

## \*-REPRESENTATIONS OF A QUANTUM HEISENBERG GROUP ALGEBRA

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ABSTRACT. In our earlier work, we constructed a specific non-compact quantum group whose quantum group structures have been constructed on a certain twisted group  $C^*$ -algebra. In a sense, it may be considered as a “quantum Heisenberg group  $C^*$ -algebra”. In this paper, we will find, up to equivalence, all of its irreducible  $*$ -representations. We will point out the Kirillov type correspondence between the irreducible representations and the so-called dressing orbits. By taking advantage of its comultiplication, we will then introduce and study the notion of *inner tensor product representations*. We will show that the representation theory satisfies a “quasitriangular” type property, which does not appear in ordinary group representation theory.

### 1. INTRODUCTION

Recently in [11], we constructed a specific non-compact  $C^*$ -algebraic quantum group, via deformation quantization of a certain non-linear Poisson–Lie bracket on an exponential solvable Lie group. The underlying  $C^*$ -algebra of this quantum group has been realized as a twisted group  $C^*$ -algebra of a nilpotent Lie group.

From its construction, it is reasonable to view it as a “quantum Heisenberg group  $C^*$ -algebra” (This observation will be made a little clearer in the third section.). Focusing on its  $C^*$ -algebra structure, we study in this paper its irreducible  $*$ -representations. It is not difficult to see that there exists a Kirillov type, one-to-one correspondence between the irreducible  $*$ -representations and the “dressing orbits” at the level of its Poisson–Lie group counterpart.

Since the object of our study is actually a Hopf  $C^*$ -algebra (i.e. quantum group), we may use its comultiplication to define and study *inner tensor product* of representations. This is a generalization to quantum case of the inner

tensor product representations of an ordinary group. We will show that unlike in the case of an ordinary group and in the cases of many earlier examples of non-compact quantum groups (e. g. [20], [26], [23], [16]), the inner tensor product representations of our Hopf  $C^*$ -algebra satisfy a certain “quasitriangularity” property.

Here, we only study representation theory of the specific example of [11]. However, our earlier results ([10, 11]) imply that this quantum group is just one example of a larger class of solvable quantum groups having twisted group  $C^*$ -algebras or (more general) twisted crossed product algebras as underlying  $C^*$ -algebras. One of the main purposes of [11] and this paper is to present a case study, so that we are later able to develop a procedure to construct and study more general class of locally compact quantum groups. Eventually, we wish to further develop a generalized orbit theory of Kirillov type, which would then be used to study the harmonic analysis of the locally compact quantum groups. This will be our forthcoming project.

Since this paper is essentially a continuation of [11], we will keep the same notation as in that paper. Some of the notations are reviewed in section 2. This section discusses the classical counterparts (Poisson–Lie groups) of our quantum groups, as well as the description of the dressing orbits. Results here will be useful in our future study.

Our main examples are briefly described at the beginning of the third section. In the rest of section 3, we discuss their representation theory: We find all the irreducible  $*$ -representations up to equivalence. We then point out the Kirillov type one-to-one correspondence between these representations and the dressing orbits.

In the last section, we study inner tensor product representations. By using the quasitriangular quantum  $R$ -matrix operator obtained in [11], we will show some interesting properties that do not appear in ordinary group representation theory.

## 2. PRELIMINARIES: POISSON–LIE GROUPS, DRESSING ACTIONS AND DRESSING ORBITS

We start with a brief review of the notations used in [11], as well as some basic results from Poisson–Lie group theory. One of the main purposes here is to calculate the dressing orbits at the classical level.

**2.1. The Poisson–Lie groups.** Let  $H$  be the  $(2n + 1)$ -dimensional Heisenberg Lie group. Its underlying space is  $\mathbb{R}^{2n+1}$  and the multiplication on it is defined

by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \beta(x, y')),$$

for  $x, y, x', y' \in \mathbb{R}^n$  and  $z, z' \in \mathbb{R}$ . Here  $\beta(, )$  is the usual inner product on  $\mathbb{R}^n$ , following the notation of [11].

Consider also the extended Heisenberg Lie group  $\tilde{H}$ , with the group law defined by

$$(x, y, z, w)(x', y', z', w') = (x + e^w x', y + e^{-w} y', z + z' + (e^{-w})\beta(x, y'), w + w').$$

The notation is similar as above, with  $w, w' \in \mathbb{R}$ . This group contains  $H$  as a normal subgroup.

In [11], we obtained the “dual Poisson–Lie group”  $G$  of  $H$ . It is determined by the multiplication law:

$$(p, q, r)(p', q', r') = (e^{\lambda r'} p + p', e^{\lambda r'} q + q', r + r'),$$

while the dual Poisson–Lie group  $\tilde{G}$  of  $\tilde{H}$  is determined by the multiplication law:

$$(p, q, r, s)(p', q', r', s') = (e^{\lambda r'} p + p', e^{\lambda r'} q + q', r + r', s + s').$$

Here  $\lambda \in \mathbb{R}$  is a fixed constant, which determines a certain non-linear Poisson structure on  $G$  (or  $\tilde{G}$ ) when  $\lambda \neq 0$ .

