes 1)E) = (\tilde{R} \circ \sigma_{-i/2}^\nu)(x)$ . Or, put another way, we have:

$$\nu(x(\text{id} \otimes \omega)(E)) = \omega((\nu \otimes \text{id})((x \otimes 1)E)) = \omega((\tilde{R} \circ \sigma_{-i/2}^\nu)(x)), \quad \text{for } \omega \in L^*, x \in \mathcal{D}(\sigma_{-i/2}^\nu).$$

So, by the same argument as in the case of  $\gamma_N$ , the map  $x \mapsto (\tilde{R} \circ \sigma_{-i/2}^\nu)(x)$  is closed, densely-defined on  $N$ , injective, and has a dense range in  $L$ . Let us define  $\gamma_L$  to be its inverse map, namely,  $\gamma_L : c \rightarrow (\sigma_{i/2}^\nu \circ \tilde{R}^{-1})(b) = (\tilde{R}^{-1} \circ \sigma_{-i/2}^\mu)(c)$ . It is clear that  $\gamma_L$  is closed, densely-defined on  $L$ , injective, has a dense range in  $N$ , as well as anti-multiplicative.



(3). For  $c, c' \in \mathcal{D}(\gamma_L)$ , we have

$$\begin{aligned} (\text{id} \otimes \mu(c' \cdot))((1 \otimes c)E) &= (\text{id} \otimes \mu)((1 \otimes c')E) = \gamma_L(c'c) = \gamma_L(c)\gamma_L(c') \\ &= \gamma_L(c)(\text{id} \otimes \mu)((1 \otimes c')E) = (\text{id} \otimes \mu)((\gamma_L(c) \otimes c')E) \\ &= (\text{id} \otimes \mu(c' \cdot))((\gamma_L(c) \otimes 1)E). \end{aligned}$$

Since  $\mu$  is faithful, and since the result is true for all  $c' \in \mathcal{D}(\gamma_L)$ , which is dense in  $L$ , we see that  $(1 \otimes c)E = (\gamma_L(c) \otimes 1)E$ .

(4). Let  $c \in \mathcal{D}(\gamma_L)$ . Then using  $(\text{id} \otimes \mu)(E) = 1$ , we have:

$$(\text{id} \otimes \mu)((1 \otimes c)E) = (\text{id} \otimes \mu)((\gamma_L(c) \otimes 1)E) = \gamma_L(c).$$

□

While we do not plan to go overly deep into this direction, the results above confirm that with our canonical idempotent  $E \in N \otimes L$ , the  $*$ -anti-isomorphism  $\tilde{R} : N \rightarrow L$ , and the weight  $\nu$  on  $N$ , the data  $(E, N, \nu)$  forms a (von Neumann algebraic) *separability triple*, in the sense of [9] (see, in particular, section 6 of that paper).

So far, we have been working in the von Neumann algebraic framework throughout this section. This has been convenient as we consider the n.s.f. weights  $\nu$  and  $\mu$ . But ultimately, we wish to formulate a  $C^*$ -algebraic structure for the base algebras. This is possible.

Consider

$$B = \overline{\{(\text{id} \otimes \omega)(E) : \omega \in L^*\}}^{\|\cdot\|} = \overline{\{(\text{id} \otimes \omega)(E) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|} (\subseteq \mathcal{B}(\mathcal{H})).$$

Similarly, consider also

$$C = \overline{\{(\theta \otimes \text{id})(E) : \theta \in N^*\}}^{\|\cdot\|} = \overline{\{(\theta \otimes \text{id})(E) : \theta \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|} (\subseteq \mathcal{B}(\mathcal{H})).$$

It is clear that  $B \subseteq N$  and  $C \subseteq L$ . We gather some results on these subspaces below:

**Proposition 4.16.** *Let  $B$  and  $C$  be as above. Then we have:*

- (1)  $\mathcal{D}(\gamma_N) \cap B$  is dense in  $B$ , and  $\gamma_N$  restricted to this space has a dense range in  $L$ .
- (2)  $B$  is a  $*$ -subalgebra of  $N$ .
- (3)  $\mathcal{D}(\gamma_L) \cap C$  is dense in  $C$ , and  $\gamma_L$  restricted to this space has a dense range in  $N$ .
- (4)  $C$  is a  $*$ -subalgebra of  $L$ .

*Proof.* (1). Let  $c_1, c_2 \in \mathcal{T}_\mu$ , the Tomita algebra, and consider  $\mu(c_2^* \cdot c_1) \in L^*$ . Note that

$$(\text{id} \otimes \mu(c_2^* \cdot c_1))(E) = (\text{id} \otimes \mu)((1 \otimes c_2^*)E(1 \otimes c_1)) = (\text{id} \otimes \mu)(E(1 \otimes c_1 \sigma_{-i}^\mu(c_2^*))) = \gamma_N^{-1}(c_1 \sigma_{-i}^\mu(c_2^*)).$$

In this way, we see that  $(\text{id} \otimes \mu(c_2^* \cdot c_1))(E) \in \mathcal{D}(\gamma_N) \cap B$ . As the Tomita algebra is dense in  $L$ , the functionals  $\mu(c_2^* \cdot c_1)$  are dense in  $L^*$ . This shows that  $\mathcal{D}(\gamma_N) \cap B$  is dense in  $B$ . In addition, the elements of the form  $c_1 \sigma_{-i}^\mu(c_2^*)$ ,  $c_1, c_2 \in \mathcal{T}_\mu$ , are dense in  $\mathcal{T}_\mu$ , so dense in  $L$ , which shows that under the map  $\gamma_N$ , the space  $\mathcal{D}(\gamma_N) \cap B$  is sent to a dense subspace in  $L$ .

