

## DRESSING ORBITS AND A QUANTUM HEISENBERG GROUP ALGEBRA

BYUNG-JAY KAHNG

ABSTRACT. In this paper, as a generalization of Kirillov's orbit theory, we explore the relationship between the dressing orbits and irreducible  $*$ -representations of the Hopf  $C^*$ -algebras  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$  we constructed earlier. We discuss the one-to-one correspondence between them, including their topological aspects.

On each dressing orbit (which are symplectic leaves of the underlying Poisson structure), one can define a Moyal-type deformed product at the function level. The deformation is more or less modeled by the irreducible representation corresponding to the orbit. We point out that the problem of finding a direct integral decomposition of the regular representation into irreducibles (Plancherel theorem) has an interesting interpretation in terms of these deformed products.

### Introduction

According to Kirillov's orbit theory [13], [14], the representation theory of a Lie group is closely related with the coadjoint action of the group on the dual vector space of its Lie algebra. The coadjoint orbits play the central role. The program is most successful for nilpotent or exponential solvable Lie groups. While it does not work as well for other types of Lie groups, the orbit theory, with some modifications, is still a very useful tool in the Lie group representation theory.

It is reasonable to expect that a generalization to some extent of the Kirillov-type orbit theory exists even for quantum groups. On the other hand, it has been known for some time that the "geometric quantization" of physical systems is very much related with the construction of irreducible unitary representations in mathematics (see [17], [3], [15]). The orbit theoretical approach is instrumental in these discussions.

Generalizing the orbit theory to the quantum group level is a quite interesting program [16], and is still on-going since late 80's ([26], [20]). In this paper, we will focus our attention to the examples of non-compact quantum

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groups  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$ , which we constructed in [10] (see also [11], [12]). Noting the fact that the classical counterparts to  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$  are exponential solvable Lie groups, we plan to study these examples from the point of view of the (generalized) orbit theory. Most of the results in this paper are not necessarily surprising. But we still believe this is a worthwhile project, especially since we later plan to explore the aspect of our examples as certain quantized spaces as well as being quantum groups.

Our goal here is twofold. One is to study the representation theory of the examples  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$ , by using the orbit theory. Although we now work with dressing actions instead of coadjoint actions, we will see that many of the results are analogous to those of the classical counterparts. A more long-term goal is to further investigate how the orbits and representation theory are related with the quantization process.

This paper is organized as follows. In Section 1, we review some preliminary results from our earlier papers. We discuss the Poisson structures, dressing actions and dressing orbits. Then we recall the definitions of our main examples  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$ . In Section 2, we study the irreducible  $*$ -representations of our examples, in connection with the dressing orbits. Going one step further than just pointing out the one-to-one correspondence between them, we will discuss the correspondence in terms of the topological structures on the set of orbits and on the set of representations.

For each dressing orbit  $\mathcal{O}$ , which is actually a symplectic leaf, we can construct a Moyal-type, deformed (quantized) product on the space of smooth functions on  $\mathcal{O}$ . We do this by considering these smooth functions as operators on a Hilbert space. We first find, in Section 3, a canonical measure on each orbit, which plays an important role in the construction of the Hilbert space. The orbit deformation is carried out in Section 4. It turns out that the deformation of an orbit  $\mathcal{O}$  is “modeled” by the irreducible  $*$ -representation corresponding to  $\mathcal{O}$ . Furthermore, we will see that the “regular representation”  $L$ , which we used in [10] to give the specific operator algebra realization of  $A$ , is equivalent to a direct integral of the irreducible  $*$ -representations. This is a version of the Plancherel theorem.

The discussion in Section 4 (concerning the deformation of orbits) is restricted to the case of  $(A, \Delta)$ , while the case of  $(\tilde{A}, \tilde{\Delta})$  is postponed to a future paper. However, in an Appendix we give a short preliminary report on the case of  $(\tilde{A}, \tilde{\Delta})$ . Although most of the general ideas do go through, there are some technical obstacles which we have to consider. We hope that the discussions in Sections 3 and 4 (as well as those in the Appendix) will be helpful in our attempt to understand the relationship between the orbits and the quantization process.

## 1. Preliminaries

Our objects of study are the Poisson–Lie groups  $H$ ,  $G$  and  $\tilde{H}$ ,  $\tilde{G}$ , as well as their quantizations (i.e., “quantum groups”)  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$ . For more complete descriptions of these objects, see [10] (and also [11]). Let us begin with a short summary.

**1.1. The Poisson–Lie groups  $H$ ,  $G$ ,  $\tilde{H}$ ,  $\tilde{G}$ . The dressing orbits.** The group  $H$  is the  $(2n+1)$ -dimensional Heisenberg Lie group. Its underlying space is  $\mathbb{R}^{2n+1}$  and the multiplication on it is defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + x \cdot y'),$$

for  $x, x', y, y' \in \mathbb{R}^n$  and  $z, z' \in \mathbb{R}$ . We also consider the extended Heisenberg group  $\tilde{H}$ , whose group law is given by

$$(x, y, z, w)(x', y', z', w') = (x + e^w x', y + e^{-w} y', z + z' + e^{-w} x \cdot y', w + w').$$

It is  $(2n+2)$ -dimensional. The notation is similar as above, with  $w, w' \in \mathbb{R}$ . It is easy to see that  $\tilde{H}$  contains  $H$  as a normal subgroup.

In [10], we obtained the “dual Poisson–Lie group”  $\tilde{G}$  of  $\tilde{H}$ . It is  $(2n+2)$ -dimensional, considered as a dual vector space of  $\tilde{H}$ , and is determined by the multiplication law

$$(p, q, r, s)(p', q', r', s') = (e^{\lambda r'} p + p', e^{\lambda r'} q + q', r + r', s + s').$$

And the dual Poisson–Lie group  $G$  of  $H$  is determined by the multiplication law

$$(p, q, r)(p', q', r') = (e^{\lambda r'} p + p', e^{\lambda r'} q + q', r + r').$$

REMARK. Above,  $\lambda \in \mathbb{R}$  is a fixed constant which determines a certain non-linear Poisson structure when  $\lambda \neq 0$ . In Section 1 of [10], we gave a discussion on how the above pairs of Poisson–Lie groups are related with a so-called “classical  $r$ -matrix” element. Meanwhile, note that by taking advantage of the fact that the groups are exponential solvable, we are considering them as vector spaces (identified with the corresponding Lie algebras).

Given a dual pair of Poisson–Lie groups, there exists the so-called *dressing action* of a Poisson–Lie group acting on its dual Poisson–Lie group. It is rather well known that the notion of a dressing action is the natural generalization of the coadjoint action of a Lie group acting on the dual space of its Lie algebra. Also by a result of Semenov-Tian-Shansky, it is known that the dressing orbits are exactly the symplectic leaves in the Poisson–Lie groups [21], [25].

It is customary to define the dressing action as a right action. But for the purpose of this paper and the future projects in our plans, it is actually more convenient to work with the “left” dressing action. It is related to our specific choice in [10] of the multiplications on  $G$  and  $(A, \Delta)$ , so that the left Haar measure naturally comes from the ordinary Lebesgue measure on  $G$  (see also

[12]). To compute the left dressing action of  $H$  on  $G$  (similarly, the action of  $\tilde{H}$  on  $\tilde{G}$ ), it is useful to consider the following “double Lie group”  $\tilde{H} \rtimes \tilde{G}$ . It is isomorphic to the definition considered in [11].

Here and throughout this paper, we denote by  $\eta_\lambda(r)$  the expression  $\eta_\lambda(r) := (e^{2\lambda r} - 1)/(2\lambda)$ . When  $\lambda = 0$ , we take  $\eta_\lambda(r) = r$ .

LEMMA 1.1. *Let  $\tilde{H} \rtimes \tilde{G}$  be defined by the following multiplication law:*

$$\begin{aligned} & (x, y, z, w; p, q, r, s)(x', y', z', w'; p', q', r', s') \\ &= (x + e^{\lambda r + w} x', y + e^{\lambda r - w} y', \\ & \quad z + z' + e^{\lambda r - w} x \cdot y' - \lambda p \cdot x' - \lambda q \cdot y' + \lambda \eta_\lambda(r) x' \cdot y', w + w'; \\ & \quad e^{\lambda r' + w'} p + p' - e^{\lambda r' + w'} \eta_\lambda(r) y', e^{\lambda r' - w'} q + q' + e^{\lambda r' - w'} \eta_\lambda(r) x', \\ & \quad r + r', s + s' - p \cdot x' + q \cdot y' + \eta_\lambda(r) x' \cdot y'). \end{aligned}$$

We recover the group structures of  $\tilde{H}$  and  $\tilde{G}$ , by identifying  $(x, y, z, w) \in \tilde{H}$  with  $(x, y, z, w; 0, 0, 0, 0)$  and  $(p, q, r, s) \in \tilde{G}$  with  $(0, 0, 0, 0; p, q, r, s)$ . It is clear that  $\tilde{H}$  and  $\tilde{G}$  are closed Lie subgroups. Note also that any element  $(x, y, z, w; p, q, r, s) \in \tilde{H} \rtimes \tilde{G}$  can be written as

$$(x, y, z, w; p, q, r, s) = (x, y, z, w; 0, 0, 0, 0)(0, 0, 0, 0; p, q, r, s).$$

In other words,  $\tilde{H} \rtimes \tilde{G}$  is the “double Lie group” of  $\tilde{H}$  and  $\tilde{G}$ . Meanwhile, if we consider only the  $(x, y, z)$  and the  $(p, q, r)$  variables, we obtain in the same way the double Lie group  $H \rtimes G$  of  $H$  and  $G$ .

*Proof.* The group is obtained by first considering the “double Lie algebra” ([21]), which comes from the Lie bialgebra  $(\tilde{\mathfrak{h}}, \tilde{\mathfrak{g}})$  corresponding to the Poisson structures on  $\tilde{H}$  and  $\tilde{G}$  (see [11] for the computation). The verification of the statements is straightforward.  $\square$

The left dressing action is defined exactly in the same manner as the usual (right) dressing action. For  $h \in \tilde{H}$  and  $\mu \in \tilde{G}$ , regarded naturally as elements in  $\tilde{H} \rtimes \tilde{G}$ , we first consider the product  $\mu \cdot h$ . Factorize the product as  $\mu \cdot h = h^\mu \cdot \mu^h$ , where  $h^\mu \in \tilde{H}$  and  $\mu^h \in \tilde{G}$ . The left dressing action,  $\delta$ , of  $\tilde{H}$  on  $\tilde{G}$  is then given by  $\delta_h(\mu) := \mu^{(h^{-1})}$ .