Although we do not explicitly mention the Poisson brackets here (see instead [11]), we can show indeed that  $\tilde{H}$  and  $\tilde{G}$  (similarly,  $H$  and  $G$ ) are mutually dual Poisson–Lie groups. For definition and some important results on Poisson–Lie groups, see for example the article by Lu and Weinstein [17] or the book by Chari and Pressley [4].

In [11], using the realization that the Poisson bracket on  $G$  is a non-linear Poisson bracket of the “cocycle perturbation” type, we have been able to construct a quantum version of  $G$ : The (non-commutative) Hopf  $C^*$ -algebra  $(A, \Delta)$ , whose underlying  $C^*$ -algebra is a twisted group  $C^*$ -algebra. Similarly for  $\tilde{G}$ , we constructed the Hopf  $C^*$ -algebra  $(\tilde{A}, \tilde{\Delta})$ . These are the main objects of study in [11] and in this paper.

**2.2. Basic definitions: Dressing actions.** Let  $G$  be a Poisson–Lie group, let  $G^*$  be its dual Poisson–Lie group, and let  $\mathfrak{g}$  and  $\mathfrak{g}^*$  be the corresponding Lie algebras. Together,  $(\mathfrak{g}, \mathfrak{g}^*)$  forms a Lie bialgebra. On the vector space  $\mathfrak{g} \oplus \mathfrak{g}^*$ , we can define a bracket operation by

$$(2.1) \quad [(X_1, \mu_1), (X_2, \mu_2)] \\ = ([X_1, X_2]_{\mathfrak{g}} - \text{ad}_{\mu_2}^* X_1 + \text{ad}_{\mu_1}^* X_2, [\mu_1, \mu_2]_{\mathfrak{g}^*} + \text{ad}_{X_1}^* \mu_2 - \text{ad}_{X_2}^* \mu_1),$$

where  $\text{ad}_X^* \mu$  and  $\text{ad}_\mu^* X$  are, respectively, the coadjoint representations of  $\mathfrak{g}$  on  $\mathfrak{g}^*$  and of  $\mathfrak{g}^*$  on  $\mathfrak{g} = (\mathfrak{g}^*)^*$ . This is a Lie bracket on  $\mathfrak{g} \oplus \mathfrak{g}^*$ , which restricts to the given Lie brackets on  $\mathfrak{g}$  and  $\mathfrak{g}^*$ . We denote the resulting Lie algebra by  $\mathfrak{g} \bowtie \mathfrak{g}^*$ , the *double Lie algebra* [17].

Let  $D = G \bowtie G^*$  be the connected, simply connected Lie group corresponding to  $\mathfrak{g} \bowtie \mathfrak{g}^*$ . There are homomorphisms of Lie groups

$$G \hookrightarrow D \hookleftarrow G^*,$$

lifting the inclusion maps of  $\mathfrak{g}$  and  $\mathfrak{g}^*$  into  $\mathfrak{g} \bowtie \mathfrak{g}^*$ . Thus we can define a product map  $G \times G^* \rightarrow D$ . We will assume from now on that the images of  $G$  and  $G^*$  are closed subgroups of  $D$  and that the map is a global diffeomorphism of  $G \times G^*$  onto  $D$ . In this case, we will say that  $D$  is a *double Lie group*. In particular, each element of  $D$  has a unique expression  $g \cdot \gamma$ , for  $g \in G$  and  $\gamma \in G^*$ .

Suppose we are given a double Lie group  $D = G \bowtie G^*$ . For  $g \in G$  and  $\gamma \in G^*$ , regarded naturally as elements in  $D$ , the product  $\gamma \cdot g$  would be factorized as

$$\gamma \cdot g = g^\gamma \cdot \gamma^g,$$

for some  $g^\gamma \in G$  and  $\gamma^g \in G^*$ . We can see without difficulty that the map  $\lambda : G^* \times G \rightarrow G$  defined by

$$\lambda_\gamma(g) = g^\gamma$$

is a left action of  $G^*$  on  $G$ . Hence,

**Definition 1.** The map  $\rho : G \times G^* \rightarrow G$  defined by

$$\rho_\gamma(g) = g^{(\gamma^{-1})}$$

is a right action, called the *drinking action* of  $G^*$  on  $G$ .

*Remark.* Sometimes, the action  $\lambda$  is called the left dressing action, while the action  $\rho$  is called the right dressing action. It is customary to call the right action  $\rho$  the dressing action. Semenov–Tian–Shansky [22] first proved that the (right) dressing action of  $G^*$  on  $G$  is a Poisson action (i. e. it preserves the respective Poisson structures). The notion of dressing action still exists (at least locally), even if the assumption that  $G \times G^* \cong D$  is not satisfied. See [17].

For any Lie algebra  $\mathfrak{h}$ , it can always be regarded as a Lie bialgebra by viewing its dual vector space  $\mathfrak{g} = \mathfrak{h}^*$  as an abelian Lie algebra. Then the dressing action of  $H$  on  $G$  actually coincides with the coadjoint action of  $H$  on  $\mathfrak{h}^*$  ( $= \mathfrak{g} \cong G$ ). In this sense, we may regard the dressing action as a generalization of the coadjoint action. This is the starting point for the attempts to generalize the Kirillov’s orbit theory, and this point of view has been helpful throughout this paper.