(2). Let  $b \in \mathcal{D}(\gamma_N) \cap B$  and let  $\tilde{b} = (\text{id} \otimes \omega)(E)$ , for  $\omega \in L^*$ . Such elements are dense in  $B$ . We have:

$$\tilde{b}b = (\text{id} \otimes \omega)(E(b \otimes 1)) = (\text{id} \otimes \omega)(E(1 \otimes \gamma_N(b))) = (\text{id} \otimes \rho)(E),$$

where  $\rho = \omega(\cdot \gamma_N(b))$ . In this way, we see that  $B$  is closed under the multiplication. It is easy to see that  $B$  is closed under the  $*$ -operation, because  $((\text{id} \otimes \omega)(E))^* = (\text{id} \otimes \bar{\omega})(E)$ , by  $E$  being self-adjoint. It follows that  $B$  is a  $*$ -subalgebra of  $N$ .

(3), (4). Proof analogous to that of (1), (2). □

*Remark.* (1). As  $B$  is a norm-closed  $*$ -subalgebra of  $N$ , that is WOT-dense in  $N$ , we conclude that  $B$  is a  $C^*$ -algebra and that its  $W^*$ -closure is  $N$ . Similarly,  $C$  is a  $C^*$ -algebra whose  $W^*$ -closure is  $L$ . As  $N$  and  $L$  act non-degenerately on  $\mathcal{H}$ , so do  $B$  and  $C$ .

(2). We point out that Proposition 4.16, obtaining the  $C^*$ -algebraic counterparts from the von Neumann algebraic separability triple  $(E, N, \nu)$ , is essentially no different from the result of Proposition 6.8 of [9]. We made some minor adjustments, to avoid working with the GNS Hilbert space  $\mathcal{H}_\nu$ , but one can see that basically the same proof strategy is being used. As such, in what follows we will often skip details and refer instead to the results in section 6 of that paper.

Here are some more results on the base  $C^*$ -algebras  $B$  and  $C$ :

**Theorem 4.17.** *Let  $B (\subseteq N)$  and  $C (\subseteq L)$  be the  $C^*$ -subalgebras obtained above, and recall the canonical idempotent  $E \in N \otimes L$ . Then*

- (1)  $B \subseteq M(A)$  and  $C \subseteq M(A)$ .
- (2) The  $\sigma_t^\mu$ ,  $t \in \mathbb{R}$ , leaves  $C$  invariant. So we may just use the same notation  $\mu$ , to denote the weight on  $C$  restricted from the n.s.f. weight  $\mu$  on  $L$ . Then  $\mu$  on  $C$  becomes a KMS weight on  $C$ , equipped with a norm-continuous automorphism group  $(\sigma_t^\mu)$ . Similarly, we may use the same notation  $\nu$ , to denote the weight on  $B$  restricted from the n.s.f. weight  $\nu$  on  $N$ . Then  $\nu$  on  $B$  becomes a KMS weight on  $B$ , equipped with a norm-continuous automorphism group  $(\sigma_t^\nu)$ .
- (3)  $E \in M(B \otimes C)$ .
- (4) The  $*$ -anti-isomorphism  $\tilde{R} : N \rightarrow L$  restricts to  $R : B \rightarrow C$ . It becomes a  $*$ -anti-isomorphism of  $C^*$ -algebras.
- (5) The data  $(E, B, \nu)$  forms a ( $C^*$ -algebraic) separability triple, in the sense of [9]. In particular, there exists a closed, densely-defined map  $\gamma_B : B \rightarrow C$ , which is none other than the restriction of  $\gamma_N$  to  $B$ , having a dense range in  $C$ , such that  $E(b \otimes 1) = E(1 \otimes \gamma_B(b))$ , for  $b \in \mathcal{D}(\gamma_B)$ . Also, we have  $\gamma_C = \gamma_L|_C$ , a closed, densely-defined map from  $C$  into  $B$ , having a dense range in  $B$ , such that  $(1 \otimes c)E = (\gamma_C(c) \otimes 1)E$ , for  $c \in \mathcal{D}(\gamma_C)$ .

*Proof.* (1). We already showed that  $B$  and  $C$  are  $C^*$ -subalgebras. As a consequence of Lemma 4.1 (1), (3), we see that  $B \subseteq M(A)$  and  $C \subseteq M(A)$ .