We are mainly interested in the dressing orbits contained in  $G$  and  $\tilde{G}$ , which will be the symplectic leaves. The following two propositions give a brief summary. For the computation of the Poisson bracket on  $G$ , see Theorem 2.2 of [10]. The Poisson bracket on  $\tilde{G}$  is not explicitly mentioned there, but we can more or less follow the proof for the case of  $G$ . The computations for the dressing orbits are straightforward from the definition of the Poisson brackets.

PROPOSITION 1.2.

(1) The Poisson bracket on  $G$  is given by the expression

$$\{\phi, \psi\}(p, q, r) = \eta_\lambda(r)(x \cdot y' - x' \cdot y),$$

for  $\phi, \psi \in C^\infty(G)$ . Here  $d\phi(p, q, r) = (x, y, z)$  and  $d\psi(p, q, r) = (x', y', z')$ , which are naturally considered as elements of  $\mathfrak{h}$ .

(2) The left dressing action of  $H$  on  $G$  is

$$(\delta(a, b, c))(p, q, r) = (p + \eta_\lambda(r)b, q - \eta_\lambda(r)a, r).$$

(3) The dressing orbits in  $G$  are:

- $\mathcal{O}_{p,q} = \{(p, q, 0)\}$ , when  $r = 0$ .
- $\mathcal{O}_r = \{(\alpha, \beta, r) : (\alpha, \beta) \in \mathbb{R}^{2n}\}$ , when  $r \neq 0$ .

The  $\mathcal{O}_{p,q}$  are 1-point orbits and the  $\mathcal{O}_r$  are  $2n$ -dimensional orbits. By standard theory, these orbits are exactly the symplectic leaves in  $G$  for the Poisson bracket on it defined above.

PROPOSITION 1.3.

(1) The Poisson bracket on  $\tilde{G}$  is given by

$$\begin{aligned} \{\phi, \psi\}(p, q, r, s) &= p \cdot (wx' - w'x) + q \cdot (w'y - wy') \\ &\quad + \eta_\lambda(r)(x \cdot y' - x' \cdot y), \end{aligned}$$

for  $\phi, \psi \in C^\infty(\tilde{G})$ . We are again using the natural identification of  $d\phi(p, q, r, s) = (x, y, z, w)$  and  $d\psi(p, q, r, s) = (x', y', z', w')$  as elements of  $\tilde{\mathfrak{h}}$ .

(2) The left dressing action of  $\tilde{H}$  on  $\tilde{G}$  is

$$\begin{aligned} (\delta(a, b, c, d))(p, q, r, s) &= (e^{-d}p + \eta_\lambda(r)b, e^d q - \eta_\lambda(r)a, r, \\ &\quad s + e^{-d}p \cdot a - e^d q \cdot b + \eta_\lambda(r)a \cdot b). \end{aligned}$$

(3) The dressing orbits in  $\tilde{G}$  are:

- $\tilde{\mathcal{O}}_s = \{(0, 0, 0, s)\}$ , when  $(p, q, r) = (0, 0, 0)$ .
- $\tilde{\mathcal{O}}_{p,q} = \{(\alpha p, \frac{1}{\alpha}q, 0, \gamma) : \alpha > 0, \gamma \in \mathbb{R}\}$ , when  $r = 0$  but  $(p, q) \neq (0, 0)$ .
- $\tilde{\mathcal{O}}_{r,s} = \{(\alpha, \beta, r, s - \frac{1}{\eta_\lambda(r)}\alpha \cdot \beta) : (\alpha, \beta) \in \mathbb{R}^{2n}\}$ , when  $r \neq 0$ .

The  $\tilde{\mathcal{O}}_s$  are 1-point orbits, the  $\tilde{\mathcal{O}}_{p,q}$  are 2-dimensional orbits, and the  $\tilde{\mathcal{O}}_{r,s}$  are  $2n$ -dimensional orbits. These are exactly the symplectic leaves in  $\tilde{G}$  for the Poisson bracket on it.

REMARK. We should point out that the notations for the 2-dimensional orbits  $\tilde{\mathcal{O}}_{p,q}$  in  $\tilde{G}$  are somewhat misleading. We can see clearly that we can have  $\tilde{\mathcal{O}}_{p,q} = \tilde{\mathcal{O}}_{p',q'}$  if  $p' = \alpha p$  and  $q' = \frac{1}{\alpha}q$  for some  $\alpha > 0$ . To avoid introducing cumbersome notations, we nevertheless chose to stay with this ambiguity. A sketch is given below (in Figure 1) to help us visualize the situation.

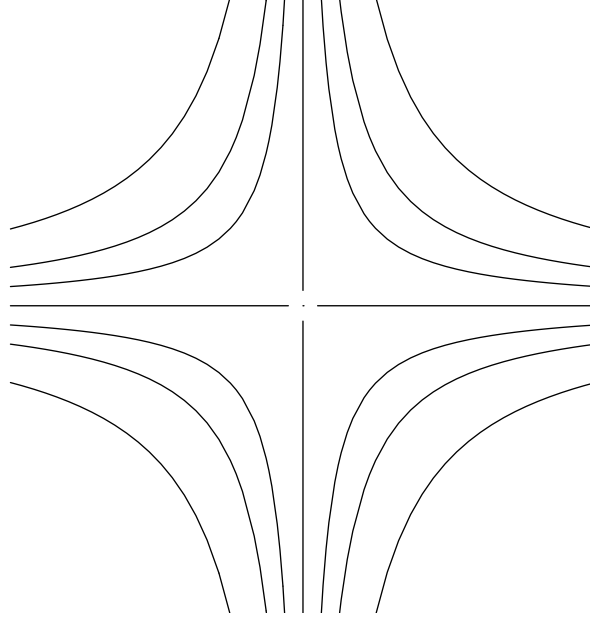


FIGURE 1. Trace curves of the orbits on the  $r = s = 0$  plane

**1.2. The Hopf  $C^*$ -algebras  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$ .** By realizing that the Poisson structure on  $G$  is a non-linear Poisson bracket of the “cocycle perturbation” type (as in [9]), we were able to construct the Hopf  $C^*$ -algebras (quantum groups)  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$  by deformation. For their precise definitions, see [10].

As a  $C^*$ -algebra,  $A$  is isomorphic to a twisted crossed product algebra. That is,  $A \cong C^*(H/Z, C_0(G/Z^\perp), \sigma)$ , where  $H$  and  $G(= H^*)$  are as in §1.1 and  $Z$  is the center of  $H$  (so  $Z = \{(0, 0, z)\}$ ). We denoted by  $\sigma$  the twisting cocycle for the group  $H/Z$ . As constructed in [10],  $\sigma$  is a continuous field of cocycles  $G/Z^\perp \ni r \mapsto \sigma^r$ , where

$$\sigma^r((x, y), (x', y')) = \bar{e}[\eta_\lambda(r)x \cdot y'].$$

Following the notation of the previous papers, we are letting  $e(t) = e^{(2\pi i)t}$  and  $\bar{e}(t) = e^{(-2\pi i)t}$ , while  $\eta_\lambda(r)$  is as before. The elements  $(x, y), (x', y')$  are group elements in  $H/Z$ . Via a certain “regular representation”  $L$ , we were able to realize the  $C^*$ -algebra  $A$  as an operator algebra in  $\mathcal{B}(\mathcal{H})$ , where  $\mathcal{H} = L^2(H/Z \times G/Z^\perp)$  is the Hilbert space consisting of the  $L^2$ -functions in the  $(x, y, r)$  variables.

In [10], we showed that the  $C^*$ -algebra  $A$  is a strict deformation quantization (in the sense of Rieffel [24]) of  $C_0(G)$ . For convenience, the deformation

parameter  $\hbar$  has been fixed ( $\hbar = 1$ ), which is the reason why we do not see it in the definition of  $A$ . When  $\hbar = 0$  (i.e., the classical limit), we take  $\sigma \equiv 1$ . Then  $A_{\hbar=0} \cong C_0(G)$ . Throughout this paper (as in our previous papers), we write  $A = A_{\hbar=1}$ . On  $A$ , an appropriate comultiplication can be defined using a certain “(regular) multiplicative unitary operator”. Actually, we can show (see [12]) that  $(A, \Delta)$  is an example of a *locally compact quantum group*, in the sense of Kustermans and Vaes [18]. All these can be done similarly for the case of  $\tilde{H}$  and  $\tilde{G}$ , obtaining  $(\tilde{A}, \tilde{\Delta})$ .

Since our goal here is to study the \*-representations of  $A$  and  $\tilde{A}$ , we will discuss of their quantum group structures only lightly. We will review the notations as the needs arise. For the most part, it will be useful to recall that we can regard  $(A, \Delta)$  as a “quantized  $C^*(H)$ ”, i.e., a “quantum Heisenberg group algebra”. Or dually, we may regard it as a “quantized  $C_0(G)$ ”. Similar comments hold for  $(\tilde{A}, \tilde{\Delta})$ , which can be considered as an “extended quantum Heisenberg group algebra” (“quantized  $C^*(\tilde{H})$ ”) or as a “quantized  $C_0(\tilde{G})$ ”.

## 2. The irreducible \*-representations

The irreducible \*-representations of  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$  have been found in [11], taking advantage of the fact that  $A$  and  $\tilde{A}$  are twisted group  $C^*$ -algebras. The results are summarized in the first two propositions below. Note that we study here the \*-representations of  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$ , instead of their corepresentations. By the observation given in the previous section, the \*-representations of  $A$  [and  $\tilde{A}$ ] more or less correspond to the group representations of  $H$  [and  $\tilde{H}$ ].

NOTATION. Below,  $\mathcal{A}$  is the space of Schwartz functions in the  $(x, y, r)$  variables having compact support in the  $r$  variable. Similarly,  $\tilde{\mathcal{A}}$  is the space of Schwartz functions in the  $(x, y, r, w)$  variables, having compact support in the variables  $r$  and  $w$ . By viewing these functions as operators (via the regular representation  $L$ ; see equation (2.4) and Example 3.6 of [10]), we saw in the previous papers that they are dense \*-subalgebras of  $A$  and  $\tilde{A}$ . Most of our specific calculations have been carried out at the level of these function algebras.