We conclude the subsection by stating the following result by Semenov–Tian–Shansky, which exhibits the close relationship between dressing actions and the geometric aspects of Poisson–Lie groups.

**Theorem 2.1.** ([22], [17]) *The dressing action  $G^*$  on  $G$  is a Poisson action. Moreover, the orbit of the dressing action through  $g \in G$  is exactly the symplectic leaf through the point  $g$  for the Poisson bracket on  $G$ .*

**2.3. Dressing orbits for the Poisson–Lie groups  $\tilde{G}$  and  $\tilde{H}$ .** Let us consider our specific Poisson–Lie groups  $\tilde{G}$  and  $\tilde{H}$ , or equivalently, the Lie bialgebra  $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ . We will construct here the double Lie group, dressing action, and dressing orbits. Along the way, we will obtain the corresponding results for  $G$  and  $H$ .

By equation (2.1) and by using the Lie brackets on  $\tilde{\mathfrak{g}}$  and on  $\tilde{\mathfrak{h}}$  (see [11]), we can construct the double Lie algebra  $\tilde{\mathfrak{g}} \bowtie \tilde{\mathfrak{h}}$ . The space for it is  $\tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{h}}$ , on which the following Lie bracket is defined:

$$\begin{aligned} & [(p, q, r, s; x, y, z, w), (p', q', r', s'; x', y', z', w')] \\ &= (\lambda(r'p - rp') + (w'p - wp') + (r'y - ry'), \\ &\quad \lambda(r'q - rq') + (wq' - w'q) + (rx' - r'x), \\ &\quad 0, (p' \cdot x - p \cdot x') + (q \cdot y' - q' \cdot y); \\ &\quad (wx' - w'x) + \lambda(rx' - r'x), (w'y - wy') + \lambda(ry' - r'y), \\ &\quad \beta(x, y') - \beta(x', y) + \lambda(p' \cdot x - p \cdot x') + \lambda(q' \cdot y - q \cdot y'), 0). \end{aligned}$$

We then calculate the corresponding Lie group  $\tilde{D}$ . Using the notation  $\eta_\lambda(r) = \frac{e^{2\lambda r} - 1}{2\lambda}$  (the function introduced in Definition 2.3 of [11]), the group  $\tilde{D}$  is given by the following multiplication law:

$$\begin{aligned} & (p, q, r, s; x, y, z, w)(p', q', r', s'; x', y', z', w') \\ &= (e^{\lambda r'} p + e^{-w} p' + e^{-\lambda r'} \eta_\lambda(r') y, e^{\lambda r'} q + e^w q' - e^{-\lambda r'} \eta_\lambda(r') x, r + r', \\ &\quad s + s' + (e^{-\lambda r' - w}) p' \cdot x - (e^{-\lambda r' + w}) q' \cdot y - \eta_\lambda(-r') \beta(x, y); \\ &\quad e^{-\lambda r'} x + e^w x', e^{-\lambda r'} y + e^{-w} y', z + z' + (e^{-\lambda r' - w}) \beta(x, y') \\ &\quad + \lambda(e^{-\lambda r' - w}) p' \cdot x + \lambda(e^{-\lambda r' + w}) q' \cdot y + \lambda \eta_\lambda(-r') \beta(x, y), w + w'). \end{aligned}$$

If we identify  $(p, q, r, s) \in \tilde{G}$  with  $(p, q, r, s; 0, 0, 0, 0)$  and  $(x, y, z, w) \in \tilde{H}$  with  $(0, 0, 0, 0; x, y, z, w)$ , it is easy to see that  $\tilde{G}$  and  $\tilde{H}$  are (closed) Lie subgroups of  $\tilde{D}$ . This defines a global diffeomorphism of  $\tilde{G} \times \tilde{H}$  onto  $\tilde{D}$ , since any element  $(p, q, r, s; x, y, z, w)$  of  $\tilde{D}$  can be written as

$$(p, q, r, s; x, y, z, w) = (p, q, r, s; 0, 0, 0, 0)(0, 0, 0, 0; x, y, z, w).$$

In other words,  $\tilde{D}$  is the *double Lie group* of  $\tilde{G}$  and  $\tilde{H}$ , which will be denoted by  $\tilde{G} \bowtie \tilde{H}$ . Meanwhile, if we consider only the  $(p, q, r)$  and the  $(x, y, z)$  variables, we obtain in the same way the double Lie group  $D = G \bowtie H$  of  $G$  and  $H$ .

By Definition 1, the *dressing action* of  $\tilde{H}$  on  $\tilde{G}$  is:

$$\begin{aligned} (\rho(x, y, z, w))(p, q, r, s) &= (e^w p - e^{-\lambda r + w} \eta_\lambda(r) y, e^{-w} q + e^{-\lambda r - w} \eta_\lambda(r) x, r, \\ &\quad s - e^{-\lambda r} p \cdot x + e^{-\lambda r} q \cdot y + e^{-2\lambda r} \eta_\lambda(r) \beta(x, y)). \end{aligned}$$

The dressing orbits, which by Theorem 2.1 are the symplectic leaves in  $\tilde{G}$  for its (non-linear) Poisson bracket, are:

- $\tilde{\mathcal{O}}_s = \{(0, 0, 0, s)\}$ , when  $(p, q, r) = (0, 0, 0)$ .
- $\tilde{\mathcal{O}}_{p,q} = \{(ap, \frac{1}{a}q, 0, c) : a > 0, c \in \mathbb{R}\}$ , when  $r = 0$  but  $(p, q) \neq (0, 0)$ .
- $\tilde{\mathcal{O}}_{r,s} = \{(a, b, r, s - \frac{1}{\eta_\lambda(r)} a \cdot b) : (a, b) \in \mathbb{R}^{2n}\}$ , when  $r \neq 0$ .