(2) – (5). For the rest, see Proposition 6.9 of [9]. Only a minor modification (like given in the proof of the previous proposition) is needed.  $\square$

Finally, we may replace  $W$  with  $\widehat{W} = \Sigma W^* \Sigma$ , work with the von Neumann algebras  $\widehat{N}$  and  $\widehat{L}$ , and the idempotent  $\widehat{E} = \Sigma W W^* \Sigma \in \widehat{N} \otimes \widehat{L}$  (see Proposition 4.6). We can introduce the distinguished weights  $\widehat{\nu}$  on  $\widehat{N}$  (similar to Definition 4.7) and  $\widehat{\mu}$  on  $\widehat{L}$ , construct the maps  $\gamma_{\widehat{N}}$  and  $\gamma_{\widehat{L}}$ .

From these, we can consider:

$$\widehat{B} = \overline{\{(\text{id} \otimes \omega)(\widehat{E}) : \omega \in \widehat{L}^*\}}^{\|\cdot\|} = \overline{\{(\text{id} \otimes \omega)(\widehat{E}) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|} (\subseteq \mathcal{B}(\mathcal{H})).$$

and

$$\widehat{C} = \overline{\{(\theta \otimes \text{id})(\widehat{E}) : \theta \in \widehat{N}^*\}}^{\|\cdot\|} = \overline{\{(\theta \otimes \text{id})(\widehat{E}) : \theta \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|} (\subseteq \mathcal{B}(\mathcal{H})).$$

We have the following theorem, analogous to Proposition 4.16 and Theorem 4.17.

**Theorem 4.18.** *Let  $\widehat{B} (\subseteq \widehat{N})$  and  $\widehat{C} (\subseteq \widehat{L})$  be as above, and recall the canonical idempotent  $\widehat{E} \in \widehat{N} \otimes \widehat{L}$ . Then*

- (1)  $\widehat{B}$  and  $\widehat{C}$  are  $C^*$ -subalgebras whose  $W^*$ -closures are  $\widehat{N}$  and  $\widehat{L}$ , respectively.

- (2) We have a KMS weight  $\hat{\nu}$  on  $\hat{B}$ , equipped with a norm-continuous automorphism group  $(\sigma_t^{\hat{\nu}})$ , such that  $(\hat{\nu} \otimes \text{id})(\hat{E}) = 1$ . We also have a KMS weight  $\hat{\mu}$  on  $\hat{C}$ , equipped with a norm-continuous automorphism group  $(\sigma_t^{\hat{\mu}})$ , such that  $(\text{id} \otimes \hat{\mu})(\hat{E}) = 1$ .
- (3) There exists a closed, densely-defined map  $\gamma_{\hat{B}} : \hat{B} \rightarrow \hat{C}$ , having a dense range in  $\hat{C}$ , such that  $\hat{E}(\hat{b} \otimes 1) = \hat{E}(1 \otimes \gamma_{\hat{B}}(\hat{b}))$ , for  $\hat{b} \in \mathcal{D}(\gamma_{\hat{B}})$ . Also there exists a closed, densely-defined map  $\gamma_{\hat{C}} : \hat{C} \rightarrow \hat{B}$ , having a dense range in  $\hat{B}$ , such that  $(1 \otimes \hat{c})\hat{E} = (\gamma_{\hat{C}}(\hat{c}) \otimes 1)\hat{E}$ , for  $\hat{c} \in \mathcal{D}(\gamma_{\hat{C}})$ .
- (4) There exists a  $*$ -anti-isomorphism of  $C^*$ -algebras  $\hat{R} : \hat{B} \rightarrow \hat{C}$ . We have:  $\hat{\nu} = \hat{\mu} \circ \hat{R}$ ,  $\sigma_t^{\hat{\nu}} = \hat{R}^{-1} \circ \sigma_{-t}^{\hat{\mu}} \circ \hat{R}$ ,  $\gamma_{\hat{\nu}} = \hat{R} \circ \sigma_{i/2}^{\hat{\nu}}$ ,  $\gamma_{\hat{\mu}} = \hat{R}^{-1} \circ \sigma_{-i/2}^{\hat{\mu}}$ .
- (5)  $\hat{E} \in M(\hat{B} \otimes \hat{C})$ .
- (6) The data  $(\hat{E}, \hat{B}, \hat{\nu})$  forms a  $(C^*$ -algebraic) separability triple, in the sense of [9].

*Proof.* (1). Analogous to Proposition 4.16 and the remarks following it.

(2). Analogous to Theorem 4.17 (2). See also Proposition 4.15.

(3). Analogous to Propositions 4.13 and 4.15. See also Theorem 4.17 (5).

(4). Analogous to Theorem 4.17 (4). See also Propositions 4.13 and 4.15.

(5). Analogous to Theorem 4.17 (3).

(6). Analogous to Theorem 4.17 (5). □

*Remark.* We have the  $*$ -anti-isomorphisms  $R : B \rightarrow C$  and  $\hat{R} : \hat{B} = \hat{C}$  (see Theorems 4.17 and 4.18). We know  $C = \hat{C}$  (see Proposition 4.2). It follows that  $B \cong \hat{B}$ . However, in general  $B \neq \hat{B}$ .