PROPOSITION 2.1. *Every irreducible representation of  $\mathcal{A}$  is equivalent to one of the following representations. Here,  $f \in \mathcal{A}$ .*

- For  $(p, q) \in \mathbb{R}^{2n}$ , there is a 1-dimensional representation  $\pi_{p,q}$  of  $\mathcal{A}$  defined by

$$\pi_{p,q}(f) = \int f(x, y, 0) \bar{e}(p \cdot x + q \cdot y) dx dy.$$

- For  $r \in \mathbb{R}$ ,  $r \neq 0$ , there is a representation  $\pi_r$  of  $\mathcal{A}$  acting on the Hilbert space  $\mathcal{H}_r = L^2(\mathbb{R}^n)$  defined by

$$\pi_r(f)\xi(u) = \int f(x, y, r) \bar{e}(\eta_\lambda(r)u \cdot y) \xi(u + x) dx dy.$$

We thus obtain all the irreducible  $*$ -representations of  $A$  by naturally extending these representations. We will use the same notation,  $\pi_{p,q}$  and  $\pi_r$ , for the representations of  $A$  constructed in this way.

PROPOSITION 2.2. *The irreducible  $*$ -representations of  $\tilde{A}$  are obtained by naturally extending the following irreducible representations of the dense subalgebra  $\tilde{A}$ . Here,  $f \in \tilde{A}$ .*

- For  $s \in \mathbb{R}$ , there is a 1-dimensional representation  $\tilde{\pi}_s$  defined by

$$\tilde{\pi}_s(f) = \int f(x, y, 0, w) \bar{e}(sw) dx dy dw.$$

- For  $(p, q) \in \mathbb{R}^{2n}$ ,  $(p, q) \neq (0, 0)$ , there is a representation  $\tilde{\pi}_{p,q}$  acting on the Hilbert space  $\tilde{\mathcal{H}}_{p,q} = L^2(\mathbb{R})$  defined by

$$\tilde{\pi}_{p,q}(f)\zeta(d) = \int f(x, y, 0, w) \bar{e}(e^d p \cdot x + e^{-d} q \cdot y) \zeta(d + w) dx dy dw.$$

- For  $(r, s) \in \mathbb{R}^2$ ,  $r \neq 0$ , there is a representation  $\tilde{\pi}_{r,s}$  acting on the Hilbert space  $\tilde{\mathcal{H}}_{r,s} = L^2(\mathbb{R}^n)$  defined by

$$\begin{aligned} \tilde{\pi}_{r,s}(f)\xi(u) &= \int f(x, y, r, w) \bar{e}(sw) \bar{e}(\eta_\lambda(r)u \cdot y) \\ &\quad \cdot (e^{-w/2})^n \xi(e^{-w}u + e^{-w}x) dx dy dw. \end{aligned}$$

We will use the same notation,  $\tilde{\pi}_s$ ,  $\tilde{\pi}_{p,q}$ , and  $\tilde{\pi}_{r,s}$ , for the corresponding representations of  $\tilde{A}$ .

REMARK. For the construction of these representations see Section 2 of [11]. Meanwhile, a comment similar to an earlier remark has to be made about the representations  $\tilde{\pi}_{p,q}$ . It is not very difficult to see that  $\tilde{\pi}_{p,q}$  and  $\tilde{\pi}_{p',q'}$  are equivalent if and only if  $p' = e^l p$  and  $q' = e^{-l} q$  for some real number  $l$ . Again, to avoid having to introduce cumbersome notations, we are staying with the (possibly ambiguous) notation used above.

These are all the irreducible  $*$ -representations of  $A$  and  $\tilde{A}$  up to equivalence (assuming we accept the ambiguity mentioned in the above remark). We did not rely on the dressing orbits to find these representations (we constructed the irreducible representations via the machinery of induced representations [11]), but we can still observe that the irreducible representations of  $A$  and  $\tilde{A}$  are in one-to-one correspondence with the dressing orbits in  $G$  and  $\tilde{G}$ ,



respectively. To emphasize the correspondence, we used the same subscripts for the orbits and the related irreducible representations.

Let us denote by  $\mathcal{O}(A)$  the set of dressing orbits contained in  $G$ . Since  $G$  is being identified with its Lie algebra  $\mathfrak{g}$ , it is equipped with the  $(2n + 1)$ -dimensional vector space topology. Note that on  $G$  we have an equivalence relation such that  $(p, q, r) \sim (p', q', r')$  if they are contained in the same orbit. By viewing  $\mathcal{O}(A) = G/\sim$ , we can give  $\mathcal{O}(A)$  a natural quotient topology.

Meanwhile, let  $\text{Irr}(A)$  be the set of equivalence classes of irreducible  $*$ -representations of  $A$  (Proposition 2.1). For every representation  $\pi \in \text{Irr}(A)$ , its kernel is a primitive ideal of  $A$ . Consider the ‘‘Jacobson topology’’ [6] on  $\text{Prim}(A)$ , that is, the closure of a subset  $\mathcal{U} \subseteq \text{Prim}(A)$  is defined to be the set of all ideals in  $\text{Prim}(A)$  containing the intersection of the elements of  $\mathcal{U}$ . Since the map  $\pi \mapsto \text{Ker}(\pi)$  is a canonical surjective map from  $\text{Irr}(A)$  onto  $\text{Prim}(A)$ , the Jacobson topology on  $\text{Prim}(A)$  can be carried over to  $\text{Irr}(A)$ . (In our case, we will actually have  $\text{Irr}(A) \cong \text{Prim}(A)$ , since  $A$  is a type I  $C^*$ -algebra.)

In exactly the same way, we can also define  $\mathcal{O}(\tilde{A})$  and  $\text{Irr}(\tilde{A})$  together with their respective topological structures. We already know that  $\mathcal{O}(A) \cong \text{Irr}(A)$  and  $\mathcal{O}(\tilde{A}) \cong \text{Irr}(\tilde{A})$  as sets. Let us now explore these correspondences a little further, in terms of their respective topologies.

For ordinary Lie groups, the question of establishing a topological homeomorphism between the (coadjoint) orbit space  $\mathcal{O}(G)$  and the representation space  $\text{Irr}(C^*(G))$  is called by some authors as the ‘‘Kirillov conjecture’’. It is certainly well known to be true in the case of nilpotent Lie groups. It is also true in the case of exponential solvable Lie groups. (This was established rather recently [19].)

In our case, the objects of our study are Hopf  $C^*$ -algebras (quantum groups), and we consider dressing orbits instead of coadjoint orbits. On the other hand, their classical limits are exponential solvable Lie groups. Because of this, for the proof of  $\text{Irr}(\tilde{A}) \cong \mathcal{O}(\tilde{A})$  and  $\text{Irr}(A) \cong \mathcal{O}(A)$ , it is possible to take advantage of the general result at the Lie group setting. Here is the main result, which is not really surprising:

**THEOREM 2.3.**

- (1)  $\text{Irr}(\tilde{A}) \cong \mathcal{O}(\tilde{A})$ , as topological spaces.
- (2)  $\text{Irr}(A) \cong \mathcal{O}(A)$ , as topological spaces.

*Proof.* If we look at the  $*$ -representations in  $\text{Irr}(\tilde{A})$  computed earlier, we can see that they closely resemble the  $*$ -representations in  $\text{Irr}(C^*(\tilde{H}))$ , the equivalence classes of unitary group representations of the (exponential) Lie group  $\tilde{H}$ . It actually boils down to replacing ‘‘ $\eta_\lambda(r)$ ’’ with ‘‘ $r$ ’’. The two sets are certainly different, but noting that  $r \mapsto \eta_\lambda(r) (= (e^{2\lambda r} - 1)/(2\lambda))$  is one-to-one and onto, it is clear that the topological structures on  $\text{Irr}(\tilde{A})$  and

on  $\text{Irr}(C^*(\tilde{H}))$  are the same. A similar comment holds for the dressing orbit space  $\mathcal{O}(\tilde{A})$  and the coadjoint orbit space  $\mathcal{O}(\tilde{H})$ .

By the general theory of representations of exponential Lie groups (see Section 3 of [19]), we have  $\mathcal{O}(\tilde{H}) \cong \text{Irr}(C^*(\tilde{H}))$ . Therefore, it follows from the observation in the previous paragraph that  $\mathcal{O}(\tilde{A}) \cong \text{Irr}(\tilde{A})$ . The proof for the homeomorphism  $\mathcal{O}(A) \cong \text{Irr}(A)$  can be carried out in exactly the same way, this time considering the (nilpotent) Lie group  $H$ .  $\square$

This is the general result we wanted to establish. But one drawback of the above proof is that it is rather difficult to see what is actually going on. To illustrate and for possible future use, we collect below a few specific results (with proofs given by direct computations) showing the topological properties of  $\text{Irr}(\tilde{A})$ . We do not mention the case of  $A$  here, but it would be obviously simpler.

Let us begin by describing the quotient topology on  $\mathcal{O}(\tilde{A})$ . Consider the ‘‘points’’ (i.e., orbits) in  $\mathcal{O}(\tilde{A})$ . For  $r \neq 0$ , the topology on the set of the points  $\tilde{\mathcal{O}}_{r,s}$  is the standard one, which is just the topology on the  $(r, s)$  plane excluding the  $r$ -axis. When  $r = 0$  and  $(p, q) = (0, 0)$ , in which case the points consist of the orbits  $\tilde{\mathcal{O}}_s$ , the topology is exactly the standard topology on a line (the  $s$ -axis). It is non-standard in the case when  $r = 0$  and  $(p, q) \neq (0, 0)$ , where the points consist of the orbits  $\tilde{\mathcal{O}}_{p,q}$ . To visualize, the picture (Figure 1) given at the end of §1.1 will be helpful here.

Let us now turn our attention to  $\text{Irr}(\tilde{A})$ . By Theorem 2.3, we already know that the topology on it coincides with the quotient topology on  $\mathcal{O}(\tilde{A})$ , under the identification of the two sets via our one-to-one correspondence. In the following three propositions, we give direct proofs of a few selected situations that show some non-standard topological behavior, e.g., when  $r = 0$  and  $(p, q) \neq (0, 0)$ . The notation for the representations in  $\text{Irr}(\tilde{A})$  are as before.