Here  $a \cdot b$  denotes the inner product. The  $\tilde{\mathcal{O}}_s$  are 1-point orbits, the  $\tilde{\mathcal{O}}_{p,q}$  are 2-dimensional orbits, and the  $\tilde{\mathcal{O}}_{r,s}$  are  $2n$ -dimensional orbits.

Similarly, if we only consider the  $(p, q, r)$  and the  $(x, y, z)$  variables, we obtain the expression for the dressing action of  $H$  on  $G$  as follows:

$$(\rho(x, y, z))(p, q, r) = (p - e^{-\lambda r} \eta_\lambda(r) y, q + e^{-\lambda r} \eta_\lambda(r) x, r).$$

So the dressing orbits in  $G$  are:

- (1-point orbits):  $\mathcal{O}_{p,q} = \{(p, q, 0)\}$ , when  $r = 0$ .
- ( $2n$ -dimensional orbits):  $\mathcal{O}_r = \{(a, b, r) : (a, b) \in \mathbb{R}^{2n}\}$ , when  $r \neq 0$ .

Meanwhile, to calculate the dressing action of  $\tilde{G}$  on  $\tilde{H}$ , it is convenient to regard  $\tilde{D}$  as the double Lie group  $\tilde{H} \bowtie \tilde{G}$  of  $\tilde{H}$  and  $\tilde{G}$ . Indeed, there exists a global diffeomorphism between  $\tilde{H} \times \tilde{G}$  and  $\tilde{D}$  defined by

$$\begin{aligned} ((x, y, z, w), (p, q, r, s)) &\mapsto (0, 0, 0, 0; x, y, z, w)(p, q, r, s, 0, 0, 0, 0) \\ &= (e^{-w} p + e^{-\lambda r} \eta_\lambda(r) y, e^w q - e^{-\lambda r} \eta_\lambda(r) x, r, \\ &\quad s + (e^{-\lambda r - w} p \cdot x - (e^{-\lambda r + w} q \cdot y - \eta_\lambda(-r) \beta(x, y)); \\ &\quad e^{-\lambda r} x, e^{-\lambda r} y, z + \lambda(e^{-\lambda r - w} p \cdot x + \lambda(e^{-\lambda r + w} q \cdot y + \lambda \eta_\lambda(-r) \beta(x, y), w)). \end{aligned}$$

Similarly, we can show that  $D$  is the double Lie group of  $H$  and  $G$ .

Using this characterization of the double Lie group  $\tilde{H} \bowtie \tilde{G}$ , the dressing action of  $\tilde{G}$  on  $\tilde{H}$  is obtained by Definition 1. That is,

$$\begin{aligned} (\rho(p, q, r, s))(x, y, z, w) \\ &= (e^{-\lambda r} x, e^{-\lambda r} y, z + \lambda(e^{-\lambda r} p \cdot x + \lambda(e^{-\lambda r} q \cdot y - \lambda(e^{-2\lambda r} \eta_\lambda(r) \beta(x, y), w)). \end{aligned}$$

The dressing orbits in  $\tilde{H}$  are:

- (1-point orbits):  $\tilde{\mathcal{O}}_{z,w} = \{(0, 0, z, w)\}$ , when  $(x, y) = (0, 0)$ .
- (2-dimensional orbits):  $\tilde{\mathcal{O}}_{x,y,w} = \{(\alpha x, \alpha y, \gamma, w) : \alpha > 0, \gamma \in \mathbb{R}\}$ , when  $(x, y) \neq (0, 0)$ .

Similarly, the dressing action of  $G$  on  $H$  is given by

$$\begin{aligned}
 &(\rho(p, q, r))(x, y, z) \\
 &= (e^{-\lambda r} x, e^{-\lambda r} y, z + \lambda(e^{-\lambda r})p \cdot x + \lambda(e^{-\lambda r})q \cdot y - \lambda(e^{-2\lambda r})\eta_\lambda(r)\beta(x, y)),
 \end{aligned}$$

and the dressing orbits in  $H$  are:

- (1-point orbits):  $\mathcal{O}_z = \{(0, 0, z)\}$ , when  $(x, y) = (0, 0)$ .
- (2-dimensional orbits):  $\mathcal{O}_{x,y} = \{(\alpha x, \alpha y, \gamma) : \alpha > 0, \gamma \in \mathbb{R}\}$ , when  $(x, y) \neq (0, 0)$ .

*Remark.* As we pointed out earlier, the dressing action is usually regarded as a generalization of the coadjoint action. In the present case, we can see easily that the dressing orbits in  $\tilde{H}$  are exactly the coadjoint orbits in  $\tilde{\mathfrak{h}} \cong \tilde{H}$ . This illustrates the point that the Poisson bracket on  $\tilde{H}$  is just the linear Poisson bracket. On the other hand, for  $\tilde{G}$ , which has a non-linear Poisson bracket, this is no longer the case. The orbits  $\tilde{\mathcal{O}}_{r,s}$  are different from the coadjoint orbits in  $\tilde{\mathfrak{g}}$ . Nevertheless, we can still see close resemblance.