## 5. ANTIPODE

**5.1. The antipode map.** So far, from a multiplicative partial isometry  $W$ , satisfying certain conditions including the *manageability*, we have constructed a  $C^*$ -algebra  $A$ ; the comultiplication map  $\Delta : A \rightarrow M(A \otimes A)$ ; the  $C^*$ -subalgebras  $B \subseteq M(A)$  and  $C \subseteq M(A)$ ; the canonical idempotent element  $E \in M(B \otimes C)$ ; the  $*$ -anti-isomorphism  $R : B \rightarrow C$ ; assumed the existence of the KMS weight  $\nu$  on  $B$ ; then obtained the KMS weight  $\mu$  on  $C$ , and the closed densely-defined maps  $\gamma_B : B \rightarrow C$  and  $\gamma_C : C \rightarrow B$ .

Loosely speaking, the  $C^*$ -algebra  $A$  plays the role of  $C_0(G)$ , for a (quantum) groupoid  $G$ ;  $\Delta$  is the comultiplication map; the subalgebras  $B$  and  $C$  are the source and the target algebras, based on the unit space  $G^{(0)}$ ; with the weights  $\nu$  and  $\mu$  on them; and  $E = \Delta(1)$ .

Considering the definition of a  $C^*$ -algebraic quantum groupoid of separable type (See Definition 4.8 of [10] or Definition 1.2 of [11]), we only need a pair of (left and right) invariant weights  $\varphi$  and  $\psi$  for us to have a locally compact quantum groupoid. Then, by following the steps carried out in [11], we can construct an antipode map,  $S$ , and its polar decomposition.

In this paper, we do not plan to consider the invariant weights. Instead, we point out that in [11], it was noted that while the construction of the antipode map  $S$  involves the weights  $\varphi$  and  $\psi$ , once it is constructed, it can be shown that  $S$  does not depend on the specific choice of the weights: See the Remark following Theorem 5.12 in [11]. In fact, a convenient characterization of the antipode map exists (see Proposition 4.27 of [11]). Based on these facts, we give here the following characterization of the antipode map:

**Theorem 5.1.** (1) If  $\widetilde{W}$  and  $Q$  are the operators providing the manageability property of  $W$ , as given in Definition 2.3, write  $\tau_t(a) := Q^{2it}aQ^{-2it}$ , for  $a \in A$ ,  $t \in \mathbb{R}$ . Then  $(\tau_t)_{t \in \mathbb{R}}$  determines a one-parameter group of automorphisms of  $A$ . This will be referred to as the “scaling group”.

(2) There exists a closed linear map  $S$  on  $A$ , such that  $\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}$  forms a core for  $S$ , and

$$S((\text{id} \otimes \omega)(W)) = (\text{id} \otimes \omega)(W^*), \quad \text{for } \omega \in \mathcal{B}(\mathcal{H})_*.$$

It is anti-multiplicative:  $S(ab) = S(b)S(a)$ , for any  $a, b \in \mathcal{D}(S)$ , and we have:  $S(S(a)^*)^* = a$  for any  $a \in \mathcal{D}(S)$ .

Moreover, there exists an involutive  $*$ -anti-automorphism  $R_A : A \rightarrow A$ , called the “unitary antipode”, such that the following polar decomposition result holds:

$$S = R_A \circ \tau_{-i/2} = \tau_{-i/2} \circ R_A,$$

where  $\tau_{-i/2}$  is the analytic generator for the automorphism group  $(\tau_t)$  at  $z = -\frac{i}{2}$ . The map  $S$  will be called the “antipode” map.

*Proof.* (1). Consider  $(\text{id} \otimes \omega)(W) \in \mathcal{A}$ , for  $\omega \in \mathcal{B}(\mathcal{H})_*$ ,  $t \in \mathbb{R}$ . By Proposition 2.5, we know that  $W = (Q^{-2it} \otimes Q^{-2it})W(Q^{2it} \otimes Q^{2it})$ , for  $t \in \mathbb{R}$ . So we have:

$$\begin{aligned} \tau_t((\text{id} \otimes \omega)(W)) &= Q^{2it} [(\text{id} \otimes \omega)(W)] Q^{-2it} \\ &= Q^{2it} (\text{id} \otimes \omega)((Q^{-2it} \otimes Q^{-2it})W(Q^{2it} \otimes Q^{2it})) Q^{-2it} = (\text{id} \otimes \omega_t)(W), \end{aligned} \quad (5.1)$$

where  $\omega_t \in \mathcal{B}(\mathcal{H})_*$  is such that  $\omega_t(\cdot) = \omega(Q^{-2it} \cdot Q^{2it})$ . We can see that  $\|\omega_t - \omega\|_{\mathcal{B}(\mathcal{H})_*} \rightarrow 0$ , as  $t \rightarrow 0$ .

From Equation (5.1), we observe that  $\tau_t(a) \in A$  for any  $a \in A$ . In fact, as  $\omega(Q^{-2it} \cdot Q^{2it})$  is dense in  $\mathcal{B}(\mathcal{H})_*$  for any  $t \in \mathbb{R}$ , we actually have  $\tau_t(\mathcal{A}) = \mathcal{A}$ , for all  $t \in \mathbb{R}$ . We note that  $\tau_t(a)$  is a norm-continuous function on  $t$ . In this way, we have a one-parameter group of automorphisms  $(\tau_t)_{t \in \mathbb{R}}$  of  $A$ .