**PROPOSITION 2.4.** *Consider the sequence of representations  $\{\tilde{\pi}_{p,q}\}$ , letting  $(p, q)$  approach  $(0, 0)$ . Then the limit points of the sequence are the representations  $\tilde{\pi}_s$ .*

*Proof.* As  $(p, q) \rightarrow (0, 0)$ , the  $\tilde{\pi}_{p,q}$  approach the (reducible) representation  $S$ , which acts on the Hilbert space  $L^2(\mathbb{R})$  and is defined by

$$S(f)\zeta(d) = \int f(x, y, 0, w)\zeta(d+w) dx dy dw.$$

To see how this representation  $S$  decomposes, consider the unitary map (Fourier transform) on  $L^2(\mathbb{R})$  given by

$$\mathcal{F}\zeta(s) = \int \zeta(\tilde{d})\bar{e}(s\tilde{d}) d\tilde{d}.$$

Using  $\mathcal{F}$ , we can define the representation  $\tilde{S}$  which is equivalent to  $S$ . By a straightforward calculation using the Fourier inversion theorem we have

$$\tilde{S}(f)\zeta(s) = \mathcal{F}S(f)\mathcal{F}^{-1}\zeta(s) = \int f(x, y, 0, w)e(sw)\zeta(s) dx dy dw.$$

We can see that  $\tilde{S}$  is the direct integral of the irreducible representations  $\tilde{\pi}_s$ . In other words,

$$S \cong \tilde{S} = \int^{\oplus} \tilde{\pi}_s ds.$$

Therefore,  $\text{Ker } S \subseteq \bigcap_s \text{Ker } \tilde{\pi}_s$ . It follows that all the  $\tilde{\pi}_s$  are limit points of the sequence  $\{\tilde{\pi}_{p,q}\}_{(p,q) \rightarrow (0,0)}$ , under the topology on  $\text{Irr}(\tilde{A})$ . In view of the result of Theorem 2.3 that  $\text{Irr}(\tilde{A}) \cong \mathcal{O}(\tilde{A})$ , they will exhaust all the limit points.  $\square$

**PROPOSITION 2.5.** *Consider the sequence of representations  $\{\tilde{\pi}_{p,q}\}$ , letting  $q \rightarrow 0$  while holding  $p$  fixed. More specifically, consider the sequence  $\{\tilde{\pi}_{p,c_n q}\}$ , where  $p$  and  $q$  are fixed and  $\{c_n\}$  is a sequence of positive numbers approaching 0. Then the limit points are the representations  $\tilde{\pi}_{p,0}$  and  $\tilde{\pi}_{0,q}$ .*

*Proof.* It is clear that  $\tilde{\pi}_{p,0}$  is a limit point of the sequence  $\{\tilde{\pi}_{p,c_n q}\}$ . Meanwhile, note that  $\tilde{\pi}_{p',q'}$  is equivalent to  $\tilde{\pi}_{e^{l}p',e^{-l}q'}$  (see the Remark following Proposition 2.2). Let us choose a sequence of real numbers  $\{l_n\}$  such that  $e^{-l_n}c_n = 1$ . Then each  $\tilde{\pi}_{p,c_n q}$  is equivalent to  $\tilde{\pi}_{e^{l_n}p,q}$ . Since the  $e^{l_n} = c_n$  obviously approach 0, we conclude that  $\tilde{\pi}_{0,q}$  is also a limit point of the sequence  $\{\tilde{\pi}_{p,c_n q}\}$ .  $\square$

**REMARK.** Similarly, we may consider the sequence  $\{\tilde{\pi}_{c_n p,q}\}$ , where  $p$  and  $q$  are fixed and  $\{c_n\}$  is a sequence of positive numbers approaching 0. In exactly the same way as above, we can show that the limit points are the representations  $\tilde{\pi}_{p,0}$  and  $\tilde{\pi}_{0,q}$ . By modifying the proof a little, we can also obtain various results of similar flavor.

**PROPOSITION 2.6.** *Consider the sequence of representations  $\{\tilde{\pi}_{r,s}\}$ , letting  $r \rightarrow 0$  while  $s$  is fixed. Then all representations  $\tilde{\pi}_{p,q}$  and  $\tilde{\pi}_s$  are limit points of the sequence.*

*Proof.* As  $r \rightarrow 0$ , the  $\tilde{\pi}_{r,s}$  approach the following (reducible) representation,  $T$ , acting on the Hilbert space  $L^2(\mathbb{R}^n)$ :

$$T(f)\xi(u) = \int f(x, y, 0, w)\bar{e}(sw)(e^{-w/2})^n \xi(e^{-w}u + e^{-w}x) dx dy dw.$$

To see how  $T$  decomposes, consider the Fourier transform on  $L^2(\mathbb{R}^n)$ ,

$$\mathcal{F}\xi(\alpha) = \int \xi(u)\bar{e}(\alpha \cdot u) du.$$

As before, we can define the representation  $\tilde{T}$  which is equivalent to  $T$ . By a straightforward calculation involving the Fourier inversion theorem, we have

$$\begin{aligned}\tilde{T}(f)\xi(\alpha) &= \mathcal{F}T(f)\mathcal{F}^{-1}\xi(\alpha) \\ &= \int f(x, y, 0, w)e(\alpha \cdot x)\bar{e}(sw)(e^{w/2})^n \xi(e^w \alpha) dx dy dw.\end{aligned}$$

If we express  $\alpha$  as  $\alpha = e^d p$ , for some  $p \in \mathbb{R}^n$  and  $d \in \mathbb{R}$ , then this becomes

$$\tilde{T}(f)\xi(e^d p) = \int f(x, y, 0, w)e(e^d p \cdot x)\bar{e}(sw)(e^{w/2})^n \xi(e^{d+w} p) dx dy dw.$$

Fix  $p (\neq 0)$ , and let us write  $\xi_p(d) := (e^{d/2})^n \xi(e^d p)$ . Then we have

$$\tilde{T}(f)\xi_p(d) = \int f(x, y, 0, w)e(e^d p \cdot x)\bar{e}(sw)\xi_p(d+w) dx dy dw.$$

This is really the expression for the inner tensor product representation,  $\tilde{\pi}_{-p,0} \boxtimes \tilde{\pi}_s$ , which is equivalent to  $\tilde{\pi}_{-p,0}$  (the proof is by straightforward calculation, similar to the one given in Proposition 4.4 of [11]). As in the proof of Proposition 2.4, we thus have  $\text{Ker } T \subseteq \bigcap_p \text{Ker } \tilde{\pi}_{p,0}$ . It follows that all the  $\tilde{\pi}_{p,0}$  are limit points of the sequence  $\{\tilde{\pi}_{r,s}\}_{r \rightarrow 0}$ , under the topology on  $\text{Irr}(\tilde{A})$ . By the result of Proposition 2.4, it also follows that all the  $\tilde{\pi}_s$  are limit points of the sequence.

Meanwhile, to look for more limit points of the sequence, let us define the representations  $Q_{r,s}$ , which are equivalent to the  $\tilde{\pi}_{r,s}$ , as follows. We consider  $\tilde{\pi}_{r,s}$  and consider the Hilbert space  $L^2(\mathbb{R}^n)$  on which  $\tilde{\pi}_{r,s}(\tilde{A})$  acts. In  $L^2(\mathbb{R}^n)$ , define the unitary map  $\mathcal{F}_r$  by

$$\mathcal{F}_r \xi(v) = \int \xi(u)\bar{e}(\eta_\lambda(r)u \cdot v) \|\eta_\lambda(r)\|^{1/2} du.$$

This is again a kind of Fourier transform, taking advantage of the existence of the bilinear form  $(u, v) \mapsto \eta_\lambda(r)u \cdot v$  in  $\mathbb{R}^n$ . Its inverse is given by

$$\mathcal{F}_r^{-1} \xi(u) = \int \xi(v)e(\eta_\lambda(r)u \cdot v) \|\eta_\lambda(r)\|^{1/2} dv.$$

We define the representation  $Q_{r,s}$  by  $Q_{r,s}(f)\xi := \mathcal{F}_r \tilde{\pi}_{r,s}(f)\mathcal{F}_r^{-1}\xi$ . We then have

$$\begin{aligned}Q_{r,s}(f)\xi(v) &= \int f(x, y, r, w)\bar{e}(sw)e[\eta_\lambda(r)(v+y) \cdot x] \\ &\quad \cdot (e^{w/2})^n \xi(e^w v + e^w y) dx dy dw.\end{aligned}$$

Since the  $Q_{r,s}$  are equivalent to the  $\tilde{\pi}_{r,s}$ , we may now consider the sequence  $\{Q_{r,s}\}_{r \rightarrow 0}$ . We can see right away that as  $r \rightarrow 0$ , the sequence approaches the following (reducible) representation,  $Q$ , acting on  $L^2(\mathbb{R}^n)$ :

$$Q(f)\xi(v) = \int f(x, y, 0, w)\bar{e}(sw)(e^{w/2})^n \xi(e^w v + e^w y) dx dy dw.$$

We can follow exactly the same method we used in the case of the representation  $T$  to show that we now have  $\text{Ker } Q \subseteq \bigcap_q \text{Ker } \tilde{\pi}_{0,q}$ . It follows that all the  $\tilde{\pi}_{0,q}$  are limit points of the sequence  $\{Q_{r,s}\}_{r \rightarrow 0}$ , or equivalently the sequence  $\{\tilde{\pi}_{r,s}\}_{r \rightarrow 0}$ .

We have shown so far that the representations  $\tilde{\pi}_{p,0}$ , the  $\tilde{\pi}_{0,q}$ , as well as the  $\tilde{\pi}_s$ , are limit points of the sequence. Although it is somewhat cumbersome, by choosing some suitable realizations of the representations  $\tilde{\pi}_{r,s}$  it is possible to show that all representations  $\tilde{\pi}_{p,q}$  are also limit points.  $\square$

Observe that the topological behavior of  $\text{Irr}(\tilde{A})$  manifested in the above three propositions is exactly that of the quotient topology on  $\mathcal{O}(\tilde{A})$ , as was to be expected from Theorem 2.3. On the other hand, it is not quite sufficient to claim only from these types of propositions that the one-to-one correspondences  $\text{Irr}(\tilde{A}) \cong \mathcal{O}(\tilde{A})$  and  $\text{Irr}(A) \cong \mathcal{O}(A)$  are topological homeomorphisms. For this reason, the actual proof was given indirectly.

These results confirm that there is a strong analogy between our “quantum” case and the “classical” case of ordinary groups. This is the underlying theme of this article. On the other hand, in [11] we discuss an interesting type of “quantum” behavior enjoyed by the  $*$ -representations (e.g., the *quasi-triangular* property), due to the role played by the comultiplications on  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$ .

### 3. Canonical measure on an orbit

We will work mostly with the dressing orbits in  $\mathcal{O}(\tilde{A})$ . The case of orbits in  $\mathcal{O}(A)$  will be simpler.