### 3. QUANTUM HEISENBERG GROUP ALGEBRA REPRESENTATIONS

Our main objects of study are the Hopf  $C^*$ -algebras  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$  constructed in [11]. As a  $C^*$ -algebra,  $A$  is isomorphic to a twisted group  $C^*$ -algebra. That is,  $A \cong C^*(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma)$ , where  $H$  is the  $(2n + 1)$  dimensional Heisenberg Lie group (see section 2) and  $Z$  is the center of  $H$ . Whereas,  $\mathfrak{g} = \mathfrak{h}^*$  is the dual space of the Lie algebra  $\mathfrak{h}$  of  $H$  and  $\mathfrak{q} = \mathfrak{z}^\perp$ , for  $\mathfrak{z} \subseteq \mathfrak{h}$  corresponding to  $Z$ . We denoted by  $\sigma$  the twisting cocycle for the group  $H/Z$ . As constructed in [11],  $\sigma$  is a continuous field of cocycles  $\mathfrak{g}/\mathfrak{q} \ni r \mapsto \sigma^r$ , where

$$(3.1) \quad \sigma^r((x, y), (x', y')) = \bar{e}[\eta_\lambda(r)\beta(x, y')].$$

Following the notation of the previous paper, we used:  $\bar{e}(t) = e^{(-2\pi i)t}$  and  $\eta_\lambda(r) = \frac{e^{2\lambda r} - 1}{2\lambda}$ . The elements  $(x, y), (x', y')$  are group elements in  $H/Z$ .

In [11], we showed that the  $C^*$ -algebra  $A$  is a strict deformation quantization (in the sense of Rieffel) of  $C_\infty(G)$ , the commutative algebra of continuous functions on  $G$  vanishing at infinity. For convenience, the deformation parameter  $\hbar$  has been fixed ( $\hbar = 1$ ), which is the reason why we do not see it in the definition of  $A$ . When  $\hbar = 0$  (i. e. classical limit), it turns out that  $\sigma \equiv 1$ . So  $A_{\hbar=0} \cong C_\infty(G)$ . Throughout this paper (as in [11]), we write  $A = A_{\hbar=1}$ .

We could also construct a “(regular) multiplicative unitary operator”  $U \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  associated with  $A$ , in the sense of Baaĳ and Skandalis. Thus  $(A, \Delta)$  becomes a Hopf  $C^*$ -algebra, whose comultiplication  $\Delta$  is determined by  $U$ . Similar realization as a twisted group  $C^*$ -algebra also exists for the “extended” Hopf  $C^*$ -algebra  $(\tilde{A}, \tilde{\Delta})$ . Again, its comultiplication  $\tilde{\Delta}$  is determined by a certain regular multiplicative unitary operator  $\tilde{U}$ .

Actually,  $(A, \Delta)$  is an example of a *locally compact quantum group*, equipped with the counit, antipode, and Haar weight, as constructed in [11]. We do not intend to give here the correct definition of a locally compact quantum group, which is still at a primitive stage (But see [15], [18], [29] for some recent developments.). Since the main goal of the present paper is in the study of  $*$ -representations of  $A$ , it would be rather sufficient to focus on the Hopf  $C^*$ -algebra structure of  $(A, \Delta)$ . For this reason, our preferred terminology throughout this paper for  $(A, \Delta)$  will be the “Hopf  $C^*$ -algebra”, although much stronger notion of the “locally compact quantum group” is still valid. Similar comments holds also for the extended Hopf  $C^*$ -algebra  $(\tilde{A}, \tilde{\Delta})$ .

From its construction, we can see that  $(A, \Delta)$  is at the same time a “quantum  $C_\infty(G)$ ” and “quantum  $C^*(H)$ ”. In the previous paper [11], the first viewpoint has been exploited: As we already mentioned,  $(A, \Delta)$  has been constructed as a deformation quantization of  $C_\infty(G)$ . The construction of the counit, antipode, and Haar weight for  $(A, \Delta)$  all comes from the corresponding structures on  $G$ .

In this article, we wish to focus our attention to the second viewpoint. Recall that  $A \cong C^*(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma)$ , where the twisting cocycle  $\sigma$  for  $H/Z$  is defined as in (3.1). Since we may put  $\eta_\lambda(r) = r$  for  $\lambda = 0$ , it is a simple exercise using Fourier inversion theorem that  $A \cong C^*(H)$  when  $\lambda = 0$  (The reader may refer to the article [10] or [19] for the definition of a twisted group  $C^*$ -algebra.). For this reason, we will on occasion call  $(A, \Delta)$  the *quantum Heisenberg group  $C^*$ -algebra*.

*Remark.* The notion of the “quantum Heisenberg group ( $C^*$ -)algebra” introduced above is different from the notion of the “quantum Heisenberg algebra” used in some physics literatures [9], [12]. They are different as algebras. Another significant distinction is that ours is equipped with a (non-cocommutative) comultiplication, while the other one does not consider any coalgebra structure. Hence the slight difference in the choice of the terminologies.