(2). Let  $\omega \in \mathcal{B}(\mathcal{H})_*$ . Without loss of generality, we can take  $\omega = \omega_{v,u}$ , where  $v \in \mathcal{D}(Q)$ ,  $u \in \mathcal{D}(Q^{-1})$ . Then  $\omega_t(\cdot) = \omega_{v,u}(Q^{-2it} \cdot Q^{2it}) = \omega_{Q^{2it}v, Q^{-2it}u}$ . By analytic continuation, we have:  $\omega_{-i/2} = \omega_{Qv, Q^{-1}u}$ . It follows from Equation (5.1) that

$$\tau_{-i/2}((\text{id} \otimes \omega)(W)) = (\text{id} \otimes \omega_{-i/2})(W) = (\text{id} \otimes \omega_{Qv, Q^{-1}u})(W) = (\text{id} \otimes \omega_{v,u})(\widetilde{W})^\top,$$

by Lemma 2.9. In particular, note that  $(\text{id} \otimes \omega)(W) \in \mathcal{D}(\tau_{-i/2})$ . This also shows that  $(\text{id} \otimes \omega)(\widetilde{W})^\top \in \mathcal{A}$ , for any  $\omega \in \mathcal{B}(\mathcal{H})_*$ .

Define a (linear) map  $R_A : \mathcal{A} \rightarrow \mathcal{A}$ , by

$$R_A : (\text{id} \otimes \omega)(W^*) \mapsto (\text{id} \otimes \omega)(\widetilde{W})^\top, \quad \text{for } \omega \in \mathcal{B}(\mathcal{H})_*. \quad (5.2)$$

We will show that  $R_A$  extends to a  $*$ -anti-automorphism on  $A$ . See (i), (ii), (iii) below:

(i). Write  $a = (\text{id} \otimes \omega_{v,u})(W^*) \in \mathcal{A}$ , for  $v \in \mathcal{D}(Q)$ ,  $u \in \mathcal{D}(Q^{-1})$ . By the definition of  $R_A$  above, we have:

$$R_A(a) = (\text{id} \otimes \omega_{v,u})(\widetilde{W})^\top = (\text{id} \otimes \omega_{Qv, Q^{-1}u})(W). \quad (5.3)$$

Consider  $a^* = (\text{id} \otimes \omega_{u,v})(W) = (\text{id} \otimes \omega_{u,v})((W^*)^*)$ . To apply  $R_A$  here, we need to know  $\widetilde{W}^*$ . But, from the characterizing equation for the manageability given in Definition 2.3, we can write:

$$\langle W^*(\eta \otimes s), \xi \otimes r \rangle = \langle \widetilde{W}^*(\bar{\xi} \otimes Qs), \bar{\eta} \otimes Q^{-1}r \rangle,$$

for any  $\xi, \eta \in \mathcal{H}$ , and any  $r \in \mathcal{D}(Q^{-1})$ ,  $s \in \mathcal{D}(Q)$ . So it is easy to see that  $\widetilde{W}^* = \widetilde{W}^*$ , and the associated closed operator is now  $Q^{-1}$ . Therefore, we have:

$$R_A(a^*) = R_A((\text{id} \otimes \omega_{u,v})(W)) = (\text{id} \otimes \omega_{u,v})(\widetilde{W}^*)^\top = (\text{id} \otimes \omega_{Q^{-1}u, Qv})(W^*). \quad (5.4)$$

This shows  $R_A(a^*) = R_A(a)^*$ , so  $R_A$  is a  $*$ -map.

(ii). Consider  $a = (\text{id} \otimes \omega_{v,u})(W^*)$  and  $b = (\text{id} \otimes \omega_{s,r})(W^*)$ . By the definition of  $R_A$ , we have  $R_A(a) = (\text{id} \otimes \omega_{Qv,Q^{-1}u})(W)$  and  $R_A(b) = (\text{id} \otimes \omega_{Qs,Q^{-1}r})(W)$ . Then from Proposition 1.3, we know

$$\begin{aligned} R_A(b)R_A(a) &= (\text{id} \otimes \omega_{Qs,Q^{-1}r} \otimes \omega_{Qv,Q^{-1}u})(W_{23}W_{12}W_{23}^*) \\ &= (\text{id} \otimes \omega_{s,r} \otimes \omega_{v,u})((1 \otimes Q^{-1} \otimes Q^{-1})W_{23}W_{12}W_{23}^*(1 \otimes Q \otimes Q)). \\ &= (\text{id} \otimes \omega_{s,r} \otimes \omega_{v,u})(W_{23}(1 \otimes Q^{-1} \otimes 1)W_{12}(1 \otimes Q \otimes 1)W_{23}^*), \end{aligned}$$

because  $W(Q \otimes Q) \subseteq (Q \otimes Q)W$ .