Recall first the (left) dressing action,  $\delta$ , of  $\tilde{H}$  on  $\tilde{G}$ , as defined in Proposition 1.3. By our identification of  $\tilde{G}$  with its Lie algebra  $\tilde{\mathfrak{g}}$ , the dressing action  $\delta$  can be viewed as an action of  $\tilde{H}$  on  $\tilde{\mathfrak{g}}$ . It has a corresponding Lie algebra representation,  $d\delta : \tilde{\mathfrak{h}} \rightarrow \text{End}(\tilde{\mathfrak{g}})$ , defined by

$$(d\delta(X))(\mu) = \left. \frac{d}{dt} \right|_{t=0} \delta(tX)(\mu).$$

From now on, let us fix a dressing orbit  $\mathcal{O}$  in  $\tilde{\mathfrak{g}}$ . Let  $\mu \in \mathcal{O}$  be a typical element in the orbit (so  $\mathcal{O} = \delta(\tilde{H})(\mu)$ ). We define the stabilizer subgroup by

$$R_\mu = \{x \in \tilde{H} : (\delta(x))\mu = \mu\} \subseteq \tilde{H}.$$

The corresponding Lie subalgebra is

$$\mathfrak{r}_\mu = \{X \in \tilde{\mathfrak{h}} : (d\delta(X))\mu = 0\} \subseteq \tilde{\mathfrak{h}}.$$

Then the map  $\Psi^\mu : x \mapsto \delta(x)\mu$  induces a diffeomorphism  $\tilde{H}/R_\mu \cong \mathcal{O}$ . The range of the differential map  $d\Psi^\mu = (d\Psi^\mu)_e : \tilde{\mathfrak{h}} \rightarrow (T\mathcal{O})_\mu$  gives us the tangent space  $(T\mathcal{O})_\mu$  at  $\mu$ . Since  $d\Psi^\mu(\tilde{\mathfrak{h}}) = d\delta(\tilde{\mathfrak{h}})(\mu)$ , we have  $\text{Ker}(d\Psi^\mu) = \mathfrak{r}_\mu$  and

$(T\mathcal{O})_\mu = \mathfrak{r}_\mu^\perp$ . And we have the diffeomorphism  $\tilde{\mathfrak{h}}/\mathfrak{r}_\mu \cong (T\mathcal{O})_\mu$ . All this is more or less the same as in the case of the coadjoint orbits (see [4]).

Meanwhile, recall that  $\mathcal{O}$  is a symplectic leaf, whose symplectic structure is given by restricting the Poisson bracket on  $\tilde{G}$ . In this way, we obtain a (non-degenerate) skew 2-form  $\omega^\mu \in \Lambda^2(T\mathcal{O})_\mu$ , such that for  $X, Y \in \tilde{\mathfrak{h}}$  we have

$$(3.1) \quad \begin{aligned} \omega^\mu(d\Psi^\mu(X), d\Psi^\mu(Y)) \\ = p \cdot (wx' - w'x) + q \cdot (w'y - wy') + \eta_\lambda(r)(x \cdot y' - x' \cdot y). \end{aligned}$$

Here  $\mu = (p, q, r, s)$  and  $X = (x, y, z, w)$ ,  $Y = (x', y', z', w')$ . The map  $\omega^\mu$  is easily shown (by direct computation) to be well-defined.

LEMMA 3.1. *For fixed  $\mu$ , the above map  $\omega^\mu \in \Lambda^2(T\mathcal{O})_\mu$  is well-defined.*

*Proof.* Suppose  $X' \in \tilde{\mathfrak{h}}$  is an arbitrary element such that  $d\Psi^\mu(X') = d\Psi^\mu(X)$ . Then we can write  $X' = X + (a, b, c, d)$ , for  $(a, b, c, d) \in \mathfrak{r}_\mu$ . Recalling the definition of  $\delta$  (from Proposition 1.3), we compute

$$(d\delta(a, b, c, d))(\mu) = (-dp + \eta_\lambda(r)b, dq - \eta_\lambda(r)a, 0, p \cdot a - q \cdot b),$$

where  $\mu = (p, q, r, s)$ . So  $(a, b, c, d) \in \mathfrak{r}_\mu$  is characterized by

$$dp = \eta_\lambda(r)b, \quad dq = \eta_\lambda(r)a, \quad p \cdot a - q \cdot b = 0.$$

From this and from the definition of  $\omega^\mu$  given by equation (3.1), we see easily that for any  $Y \in \tilde{\mathfrak{h}}$  we have

$$\omega^\mu(d\Psi^\mu(X'), d\Psi^\mu(Y)) = \omega^\mu(d\Psi^\mu(X), d\Psi^\mu(Y)).$$

A similar argument can be made for the second entry, and we conclude that  $\omega^\mu$  is well-defined.  $\square$

We can define  $\omega^\mu$  for each  $\mu \in \mathcal{O}$ . But we can show that  $\omega : \mu \mapsto \omega^\mu$  is  $\delta(\tilde{H})$ -invariant. To illustrate this more clearly, consider an arbitrary element  $\nu$  of  $\mathcal{O}$ . Suppose it is written as  $\nu = \delta(h)\mu$ , for some  $h \in \tilde{H}$ . The following results are true.

LEMMA 3.2. *Let the notation be as above. Define  $\alpha_h : \tilde{H} \rightarrow \tilde{H}$  by  $\alpha_h(x) = h x h^{-1}$ . Then we have*

$$\delta(h) \circ \Psi^\mu = \Psi^\nu \circ \alpha_h.$$

*Proof.* For any  $x \in \tilde{H}$ ,

$$\begin{aligned} \Psi^\nu(\alpha_h(x)) &= \Psi^\nu(h x h^{-1}) = \delta(h x h^{-1})(\nu) && \text{(by definition)} \\ &= \delta(h x) \delta(h^{-1})(\nu) && (\delta \text{ is a left action}) \\ &= \delta(h x)(\mu) && (\nu = \delta(h)\mu \text{ or } \delta(h^{-1})\nu = \mu) \\ &= \delta(h) \delta(x)(\mu) = \delta(h) \Psi^\mu(x). && \square \end{aligned}$$

PROPOSITION 3.3. *Let  $\phi_h := d(\delta(h))_\mu : (T\mathcal{O})_\mu \rightarrow (T\mathcal{O})_\nu$ . We then have, for  $X, Y \in \tilde{\mathfrak{h}}$ ,*

$$\omega^\nu(\phi_h \circ d\Psi^\mu(X), \phi_h \circ d\Psi^\mu(Y)) = \omega^\mu(d\Psi^\mu(X), d\Psi^\mu(Y)).$$

*This shows that  $\omega : \nu \mapsto \omega^\nu$  is  $\delta(\tilde{H})$ -invariant.*

*Proof.* From the previous lemma we know that  $\delta(h) \circ \Psi^\mu = \Psi^\nu \circ \alpha_h$ . By taking differentials, it follows that

$$\phi_h \circ d\Psi^\mu = d\Psi^\nu \circ d(\alpha_h).$$

For convenience we wrote  $d(\alpha_h) = (d(\alpha_h))_e$ . From this we have

$$\omega^\nu(\phi_h \circ d\Psi^\mu(X), \phi_h \circ d\Psi^\mu(Y)) = \omega^\nu(d\Psi^\nu \circ d(\alpha_h)(X), d\Psi^\nu \circ d(\alpha_h)(Y)).$$

To see that this is same as  $\omega^\mu(d\Psi^\mu(X), d\Psi^\mu(Y))$ , we will use direct computation. So let  $h = (a, b, c, d)$  and compute  $\alpha_h$ , using the multiplication law on  $\tilde{H}$  as defined earlier (in §1.1):

$$\begin{aligned} \alpha_h(x, y, z, w) &= (a, b, c, d)(x, y, z, w)(a, b, c, d)^{-1} \\ &= (a + e^d x, b + e^{-d} y, c + z + e^{-d} a \cdot y, d + w)(-e^{-d} a, -e^d b, -c + a \cdot b, -d) \\ &= (a + e^d x - e^w a, b + e^{-d} y - e^{-w} b, z + a \cdot b \\ &\quad + e^{-d} a \cdot y - e^{d-w} x \cdot b - e^{-w} a \cdot b, w). \end{aligned}$$

It follows that for  $X = (x, y, z, w) \in \tilde{\mathfrak{h}}$  we have

$$(d(\alpha_h))(X) = (e^d x - wa, e^{-d} y + wb, z + e^{-d} a \cdot y - e^d x \cdot b + wa \cdot b, w).$$

Meanwhile, let us write  $\mu = (p, q, r, s)$ . Then by definition,

$$\begin{aligned} \nu = \delta(h)\mu &= (\delta(a, b, c, d))(p, q, r, s) \\ &= (e^{-d} p + \eta_\lambda(r)b, e^d q - \eta_\lambda(r)a, r, s + e^{-d} p \cdot a - e^d q \cdot b + \eta_\lambda(r)a \cdot b). \end{aligned}$$

Therefore, by using the definition of  $\omega$  as given in (3.1) and by direct computation using the expressions we obtained above, we see that for  $X = (x, y, z, w)$  and  $Y = (x', y', z', w')$ ,

$$\begin{aligned} &\omega^\nu(d\Psi^\nu(d(\alpha_h)(X)), d\Psi^\nu(d(\alpha_h)(Y))) \\ &= (e^{-d} p + \eta_\lambda(r)b) \cdot [w(e^d x' - w'a) - w'(e^d x - wa)] \\ &\quad + (e^d q - \eta_\lambda(r)a) \cdot [w'(e^{-d} y + wb) - w(e^{-d} y' + w'b)] \\ &\quad + \eta_\lambda(r)[(e^d x - wa) \cdot (e^{-d} y' + w'b) - (e^d x' - w'a) \cdot (e^{-d} y + wb)] \\ &= p \cdot (wx' - w'x) + q \cdot (w'y - wy') + \eta_\lambda(r)(x \cdot y' - x' \cdot y) \\ &= \omega^\mu(d\Psi^\mu(X), d\Psi^\mu(Y)). \end{aligned}$$

Together with the result obtained in the first part, we conclude our proof.  $\square$

The invariance of  $\omega$  means that it is a  $C^\infty$  2-form on  $\mathcal{O}$ , in its unique symplectic manifold structure inherited from the Poisson bracket on  $\tilde{G}$ .