The study of the  $*$ -representations of  $(A, \Delta)$  (and also of  $(\tilde{A}, \tilde{\Delta})$ ) will be a generalization of the study of the Heisenberg group representation theory (which, by a standard result, is equivalent to the  $*$ -representation theory of  $C^*(H)$ ). The



topic itself is of interest to us (providing us with the properties like “quasitriangularity”). But the success of the (ordinary) Heisenberg group representation theory in various applications also suggests that this is a worthwhile topic to develop.

Before we begin our discussion, let us fix our terminology. Suppose  $(A, \Delta)$  is a general Hopf  $C^*$ -algebra (in the sense of [25], [1]). By a *representation* of  $(A, \Delta)$ , we will just mean a non-degenerate  $*$ -representation of the  $C^*$ -algebra  $A$  on a certain Hilbert space.

On the other hand, by a *coaction* of  $(A, \Delta)$  on a  $C^*$ -algebra  $B$ , we will mean a non-degenerate  $*$ -homomorphism  $\delta_B : B \rightarrow M(B, A)$  such that

$$(\text{id}_B \otimes \Delta)\delta_B = (\delta_B \otimes \text{id}_A)\delta_B.$$

Here  $M(B, A)$  is the set  $\{x \in M(B \otimes A) : x(1_{M(B)} \otimes A) + (1_{M(B)} \otimes A)x \subseteq B \otimes A\}$ , which is a  $C^*$ -subalgebra of the multiplier algebra  $M(B \otimes A)$ . Similarly, a *(unitary) corepresentation* of the Hopf  $C^*$ -algebra  $(A, \Delta)$  on a Hilbert space  $\mathcal{H}$  is a unitary  $\Pi \in M(\mathcal{K}(\mathcal{H}) \otimes A)$  such that:

$$(3.2) \quad (\text{id} \otimes \Delta)(\Pi) = \Pi_{12}\Pi_{13}.$$

Here  $\Pi_{12}$  is understood as an element in  $M(\mathcal{K}(\mathcal{H}) \otimes A \otimes A)$  such that it acts as  $\Pi$  on the first and the second variables while the remaining variable is unchanged. The notation  $\Pi_{13}$  is understood in the similar manner.

*Remark.* Corepresentations of  $(A, \Delta)$  are actually the “representations of the coalgebra structure on  $(A, \Delta)$ ”. So in many articles on quantum groups, they are often called “(unitary) representations of the locally compact quantum group  $(A, \Delta)$ ”. In particular, representation theory in this sense of *compact quantum groups* [27], which are themselves Hopf  $C^*$ -algebras, have been neatly studied by Woronowicz in [28]. However, note that we will use the terminologies “representations” and “corepresentations” of a Hopf  $C^*$ -algebra in the sense defined above. This would make things a little simpler. Moreover, this is closer to the spirit of this paper, trying to view our  $(A, \Delta)$  as a quantum Heisenberg group  $C^*$ -algebra.

In our case, the  $C^*$ -algebra  $A$  is isomorphic to the twisted group  $C^*$ -algebra  $C^*(H/Z, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma)$ . So by slightly modifying Theorem 3.3 and Proposition 3.4 of [3], we are able to obtain the representations of  $A$  from the so-called “representing pairs”  $(\mu, Q^\sigma)$ . Such a pair  $(\mu, Q^\sigma)$  consists of a nondegenerate representation  $\mu$  of  $C_\infty(\mathfrak{g}/\mathfrak{q})$  and a generalized projective representation  $Q^\sigma$  of  $H/Z$ , satisfying the following property:

$$(3.3) \quad Q_{(x,y)}^\sigma Q_{(x',y')}^\sigma = \mu(\sigma((x,y), (x',y'))) Q_{(x+x',y+y')}^\sigma,$$

for  $(x,y), (x',y') \in H/Z$ .

Given a representing pair  $(\mu, Q^\sigma)$ , we can construct its “integrated form”. On the dense subspace of Schwartz functions, it reads:

$$(3.4) \quad \pi(f) = \int_{H/Z} \mu(f(x, y; \cdot)) Q_{(x,y)}^\sigma dx dy.$$

By natural extension to the  $C^*$ -algebra level (Due to the amenability of the group involved, there is no ambiguity. See [10].), we obtain in this way a representation of  $A$ .

*Remark.* Let us from now on denote by  $\mathcal{A}$  the dense subspace  $S_{3c}(H/Z \times \mathfrak{g}/\mathfrak{q})$  of  $A$ , which is the space of Schwartz functions in the  $(x, y; r)$  variables having compact support in the  $r \in \mathfrak{g}/\mathfrak{q}$  variable. This is a dense subalgebra (under the twisted convolution) of our twisted group  $C^*$ -algebra  $A$ , and it has been used throughout [11] (However, we should point out that our usage of  $\mathcal{A}$  is slightly different from that of [11]: There,  $\mathcal{A}$  is contained in  $S(\mathfrak{g})$ , while at present we view it as functions in the  $(x, y; r)$  variables. Nevertheless, they can be regarded as the same if we consider these functions as operators contained in our  $C^*$ -algebra.). Similarly for  $\tilde{A}$ , we will consider the dense subalgebra  $\tilde{\mathcal{A}}$  of Schwartz functions in the  $(x, y, r, w)$  variables having compact support in the  $r$  and  $w$ .