Meanwhile,

$$\begin{aligned} ab &= (\text{id} \otimes \omega_{v,u})(W^*)(\text{id} \otimes \omega_{s,r})(W^*) = [(\text{id} \otimes \omega_{r,s})(W)(\text{id} \otimes \omega_{u,v})(W)]^* \\ &= [(\text{id} \otimes \omega_{r,s} \otimes \omega_{u,v})(W_{23}W_{12}W_{23}^*)]^* = (\text{id} \otimes \omega_{s,r} \otimes \omega_{v,u})(W_{23}W_{12}W_{23}^*) = (\text{id} \otimes \theta)(W^*), \end{aligned}$$

where  $\theta(T) = (\omega_{s,r} \otimes \omega_{v,u})(W(T \otimes 1)W^*)$ . Therefore,

$$\begin{aligned} R_A(ab) &= R_A((\text{id} \otimes \theta)(W^*)) = (\text{id} \otimes \theta)(\widetilde{W})^\top = (\text{id} \otimes \theta(Q^{-1} \cdot Q))(W) \\ &= (\text{id} \otimes \omega_{s,r} \otimes \omega_{v,u})(W_{23}(1 \otimes Q^{-1} \otimes 1)W_{12}(1 \otimes Q \otimes 1)W_{23}^*). \end{aligned}$$

Comparing, we have:  $R_A(ab) = R_A(b)R_A(a)$ , proving the anti-multiplicativity of  $R_A$ .

(iii). For  $a = (\text{id} \otimes \omega_{v,u})(W^*) \in \mathcal{A}$ , by combining Equations (5.3) and (5.4), we have:

$$R_A(R_A(a)) = R_A((\text{id} \otimes \omega_{Qv,Q^{-1}u})(W)) = (\text{id} \otimes \omega_{v,u})(W^*) = a.$$

This shows that  $R_A \circ R_A = \text{Id}_{\mathcal{A}}$ .

By (i), (ii), (iii), we see that  $R_A$  is a \*-anti-homomorphism (so bounded), which is one-to-one from  $\mathcal{A}$  onto  $\mathcal{A}$ , which is dense in  $A$ . Therefore, we see that  $R_A$  extends to an involutive \*-anti-automorphism on  $A$ .

Finally, define the map  $S$ , by

$$S := R_A \circ \tau_{-i/2}.$$

As  $\tau_{-i/2}$  is a closed densely-defined map having  $\mathcal{A}$  as a core (see above), so is  $S$ . As  $R_A$  is anti-multiplicative, so is  $S$ . Under this map  $S$ , we have:

$$S((\text{id} \otimes \omega_{v,u})(W)) = R_A((\text{id} \otimes \omega_{Qv,Q^{-1}u})(W)) = (\text{id} \otimes \omega_{v,u})(W^*).$$

This gives an alternative characterization of  $S$ . It is also easy to see that  $R_A \circ \tau_{-i/2} = \tau_{-i/2} \circ R_A$ .

Finally, we can also see that for any  $a = (\text{id} \otimes \omega)(W) \in \mathcal{A}$ , we have:

$$\begin{aligned} S(S(a)^*)^* &= S(S((\text{id} \otimes \omega)(W))^*)^* = S([( \text{id} \otimes \omega)(W^*)]^*)^* \\ &= S((\text{id} \otimes \bar{\omega})(W))^* = [(\text{id} \otimes \bar{\omega})(W^*)]^* = (\text{id} \otimes \omega)(W) = a. \end{aligned}$$

□

*Remark.* This construction of the antipode map is different from the way that was done in [11], which used the invariant weights. For instance, the  $Q$  (or rather  $Q^2$ ) operator that is being used here to define the scaling group is different from the  $L$  operator used in that paper. On the other hand, the characterization of  $S$  given in (2) of Theorem 5.1 is exactly same as the one obtained in Proposition 4.27 of [11]. Moreover, from  $S^2 = \tau_{-i}$ , we can see that the analytic generators of the scaling groups for the two formulations are same, meaning that the scaling groups  $(\tau_t)$  coincide, so also the unitary antipode maps  $R_A$ . This means that  $S$ ,  $R_A$ ,  $(\tau_t)$  are exactly same for the two formulations, even though the approaches to arriving at them are different.

The observation made in the remark above means that any of the results obtained in [11] regarding the antipode map will be valid in our setting as well. As such, we will refer the reader to the main papers [10] and [11] for other details. For instance, here are some results (without proof) regarding the maps  $R_A$ ,  $S$ , and the scaling group  $(\tau_t)$  at the level of the base algebras: Note that the maps  $\gamma_B$ ,  $\gamma_C$  earlier are in fact the restrictions of the antipode map  $S$ , to the level of  $B$  and  $C$ , respectively.