Our plan (to be carried out in the next section) is to construct a deformed product on a dense subspace of  $C^\infty(\mathcal{O})$ , by realizing the functions as operators on a Hilbert space. In the remainder of this section we make preparations at the level of orbits. Let us begin by pointing out that we can identify  $\tilde{H}/R_\mu$  with the vector space  $V = \tilde{\mathfrak{h}}/\mathfrak{r}_\mu$ . This means that we are regarding  $\mathcal{O} = (T\mathcal{O})_\mu$ ; see the following remark.

REMARK. In general, we do not know if  $R_\mu \cong \mathfrak{r}_\mu$  (we may have  $R_\mu$  not connected). But in our case, if we just follow the definition, it is not difficult to see that they can be identified. So we will regard  $R_\mu = \mathfrak{r}_\mu$ , for every  $\mu$ . Together with our earlier (spatial) identification  $\tilde{H} = \tilde{\mathfrak{h}}$ , we thus see that  $\tilde{H}/R_\mu = \tilde{\mathfrak{h}}/\mathfrak{r}_\mu$  as vector spaces. The advantage of having a description of  $\mathcal{O}(\cong \tilde{H}/R_\mu)$  as a vector space is clear. We can use various linear algebraic tools, as well as Fourier transforms.

Let us turn our attention to the vector space  $V = \tilde{H}/R_\mu = \tilde{\mathfrak{h}}/\mathfrak{r}_\mu$ . By the diffeomorphism  $\tilde{\mathfrak{h}}/\mathfrak{r}_\mu \cong (T\mathcal{O})_\mu$ , it is equipped with the skew, bilinear form  $B^\mu$ , defined by

$$B^\mu(\dot{X}, \dot{Y}) := \omega^\mu(d\Psi^\mu(X), d\Psi^\mu(Y)),$$

where  $X$  and  $Y$  are the representatives in  $\tilde{\mathfrak{h}}$  of the classes  $\dot{X}, \dot{Y} \in V$ . By the following lemma, we can thus construct a unique “self-dual” measure on  $V$ .

LEMMA 3.4.

- (1) *Let  $V$  be a (real) vector space, and let  $dm$  be a Euclidean measure on  $V$ . Suppose  $B(\cdot, \cdot)$  is a nondegenerate bilinear form on  $V$ . Then we may use  $B$  to identify  $V$  with its dual space  $V^*$ , and can define a Fourier transform  $\mathcal{F}_B$  by*

$$(\mathcal{F}_B f)(v) = \int_V f(v') e^{-2\pi i B(v, v')} dm(v').$$

*We say that  $dm$  is “self-dual” if  $\|\mathcal{F}_B f\|_2 = \|f\|_2$ , where  $\|f\|_2^2 = \int |f(v)|^2 dm(v)$ .*

- (2) *Suppose  $\{e_1, e_2, \dots, e_k\}$  is a basis of  $V$  and let  $dv$  be the normalized Lebesgue measure on  $V$ . If we denote by the same letter  $B$  the matrix such that  $b_{ij} = B(e_i, e_j)$ , then the self-dual measure on  $V$  determined by  $B$  is:  $dm = |\det B|^{1/2} dv$ .*

*Proof.* Linear algebra. □

In our case, on  $V$ , we have  $dm = |\det(B^\mu)|^{1/2} d\dot{X}$ , where  $d\dot{X}$  is the measure inherited from the Lebesgue measure,  $dx dy dz dw$ , on  $\tilde{\mathfrak{h}}$ . Meanwhile, recall that  $(T\mathcal{O})_\mu = \mathfrak{r}_\mu^\perp$ . So we may regard  $(T\mathcal{O})_\mu$  as the dual vector space of  $V = \tilde{\mathfrak{h}}/\mathfrak{r}_\mu$ .



Let us give it the dual measure,  $d\theta$ , of  $dm$ . Since  $(T\mathcal{O})_\mu$  is considered as a subspace  $\mathfrak{r}_\mu^\perp$  of  $\tilde{\mathfrak{g}}$ , there already exists a natural measure on it, denoted by  $dl$ , inherited from the Plancherel Lebesgue measure,  $d\mu d\nu d\mathfrak{r} ds$ , on  $\tilde{\mathfrak{g}}(= \tilde{\mathfrak{h}}^*)$ . We will have  $d\theta = |\det(B^\mu)|^{-1/2} dl$ .

Since  $dm$  and  $d\theta$  have been chosen by using the (unique) symplectic structure on  $\mathcal{O}$ , we know that they will be the canonical measures on  $V(= \tilde{\mathfrak{h}}/\mathfrak{r}_\mu)$  and  $\mathcal{O}(= (T\mathcal{O})_\mu)$ , respectively. In terms of these canonical measures, there exists the ‘‘symplectic Fourier transform’’,  $\mathcal{F}_\omega$ , from  $S_c(V)$  to  $S_c(\mathcal{O})$ , as well as its inverse. That is,

$$(\mathcal{F}_\omega f)(l) = \int_{\tilde{\mathfrak{h}}/\mathfrak{r}_\mu} f(\dot{X}) \bar{e}[\langle l, \dot{X} \rangle] dm(\dot{X}), \quad f \in S_c(V),$$

and

$$(\mathcal{F}_\omega^{-1} f)(\dot{X}) = \int_{(T\mathcal{O})_\mu} f(l) e[\langle l, \dot{X} \rangle] d\theta(l), \quad f \in S_c(\mathcal{O}).$$

The notation  $S_c(V)$  means the space of Schwartz functions on  $V$  having compact support. So  $S_c(V) \subseteq C_c^\infty(V)$ . We can actually work in spaces which are a little larger than  $S_c(V)$  and  $S_c(\mathcal{O})$ , but they are good enough for our present purposes.

Let us consider the Hilbert space  $L^2(V, dm)$ , which contains  $S_c(V)$  as a dense subspace. By the symplectic Fourier transform, we see that  $L^2(V, dm) \cong L^2(\mathcal{O}, d\theta)$ . Due to the canonical nature of our construction, it is rather easy to see that the definition of  $L^2(\mathcal{O}, d\theta)$  obtained in this way does not really depend on the choice of the representative  $\mu \in \mathcal{O}$ . Meanwhile, even though we did not explicitly mention the case of orbits in  $\mathcal{O}(A)$ , it is obvious that everything we have been discussing in this section can be carried out in exactly the same way. (All we need to do is to let  $w$  (or  $d$ ) and  $s$  variables to be zero.)

#### 4. Deformation of the orbits

Since a typical dressing orbit  $\mathcal{O}$  is a symplectic manifold, one hopes that there would be a way to define a deformed product on  $C^\infty(\mathcal{O})$ , in the spirit of Weyl quantization and Moyal products. Usually, this kind of deformation quantization is done in terms of  $*$ -products, involving formal power series [3], [27]. Indeed, Arnal and Cortet ([1], [2]) have shown that for nilpotent or some exponential solvable Lie groups there exist such quantizations on coadjoint orbits (again via  $*$ -products).

Here, we wish to achieve a similar goal of defining a Moyal-type deformed product, but without resorting to formal power series. We will define our deformed product on a dense subspace  $S_c(\mathcal{O})$  of  $C^\infty(\mathcal{O})$ , by using an operator algebra realization on a Hilbert space. The strategy is to relate the deformation with the irreducible  $*$ -representation corresponding to the given orbit. In this article, we plan to discuss only the case of  $\mathcal{O}(A)$  in relation with  $\text{Irr}(A)$ .

The case of  $\mathcal{O}(\tilde{A})$  and  $\text{Irr}(\tilde{A})$  will be our future project (but see the Appendix following this section).

Let us begin our discussion by turning our attention to the irreducible \*-representations. The next proposition is very crucial.

**PROPOSITION 4.1.** *Let  $\pi \in \text{Irr}(A)$  be an irreducible representation of  $A$  acting on  $\mathcal{H}_\pi$  (see Proposition 2.1 for the classification). Let  $\mathcal{O} \in \mathcal{O}(A)$  be the corresponding orbit. Then for  $f \in \mathcal{A}$  the operator  $\pi(f) \in \mathcal{B}(\mathcal{H}_\pi)$  turns out to be a trace-class operator. Furthermore,*

$$(4.1) \quad \text{Tr}(\pi(f)) = \int_{\mathcal{O}} \hat{f}|_{\mathcal{O}}(\nu) d\theta(\nu),$$

where  $d\theta$  is the canonical measure on  $\mathcal{O}$  defined earlier. And  $\hat{f}$  denotes the partial Fourier transform of  $f$  defined by

$$\hat{f}(p', q', r') = \int f(\tilde{x}, \tilde{y}, r') \bar{e}(p' \cdot \tilde{x} + q' \cdot \tilde{y}) d\tilde{x}d\tilde{y}.$$

**REMARK.** The partial Fourier transform  $f \mapsto \hat{f}$  is the usual one (with respect to the Lebesgue measures), not to be confused with the symplectic Fourier transform, which appeared in the previous section. This can be done without trouble, since we are (spatially) identifying  $\mathcal{O}(= (T\mathcal{O})_\mu)$  with the subspace  $\mathfrak{t}_\mu^\perp$  of  $\mathfrak{g}$ .

*Proof.* *Case of  $\pi_{p,q}$ :* This is a trivial case. Since

$$\pi_{p,q}(f) = \int f(x, y, 0) \bar{e}(p \cdot x + q \cdot y) dx dy = \hat{f}|_{\mathcal{O}_{p,q}}(p, q, 0) \in \mathbb{C},$$

we have  $\text{Tr}(\pi_{p,q}(f)) = \hat{f}|_{\mathcal{O}_{p,q}}(p, q, 0)$ .

*Case of  $\pi_r$ :* Note that, by the Fourier inversion theorem,

$$\begin{aligned} (\pi_r(f))\xi(u) &= \int f(x, y, r) \bar{e}[\eta_\lambda(r)u \cdot y] \xi(u + x) dx dy \\ &= \int \hat{f}|_{\mathcal{O}_r}(\tilde{p}, \eta_\lambda(r)u, r) \xi(u + \tilde{x}) e(\tilde{p} \cdot \tilde{x}) d\tilde{p}d\tilde{x} \\ &= \int K(u, \tilde{x}) \xi(\tilde{x}) d\tilde{x}, \end{aligned}$$

where  $K(u, \tilde{x}) = \int \hat{f}|_{\mathcal{O}_r}(\tilde{p}, \eta_\lambda(r)u, r) e[\tilde{p} \cdot (\tilde{x} - u)] d\tilde{p}$ . That is,  $\pi_r(f)$  is an integral operator whose kernel is given by  $K$ , which is clearly an  $L^2$ -function since  $\hat{f}$  is a Schwartz function. This means that  $\pi_r(f)$  is a trace-class operator.