To find irreducible representations of  $A$ , let us look for some representing pairs  $(\mu, Q^\sigma)$  consisting of irreducible  $\mu$  and  $Q^\sigma$ . Irreducible representations of the commutative algebra  $C_\infty(\mathfrak{g}/\mathfrak{q})$  are just the pointwise evaluations at  $r \in \mathfrak{g}/\mathfrak{q}$ . So let us fix  $r \in \mathfrak{g}/\mathfrak{q}$  and the corresponding 1-dimensional representation  $\mu$  of  $C_\infty(\mathfrak{g}/\mathfrak{q})$ , given by  $\mu(v) = v(r)$ , for  $v \in C_\infty(\mathfrak{g}/\mathfrak{q})$ . Then the condition for  $Q^\sigma$  becomes:

$$(3.5) \quad \begin{aligned} Q_{(x,y)}^\sigma Q_{(x',y')}^\sigma &= \sigma((x, y), (x', y'); r) Q_{(x+x', y+y')}^\sigma \\ &= \sigma^r((x, y), (x', y')) Q_{(x+x', y+y')}^\sigma. \end{aligned}$$

That is,  $Q^\sigma$  satisfies the condition for an (ordinary) projective representation of  $H/Z$  with respect to the ordinary  $\mathbb{T}$ -valued cocycle  $\sigma^r$ .

Using  $\sigma^r$ , we may define an extension group  $E$  of  $H/Z$ . Its underlying space is  $H/Z \times \mathbb{T}$  and its multiplication law is given by

$$\begin{aligned} (x, y; \theta)(x', y'; \theta') &= (x + x', y + y'; \theta\theta' \sigma^r((x, y), (x', y'))) \\ &= (x + x', y + y'; \theta\theta' \bar{e}[\eta_\lambda(r)\beta(x, y')]). \end{aligned}$$

Standard theory tells us that the unitary projective representations of  $(H/Z, \sigma^r)$  come from the unitary group representations of  $E$ , which are easier to study:

The next lemma gives us all the irreducible unitary representations of  $E$ , up to equivalence.

**Lemma 3.1.** *Let  $E$  be the extension group of  $(H/Z, \sigma^r)$  as defined above. Then its irreducible unitary representations are equivalent to one of the following:*

- For each  $(p, q) \in \mathbb{R}^{2n}$ , there exists a 1-dimensional representation  $Q_{p,q}$  defined by

$$Q_{p,q}(x, y; \theta) = \bar{e}(p \cdot x + q \cdot y).$$

- There also exists an infinite dimensional representation  $Q_r$  on  $L^2(\mathbb{R}^n)$  defined by

$$(Q_r(x, y; \theta)\xi)(u) = \theta \bar{e}[\eta_\lambda(r)\beta(u, y)]\xi(u + x).$$

PROOF. Observe that  $E$  is a semi-direct product of two abelian groups  $X = \{(x, 0, 1) : x \in \mathbb{R}^n\}$  and  $Y \times \mathbb{T} = \{(0, y, \theta) : y \in \mathbb{R}^n, \theta \in \mathbb{T}\}$ . So by using Mackey analysis, every irreducible representation of  $E$  is obtained as an “induced representation” [8], [24], [21].

Since  $[E, E] = \mathbb{T}$ , we have:  $E/[E, E] = X \times Y$ , which is abelian. So all the irreducible representations of  $E/[E, E]$  are one-dimensional. By lifting from these 1-dimensional representations, we obtain the (irreducible) representations  $\{Q_{p,q}\}_{(p,q) \in \mathbb{R}^{2n}}$  of  $E$  that are trivial on the commutator  $[E, E]$ .

The infinite dimensional representation  $Q_r$  is actually the induced representation  $\text{Ind}_{Y \times \mathbb{T}, \chi}^E$ , where  $\chi$  is the representation of  $Y \times \mathbb{T}$  defined by:  $\chi(y, \theta) = \theta$ . It turns out (by using standard Mackey theory) that  $\{Q_{p,q}\}_{(p,q) \in \mathbb{R}^{2n}}$  and  $Q_r$  exhaust all the irreducible representations of  $E$ , up to equivalence.  $\square$

We are now able to find the irreducible projective representations of  $(H/Z, \sigma)$ . Check equation (3.5) and we obtain the following representing pairs consisting of irreducible  $\mu$  and  $Q^\sigma$ .

- (1) (When  $r = 0 \in \mathfrak{g}/\mathfrak{q}$ ): For each  $(p, q) \in \mathbb{R}^{2n}$ , there is a pair  $(\mu, Q^\sigma)$  given by
  - $\mu(v) = v(0), v \in C_\infty(\mathfrak{g}/\mathfrak{q})$ .
  - $Q^\sigma(x, y) = \bar{e}(p \cdot x + q \cdot y), (x, y) \in H/Z$ .
- (2) (When  $r \neq 0 \in \mathfrak{g}/\mathfrak{q}$ ): There is a pair  $(\mu, Q^\sigma)$  given by
  - $\mu(v) = v(r), v \in C_\infty(\mathfrak{g}/\mathfrak{q})$ .
  - On  $L^2(\mathbb{R}^n)$ ,

$$(Q^\sigma(x, y)\xi)(u) = \bar{e}[\eta_\lambda(r)\beta(u, y)]\xi(u + x), \quad (x, y) \in H/Z.$$

Therefore, we obtain the following proposition. Observe the similarity between this result and the representation theory of the Heisenberg group  $H$  or the Heisenberg group  $C^*$ -algebra  $C^*(H)$  (This is not surprising, given the point we made earlier in this section.).