**Proposition 5.2.** (1). *The scaling group  $(\tau_t)$  leaves both  $B$  and  $C$  invariant. Moreover, we have:  $\tau_t|_B = \sigma_{-t}^\nu$  and  $\tau_t|_C = \sigma_t^\mu$ .*

(2).  *$S|_B = \gamma_B : B \rightarrow C$  and  $S|_C = \gamma_C : C \rightarrow B$ .*

*Proof.* See Propositions 5.23 and 5.24 (and its Corollary) in [11].  $\square$

*Remark.* In particular, the proposition confirms that restricted to  $B$  (or  $N$ ), we have:  $\sigma_{i/2}^\nu(\cdot) = \tau_{-i/2}(\cdot) = Q(\cdot)Q^{-1} = T(\cdot)$ , where  $T$  is the operator considered in Proposition 4.11. We had  $\kappa = R_\kappa \circ T$  at the von Neumann algebra level. We saw that  $\kappa = \gamma_N$  (Proposition 4.12) and that  $\gamma_N = R \circ \sigma_{i/2}^\nu$  (Proposition 4.13). The fact that  $T = \sigma_{i/2}^\nu$  means  $R_\kappa|_B = R$ , actually. We can see that what we learned at the base algebra level in Section 4 agree well with the discussion given in this section.

There exist corresponding results to Theorem 5.1 and Proposition 5.2 for the case of the dual object  $(\widehat{A}, \widehat{\Delta})$ , which is another quantum groupoid obtained by working with  $\widehat{W} = \Sigma W^* \Sigma$ . We can construct an antipode map,  $\widehat{S}$ , which can be characterized as follows:

**Theorem 5.3.** (1) *Write  $\hat{\tau}_t(a) := Q^{2it} a Q^{-2it}$ , for  $a \in A$ ,  $t \in \mathbb{R}$ . Then  $(\hat{\tau}_t)_{t \in \mathbb{R}}$  determines a one-parameter group of automorphisms of  $\widehat{A}$ , which will be the “scaling group”.*

(2) *There exists a closed linear map (the “antipode”)  $\widehat{S}$  on  $\widehat{A}$ , such that  $\{(\omega \otimes \text{id})(W^*) : \omega \in \mathcal{B}(\mathcal{H})_*\}$  forms a core for  $S$ , and*

$$\widehat{S}((\omega \otimes \text{id})(W^*)) = (\omega \otimes \text{id})(W), \quad \text{for } \omega \in \mathcal{B}(\mathcal{H})_*.$$

*It is anti-multiplicative, and we have:  $\widehat{S}(\widehat{S}(a)^*)^* = a$  for any  $a \in \mathcal{D}(\widehat{S})$ .*

*Moreover, there exists an involutive  $*$ -anti-automorphism  $R_{\widehat{A}} : \widehat{A} \rightarrow \widehat{A}$ , called the “unitary antipode”, such that the following polar decomposition result holds:*

$$\widehat{S} = R_{\widehat{A}} \circ \hat{\tau}_{-i/2} = \hat{\tau}_{-i/2} \circ R_{\widehat{A}},$$

*where  $\hat{\tau}_{-i/2}$  is the analytic generator for the automorphism group  $(\hat{\tau}_t)$  at  $z = -\frac{i}{2}$ .*

*Proof.* Work with  $\widehat{W} = \Sigma W^* \Sigma$ , which is also a manageable multiplicative partial isometry, associated with  $\widetilde{W} = (\Sigma \widehat{W}^* \Sigma)^{\top \otimes \top}$  and  $Q$  (see Proposition 2.8). Then we can just apply the results of Theorem 5.1.

Definition of  $\hat{\tau}$  easily follows from Theorem 5.1, as the same  $Q$  operator is associated with  $\widehat{W}$ .

By Theorem 5.1, we know  $\widehat{S}$  can be characterized by  $\widehat{S} : (\text{id} \otimes \omega)(\widehat{W}) \mapsto (\text{id} \otimes \omega)(\widehat{W}^*)$ , for  $\omega \in \mathcal{B}(\mathcal{H})_*$ , which will satisfy the desired properties such as the anti-multiplicativity, the relationship with the adjoint, and the like. But from  $\widehat{W} = \Sigma W^* \Sigma$ , it becomes  $\widehat{S} : (\omega \otimes \text{id})(W^*) \mapsto (\omega \otimes \text{id})(W)$ ,  $\omega \in \mathcal{B}(\mathcal{H})_*$ .

From the proof of Theorem 5.1, more specifically from Equation (5.2), we know that there exists an involutive  $*$ -anti-automorphism  $R_{\widehat{A}} : \widehat{A} \rightarrow \widehat{A}$ , given by

$$R_{\widehat{A}} : (\text{id} \otimes \omega)(\widehat{W}^*) \mapsto (\text{id} \otimes \omega)(\widetilde{W})^\top, \quad \text{for } \omega \in \mathcal{B}(\mathcal{H})_*.$$

Use now the fact that  $\widehat{W} = \Sigma W^* \Sigma$  and that  $\widetilde{W} = (\Sigma \widetilde{W}^* \Sigma)^{\top \otimes \top}$ . Then this characterization becomes:

$$R_{\widehat{A}} : (\omega \otimes \text{id})(W) \mapsto [(\omega \otimes \text{id})(\widetilde{W}^*)^{\top \otimes \top}]^{\top \otimes \top} = (\omega^\top \otimes \text{id})(\widetilde{W}^*) \quad \text{for } \omega \in \mathcal{B}(\mathcal{H})_*. \quad (5.5)$$