Moreover,

$$\begin{aligned} \mathrm{Tr}(\pi_r(f)) &= \int K(u, u) du = \int \hat{f}|_{\mathcal{O}_r}(\tilde{p}, \eta_\lambda(r)u, r) d\tilde{p}du \\ &= \int \hat{f}|_{\mathcal{O}_r}(\tilde{p}, u, r) \frac{1}{|\eta_\lambda(r)|^n} d\tilde{p}du. \end{aligned}$$

But the measure  $|\eta_\lambda(r)|^{-n} d\tilde{p}du$  is none other than the canonical measure  $d\theta(\tilde{p}, u, r)$  on  $\mathcal{O}_r$ . To see this more clearly, recall the definition of the bilinear map  $B^\mu$  on  $\mathfrak{h}/\mathfrak{t}_\mu \cong \mathcal{O}$ . In our case, we may choose  $\mu = (0, 0, r)$  and  $\mathfrak{h}/\mathfrak{t}_\mu = \mathfrak{h}/\mathfrak{z}$ . We then have

$$B^\mu(\dot{X}, \dot{Y}) = \eta_\lambda(r)(x \cdot y' - x' \cdot y),$$

where  $\dot{X}$  and  $\dot{Y}$  are the classes in  $\mathfrak{h}/\mathfrak{z}$  represented by  $X = (x, y, 0)$  and  $Y = (x', y', 0)$ . By a simple calculation we have  $\det(B^\mu) = (\eta_\lambda(r))^{2n}$ . From this it follows from the discussion following Lemma 3.4 that the canonical measure on  $\mathcal{O}_r$  is

$$d\theta(p', q', r) = |\det(B^\mu)|^{-1/2} dp' dq' = |\eta_\lambda(r)|^{-n} dp' dq'.$$

This verifies the trace formula, equation (4.1).  $\square$

**COROLLARY.** *Let  $\pi \in \mathrm{Irr}(A)$ . Since each  $\pi(f)$ ,  $f \in \mathcal{A}$ , is a trace-class operator, it is also Hilbert–Schmidt. In our case, the Hilbert–Schmidt norms are given by*

$$\begin{aligned} \|\pi_{p,q}(f)\|_{\mathrm{HS}}^2 &= |\hat{f}(p, q, 0)|^2, \\ \|\pi_r(f)\|_{\mathrm{HS}}^2 &= \int \overline{\hat{f}(p', q', r)} \hat{f}(p', q', r) |\eta_\lambda(r)|^{-n} dp' dq' = \int |\hat{f}(p', q', r)|^2 d\theta. \end{aligned}$$

Indeed, we actually have, for  $f, g \in \mathcal{A}$ ,

$$\mathrm{Tr}(\pi(g)^* \pi(f)) = \mathrm{Tr}(\pi(g^* \times f)) = \langle \hat{f}|_{\mathcal{O}_r}, \hat{g}|_{\mathcal{O}_r} \rangle_{\mathcal{O}},$$

where  $\times$  is the multiplication on  $\mathcal{A}$  and  $g \mapsto g^*$  is the involution on  $\mathcal{A}$ , while  $\langle \cdot, \cdot \rangle_{\mathcal{O}}$  denotes the inner product on  $L^2(\mathcal{O}, d\theta)$ .

*Proof.* We just need to remember the definitions of the multiplication and involution on  $\mathcal{A}$  (for instance, see Propositions 2.8 and 2.9 of [10]), and use the trace formula obtained in the previous proposition. The result follows by straightforward computation.  $\square$

Since the elements  $\pi(f)$ ,  $f \in \mathcal{A}$ , form a dense subspace in  $HS(\mathcal{H}_\pi)$ , the above Corollary implies that we have a Hilbert space isomorphism between  $HS(\mathcal{H}_\pi)$  and  $L^2(\mathcal{O}, d\theta)$ . To be a little more precise, let us consider the map from  $HS(\mathcal{H}_\pi)$  to  $L^2(\mathcal{O}, d\theta)$  by naturally extending the map  $\pi(f) \mapsto \hat{f}|_{\mathcal{O}}$ . By the result we just obtained, the map is an isometry, preserving the inner product. It is clearly onto. Let us from now on consider its inverse map and

denote it by  $S_\pi$ . In this way, we have the spatial isomorphism  $S_\pi : L^2(\mathcal{O}, d\theta) \cong HS(\mathcal{H}_\pi)$ .

We are now ready to discuss the deformation of the orbits  $\mathcal{O}$ . The point is that through the map  $S_\pi$  an arbitrary element  $\phi \in S_c(\mathcal{O})$  can be considered as a (Hilbert–Schmidt) operator on  $\mathcal{H}_\pi$ . Let us denote this correspondence by  $Q_\pi$ . It is the same as the map  $S_\pi$  above, but now we work with the operator norm on  $\mathcal{B}(\mathcal{H}_\pi)$  instead of the Hilbert–Schmidt norm on  $HS(\mathcal{H}_\pi)$ . The result can be summarized as follows.

**PROPOSITION 4.2.** *Let  $Q_\pi : S_c(\mathcal{O}) \rightarrow \mathcal{B}(\mathcal{H}_\pi)$  be defined as above. Then  $Q_\pi$  determines a deformed product on  $S_c(\mathcal{O})$  by  $Q_\pi(\phi \times_Q \psi) = Q_\pi(\phi)Q_\pi(\psi)$ . The involution on  $S_c(\mathcal{O})$  will be given by  $Q_\pi(\psi^*) = Q_\pi(\psi)^*$ . They are described as follows:*

- (1) (Case of the 1-point orbit  $\mathcal{O}_{p,q}$ ): For  $\phi, \psi \in S_c(\mathcal{O}_{p,q})$ ,

$$\begin{aligned} (\phi \times_Q \psi)(p, q, 0) &= \phi(p, q, 0)\psi(p, q, 0), \\ \psi^*(p, q, 0) &= \overline{\psi(p, q, 0)}. \end{aligned}$$

- (2) (Case of the 2n-dimensional orbit  $\mathcal{O}_r$ ): For  $\phi, \psi \in S_c(\mathcal{O}_r)$ ,

$$\begin{aligned} (\phi \times_Q \psi)(\alpha, \beta, r) &= \int \phi(\tilde{p}, \beta, r)\psi(\alpha, \beta + \eta_\lambda(r)\tilde{x}, r)e[(\tilde{p} - \alpha) \cdot \tilde{x}]d\tilde{p}d\tilde{x}, \\ \psi^*(\alpha, \beta, r) &= \int \overline{\psi(\tilde{p}, \tilde{q}, r)}\bar{e}[(\alpha - \tilde{p}) \cdot x + (\beta - \tilde{q}) \cdot y]\bar{e}[\eta_\lambda(r)x \cdot y]d\tilde{p}d\tilde{q}dx dy. \end{aligned}$$

*Proof.* The computations are straightforward from the definitions, so we will only verify the multiplication formula  $\phi \times_Q \psi$ , for the case of the 2n-dimensional orbit. Indeed, for  $\xi \in \mathcal{B}(L^2(\mathbb{R}^n))$ , we have

$$\begin{aligned} &Q_{\pi_r}(\phi)Q_{\pi_r}(\psi)\xi(u) \\ &= \int \phi(\tilde{p}, \tilde{q}, r)e(\tilde{p} \cdot x + \tilde{q} \cdot y)\bar{e}[\eta_\lambda(r)u \cdot y]Q_{\pi_r}(\psi)\xi(u+x)d\tilde{p}d\tilde{q}dx dy \\ &= \int \phi(\tilde{p}, \eta_\lambda(r)u, r)e(\tilde{p} \cdot x)Q_{\pi_r}(\psi)\xi(u+x)d\tilde{p}dx \\ &= \int \phi(\tilde{p}, \eta_\lambda(r)u, r)e(\tilde{p} \cdot x)\psi(\tilde{p}, \eta_\lambda(r)(u+x), r) \\ &\quad \cdot e(\tilde{p} \cdot x')\xi(u+x+x')d\tilde{p}dx d\tilde{p}dx' \\ &= \int (\phi \times_Q \psi)(\tilde{p}, \eta_\lambda(r)u, r)e(\tilde{p} \cdot x')\xi(u+x')d\tilde{p}dx' \\ &= Q_{\pi_r}(\phi \times_Q \psi)\xi(u), \end{aligned}$$

where

$$(\phi \times_Q \psi)(\tilde{p}, \eta_\lambda(r)u, r) = \int \phi(\tilde{p}, \eta_\lambda(r)u, r) \psi(\tilde{p}, \eta_\lambda(r)(u+x), r) \cdot e[(\tilde{p} - \tilde{p}) \cdot x] d\tilde{p}dx. \quad \square$$

REMARK. Since the  $*$ -algebra structure on  $S_c(\mathcal{O})$  has been defined via a  $*$ -representation, the properties like associativity of the multiplication are immediate. Although we do not plan to describe the actual deformation process, the product on  $S_c(\mathcal{O})$  as obtained above can be shown to be a deformation quantization of the pointwise product on  $S_c(\mathcal{O})$ , in the direction of the symplectic structure (given by  $\omega^\mu$  or  $B^\mu$  in our case) on the orbit  $\mathcal{O}$ : For instance, just as in [23] or in [9], we can replace  $B^\mu(\dot{X}, \dot{Y})$  by  $\frac{1}{\hbar} B^\mu(\hbar\dot{X}, \hbar\dot{Y})$  and proceed, with  $\hbar$  being the deformation parameter.

In this sense, we call  $\phi \times_Q \psi$  a Moyal-type product, because it resembles the process of Weyl quantization of  $C^\infty(\mathbb{R}^{2k})$  and the Moyal product. Recall that in Weyl quantization (e.g., see [8]), functions  $\phi, \psi \in C_c^\infty(\mathbb{R}^{2k})$  are associated to certain operators  $F_\phi$  and  $F_\psi$  involving Schrödinger's  $P, Q$  operators, and the operator multiplication  $F_\phi F_\psi$  is defined to be  $F_{\phi \times \psi}$ , giving us the Moyal product. Several authors since have modified this process to obtain various versions of Moyal-type products (mostly in terms of  $*$ -products and formal power series). The above formulation is just one such product. On the other hand, note that in our case we do not have to resort to the formal power series. Our quantization is carried out using the  $C^*$ -algebra framework.