By Theorem 5.1, we should be able to express  $\widehat{S}$  in the polar decomposition form,  $\widehat{S} = R_{\widehat{A}} \circ \widehat{\tau}_{-i/2} = \widehat{\tau}_{-i/2} \circ R_{\widehat{A}}$ .  $\square$

**Corollary.** *Consider the inverse of  $\widehat{S}$ . The space  $\{(\omega \otimes \text{id})(W) : \omega \in \mathcal{B}(\mathcal{H})_*\} \subset \widehat{A}$  forms a core for  $\widehat{S}^{-1}$ , and we have:*

$$\widehat{S}^{-1}((\omega \otimes \text{id})(W)) = (\omega \otimes \text{id})(W^*), \quad \text{for } \omega \in \mathcal{B}(\mathcal{H})_*.$$

*It is evident that  $\widehat{S}$  is anti-multiplicative, satisfies  $\widehat{S}^{-1}(\widehat{S}^{-1}(a)^*)^* = a$ ,  $a \in \mathcal{D}(\widehat{S}^{-1})$ , and we have the following polar decomposition result:*

$$\widehat{S}^{-1} = R_{\widehat{A}} \circ \widehat{\tau}_{i/2} = \widehat{\tau}_{i/2} \circ R_{\widehat{A}}.$$

*Remark.* Proof is straightforward from Proposition 5.3. Note that the result of this Corollary is analogous to Theorem 1.5 (4) of [28], when  $W$  is a manageable multiplicative unitary.

We do not write the dual counterpart to Proposition 5.2 here, but it is evident that an analogous result will hold for the restriction of the scaling group  $(\widehat{\tau}_t)$  to the level of the base algebras  $\widehat{B}$  and  $\widehat{C}$ . See a future paper [12], for more discussion on the duality picture.

Meanwhile, in terms of the unitary antipode, in the below is a result that gives an alternative characterization of the operator  $\widetilde{W}$ . An analogous result was obtained for a manageable multiplicative unitary  $W$ , in [28].

**Proposition 5.4.** *For convenience write  $\widehat{R}$  to mean the unitary antipode map  $R_{\widehat{A}}$ , and also for convenience, use the exponent notation. That is, write  $x^{\widehat{R}}$  instead of  $\widehat{R}(x) = R_{\widehat{A}}(x)$ ,  $x \in \widehat{A}$ . Then we have:*

$$W^{\top \otimes \widehat{R}} = \widetilde{W}^*.$$

*Proof.* Observe from Equation (5.5) that  $[(\omega \otimes \text{id})(W)]^{\widehat{R}} = (\omega^\top \otimes \text{id})(\widetilde{W}^*)$ . Or equivalently,  $(\omega \otimes \text{id})(W) = [(\omega^\top \otimes \text{id})(\widetilde{W}^*)]^{\widehat{R}}$ . This is in turn equivalent to

$$(\omega \otimes \text{id})(W) = (\omega \otimes \text{id})([\widetilde{W}^*]^{\top \otimes \widehat{R}}),$$

true for any  $\omega \in \mathcal{B}(\mathcal{H})_*$ . It follows that  $W = [\widetilde{W}^*]^{\top \otimes \widehat{R}}$ , or equivalently, we have  $W^{\top \otimes \widehat{R}} = \widetilde{W}^*$ .  $\square$

**Corollary.**  *$\widetilde{W}$  is also a partial isometry.*

*Proof.* While it seemed implicitly true, we never proved that the operator  $\widetilde{W}$  for a multiplicative partial isometry  $W$  is itself a partial isometry as well.

Now, as a consequence of the characterization given in Proposition 5.4, it is easy to see that

$$\widetilde{W}^* \widetilde{W} \widetilde{W}^* = [W^{\top \otimes \widehat{R}}]^* [W^{\top \otimes \widehat{R}}] [W^{\top \otimes \widehat{R}}]^* = [W^* W W^*]^{\top \otimes \widehat{R}} = W^{\top \otimes \widehat{R}} = \widetilde{W}^*,$$

as the transpose map and  $\widehat{R}$  are both  $*$ -anti-isomorphisms.  $\square$

**5.2. Final remark.** We have shown in this paper that from a manageable multiplicative partial isometry, under some additional conditions such as the *manageability*, we can essentially construct a  $C^*$ -algebraic quantum groupoid of separable type. Going the other way, if we begin with such a quantum groupoid (see Definition 1.2 in [11]), we can construct from the defining axioms a multiplicative partial isometry  $W$ , as well as the antipode map and its polar decomposition. We expect that we can always find a positive operator  $P$  implementing the scaling group, while  $P^{-\frac{1}{2}}$  behaves quite like a  $Q$  operator arising in the manageability axioms (Definition 2.3). An analogous result is known in the quantum group case [13], and this aspect will be discussed more carefully in our future paper [12].

Since any multiplicative partial isometry constructed from the axioms for quantum groupoids of separable type would turn out to be manageable, the results from the current paper will allow us to find a convenient way to construct a dual quantum groupoid of the same type.

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