DEFINITION 4.3. We will write  $A_\pi := \overline{Q_\pi(S_c(\mathcal{O}_\pi))}^{\|\cdot\|_{\text{op}}}$ , the norm closure in  $\mathcal{B}(\mathcal{H}_\pi)$  of the  $*$ -algebra  $S_c(\mathcal{O}_\pi)$  considered above. The  $C^*$ -algebras  $A_\pi$  will be considered as the quantizations of the orbits  $\mathcal{O}_\pi$ .

In view of Proposition 4.2, we may as well say that each irreducible representation of  $(A, \Delta)$  “models” the deformed multiplication on each  $S_c(\mathcal{O})$ . This is not necessarily a very rigorous statement, but it does give us a helpful insight: Note that in the geometric quantization program, especially in the program introduced in [3], one studies the representation theory of Lie groups (or more general objects) via deformed products of functions.

Meanwhile, from the Corollary to Proposition 4.1, the following Plancherel-type result is immediate. Here,  $\mathcal{A}$  is viewed as a dense subspace of  $\mathcal{H}$ , which is the Hilbert space consisting of the  $L^2$ -functions in the  $(x, y, r)$  variables. It is the Hilbert space on which our Hopf  $C^*$ -algebra  $(A, \Delta)$  acts (see §1.2, as well as our previous paper [10]).

PROPOSITION 4.4. For  $f \in \mathcal{A}$ , viewed as a Schwartz function contained in the Hilbert space  $\mathcal{H}$ , we have

$$(4.2) \quad \|f\|_2^2 = \int \|\pi_r(f)\|_{\text{HS}}^2 |\eta_\lambda(r)|^n dr.$$

*Proof.* For  $f \in \mathcal{A}$ ,

$$\begin{aligned} \|f\|_2^2 &= \|\hat{f}\|_2^2 = \int |\hat{f}(p, q, r)|^2 dpdqdr \\ &= \int \left( \int |\hat{f}(p, q, r)|^2 |\eta_\lambda(r)|^{-n} dpdq \right) |\eta_\lambda(r)|^n dr \\ &= \int \|\pi_r(f)\|_{\text{HS}}^2 |\eta_\lambda(r)|^n dr. \quad \square \end{aligned}$$

In  $\text{Irr}(A) \cong \mathcal{O}(A)$ , the topology on the subset  $\{\pi_r\}$  (consisting of generic representations) is the standard one, which is just the vector space topology on a line (the  $r$ -axis) with one point removed (at  $r = 0$ ). And, the  $\{\pi_r\}$  form a dense subset in  $\text{Irr}(A)$ . So  $d\mu = |\eta_\lambda(r)|^n dr$  in (4.2) may be regarded as a measure on  $\text{Irr}(A)$ , which is (densely) supported on  $\{\pi_r\} \subseteq \text{Irr}(A)$ . Keeping the analogy with the representation theory of nilpotent Lie groups [4], we will call  $d\mu$  the *Plancherel measure* on  $\text{Irr}(A)$ .

Relative to the Plancherel measure  $d\mu$ , we can construct the following direct integral of Hilbert spaces:

$$\int_{\text{Irr}(A)}^{\oplus} \text{HS}(\mathcal{H}_\pi) d\mu(\pi),$$

where  $\text{HS}(\mathcal{H}_\pi)$  denotes the space of Hilbert–Schmidt operators on  $\mathcal{H}_\pi$ . The Plancherel formula (4.2) implies that the map

$$T : \mathcal{A} \rightarrow \int_{\text{Irr}(A)}^{\oplus} \text{HS}(\mathcal{H}_\pi) d\mu(\pi)$$

defined by  $T(f)(\pi) := \pi(f)$ ,  $\pi \in \text{Irr}(A)$ , is an isometry. Clearly,  $T$  will extend to an isometry on  $\mathcal{H}$ . Actually,  $T$  is an onto isometry.

THEOREM 4.5 (The Plancherel Theorem). *The map defined above,  $T : \mathcal{H} \rightarrow \int_{\text{Irr}(A)}^{\oplus} \text{HS}(\mathcal{H}_\pi) d\mu(\pi)$ , is an onto isometry. In other words, the spatial isomorphism  $T$  gives a direct integral decomposition of  $\mathcal{H}$ , via the Plancherel formula (4.2).*

*Proof.* This is a version of the Plancherel Theorem. The proof can be given following the direct integral analysis of Dixmier [6, §18]. We can further say that the Plancherel measure is actually unique. This result illustrates the fact that  $A$  is a type I  $C^*$ -algebra.  $\square$

Recall that on  $\mathcal{H}$ , the algebra  $\mathcal{A}$  (or  $A$ ) acts by regular representation  $L$ . The precise definition can be found in our previous papers, but  $L(f)$ ,  $f \in \mathcal{A}$ , is none other than the multiplication operator defined by  $L(f)\xi = f \times \xi$ , where  $\xi \in \mathcal{A} \subseteq \mathcal{H}$  and  $\times$  is the multiplication on  $\mathcal{A}$ . Meanwhile, on each fiber  $\text{HS}(\mathcal{H}_\pi)$  of the decomposition,  $f \in \mathcal{A}$  acts by  $F \mapsto \pi(f)F$ , where the right hand side means the operator multiplication between the two Hilbert–Schmidt operators on  $\mathcal{H}_\pi$ .

It is easy to see that  $T$  intertwines these actions. That is, if  $f, \xi \in \mathcal{A}(\subseteq \mathcal{H})$ , then

$$T(L(f)\xi)(\pi) = T(f \times \xi)(\pi) = \pi(f \times \xi) = \pi(f)\pi(\xi) = \pi(f)(T(\xi)(\pi)),$$

which holds for all  $\pi$ . In other words, we have the following equivalence of representations:

$$L \cong \int_{\text{Irr}(A)}^\oplus \pi d\mu(\pi).$$

This means that the regular representation has a direct integral decomposition into irreducible representations.

REMARK. Even though the Plancherel measure is supported only on the set of generic irreducible representations  $\{\pi_r\}$ , note that since  $\mu(\text{Irr}(A) \setminus \{\pi_r\}) = 0$  and closure of the  $\{\pi_r\}$  is all of  $\text{Irr}(A)$ , the above result is consistent with the observation made in our earlier paper (in [10]) on the amenability of  $A$ . Meanwhile, we point out here that very recently Desmedt [5] has studied the amenability problems and a generalized version of Plancherel theorem, in the setting of locally compact quantum groups. (The author thanks the referee for this information.)

Before we wrap up, let us point out the following interesting observation: By our quantization map  $Q_\pi$ , we saw that the deformed product on  $S_c(\mathcal{O})$  is actually an operator multiplication, and we summarized this situation by saying that “each irreducible representation models the deformed multiplication on each  $S_c(\mathcal{O})$ ”. Now by the Plancherel theorem, the regular representation  $L$  of  $A$  has a direct integral decomposition into the irreducible representations. Since the regular representation is given by the left multiplication on  $\mathcal{A}$  (and  $A$ ), and since each irreducible representation models the Moyal-type deformed multiplication on each  $S_c(\mathcal{O})$ , the Plancherel theorem can be loosely stated as follows: “The twisted product on  $A$  is patched up from the deformed multiplications on the  $S_c(\mathcal{O})$ .”

In our case, the twisted product on  $A$  was obtained directly as a deformation quantization of the Poisson bracket, via a certain (continuous) cocycle [9], [10]. The above paragraph suggests a more geometric approach such that one may try to construct the twisted product by first studying the individual dressing orbits (symplectic leaves), find Moyal-type products on them, and

then “patch up” these deformed products. A similar idea is being used in [22], although the settings are different from ours.

In general, we do not expect this to work fully, due to various obstructions caused by the complexities of the symplectic leaves themselves and of the way the leaves lie inside the Poisson manifold. Nevertheless, this observation underlines the point that the dressing orbits and the representation theoretical analysis play a very useful role in the development of quantization methods.

### Appendix: Deformation of the orbits in $\mathcal{O}(\tilde{A})$

Finally, a short remark is in order for the case of the orbits in  $\mathcal{O}(\tilde{A})$ , in relation with the representations in  $\text{Irr}(\tilde{A})$ . Generally speaking, the ideas we followed in Section 4 do go through, in the sense that we can define a deformed multiplication on an orbit  $\mathcal{O} \in \mathcal{O}(\tilde{A})$ , which is modeled by the irreducible representation corresponding to the orbit. In addition, the Plancherel type result exists, giving us an interpretation as above that the regular representation is “patched up” from the deformed multiplications on the orbits.

On the other hand, not all the steps go through and some modifications should be made. We do not plan to give a detailed discussion here (since it would be rather lengthy and is still in the works), but we will briefly indicate below where the modifications should occur.

For the cases of the representations  $\tilde{\pi}_s$  and  $\tilde{\pi}_{p,q}$  essentially the same results hold as in Propositions 4.1 and 4.2. That is, we do have the spatial isomorphism  $L^2(\mathcal{O}_\pi, d\theta) \cong \text{HS}(\mathcal{H}_\pi)$ , and from this the Moyal-type deformed multiplication on the orbits can be obtained. The case of the representations  $\tilde{\pi}_{r,s}$  is when we need some care: The operators  $\tilde{\pi}_{r,s}(f)$ ,  $f \in \tilde{A}$ , are no longer trace-class, and it seems that we need to incorporate a kind of a “formal degree” operator (as in [7]) for  $\tilde{\pi}_{r,s}$ , to define the Hilbert-Schmidt operators. Even with this adjustment, the Hilbert-Schmidt operator space is not isomorphic to  $L^2(\mathcal{O}_{r,s}, d\theta)$ .

Some of these are serious obstacles, but it turns out that there is still a way to define a deformed multiplication on each orbit, again modeled by the irreducible representations. We do not seem to have spatial isomorphisms between the  $L^2(\mathcal{O}_{r,s}, d\theta)$  and the  $\text{HS}(\mathcal{H}_{\pi_{r,s}})$ , but we can still show that  $\int^\oplus L^2(\mathcal{O}_{r,s}, d\theta) ds \cong \int^\oplus \text{HS}(\mathcal{H}_{\pi_{r,s}}) ds$ . From this it follows that  $\int^\oplus L^2(\mathcal{O}_{r,s}, d\theta) |\eta_\lambda(r)|^{-1} dr ds \cong \tilde{\mathcal{H}}$ , where  $\tilde{\mathcal{H}}$  is the Hilbert space on which our  $C^*$ -algebra  $\tilde{A}$  acts by regular representation (as in Example 3.6 of [10]). This would be the result taking the place of Theorem 4.5 above.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KS 66045, USA  
*E-mail address*: `bjkahng@math.ku.edu`