**Proposition 3.2.** *Every irreducible representation of the twisted convolution algebra  $\mathcal{A}$  is equivalent to one of the following representations, which have been obtained by integrating the representing pairs  $(\mu, Q^\sigma)$  of the preceding paragraph.*

- For  $(p, q) \in \mathbb{R}^{2n}$ , there is a 1-dimensional representation  $\pi_{p,q}$  of  $\mathcal{A}$ , defined by

$$\pi_{p,q}(f) = \int f(x, y, 0) \bar{e}(p \cdot x + q \cdot y) dx dy.$$

- For  $r \in \mathbb{R}$ , there is a representation  $\pi_r$  of  $\mathcal{A}$ , acting on the Hilbert space  $\mathcal{H}_r = L^2(\mathbb{R}^n)$  and is defined by

$$(\pi_r(f)\xi)(u) = \int f(x, y, r) \bar{e}[\eta_\lambda(r)\beta(u, y)] \xi(u + x) dx dy.$$

Since  $\mathcal{A}$  is a dense subalgebra of our  $C^*$ -algebra  $A$ , we thus obtain all the irreducible representations (up to equivalence) of  $A$  by naturally extending these representations. We will use the same notation,  $\pi_{p,q}$  and  $\pi_r$ , for the representations of  $A$  constructed in this way.

Let us now consider the representations of the  $C^*$ -algebra  $\tilde{A}$ . They are again obtained from representations of the dense subalgebra  $\tilde{\mathcal{A}}$ , which have been identified with the twisted convolution algebra of functions in the  $(x, y, r, w)$  variables (where  $(x, y, w) \in \tilde{H}/Z$  and  $r \in \tilde{\mathfrak{g}}/\mathfrak{z}^\perp$ ) having compact support in the  $r$  and  $w$  variables. We may employ the same argument as above to find (up to equivalence) the irreducible representations of  $\tilde{A}$ . The result is given in the following proposition:

**Proposition 3.3.** *The irreducible representations of  $\tilde{A}$  are obtained by naturally extending the following irreducible representations of the dense subalgebra  $\tilde{\mathcal{A}}$ .*

- For  $s \in \mathbb{R}$ , there is a 1-dimensional representation  $\tilde{\pi}_s$  defined by

$$\tilde{\pi}_s(f) = \int f(x, y, 0, w) \bar{e}(sw) dx dy dw.$$

- For  $(p, q) \in \mathbb{R}^{2n}$ , there is a representation  $\tilde{\pi}_{p,q}$  acting on the Hilbert space  $\tilde{\mathcal{H}}_{p,q} = L^2(\mathbb{R})$  defined by

$$(\tilde{\pi}_{p,q}(f)\zeta)(d) = \int f(x, y, 0, w) \bar{e}(e^d p \cdot x + e^{-d} q \cdot y) \zeta(d + w) dx dy dw.$$

- For  $(r, s) \in \mathbb{R}^2$ , there is a representation  $\tilde{\pi}_{r,s}$  acting on the Hilbert space  $\tilde{\mathcal{H}}_{r,s} = L^2(\mathbb{R}^n)$  defined by

$$(\tilde{\pi}_{r,s}(f)\xi)(u) = \int f(x, y, r, w) \bar{e}(sw) \bar{e}[\eta_\lambda(r)\beta(u, y)] (e^{-\frac{w}{2}})^n \xi(e^{-w}u + e^{-w}x) dx dy dw.$$

We will use the same notation,  $\tilde{\pi}_s$ ,  $\tilde{\pi}_{p,q}$  and  $\tilde{\pi}_{r,s}$ , for the corresponding representations of  $\tilde{A}$ .

PROOF. As before, let us first fix  $r \in \mathfrak{g}/\mathfrak{z}^\perp$ . Look for the irreducible projective representations of  $\tilde{H}/Z$ , with respect to the ( $\mathbb{T}$ -valued) cocycle for  $\tilde{H}/Z$  defined by

$$\tilde{\sigma}^r : ((x, y, w), (x', y', w')) \mapsto \bar{e}[e^{-w}\eta_\lambda(r)\beta(x, y')].$$

To do this, we consider the extension group  $\tilde{E}$  of  $\tilde{H}/Z$ , whose underlying space is  $\tilde{H}/Z \times \mathbb{T}$  and whose multiplication law is given by

$$(x, y, w; \theta)(x', y', w'; \theta') = (x + e^w x', y + e^{-w} y', w + w'; \theta\theta' \bar{e}[e^{-w}\eta_\lambda(r)\beta(x, y')]).$$

Again, all the irreducible representations of  $\tilde{E}$  are obtained by “inducing”. Up to equivalence, they are:

- For each  $s \in \mathbb{R}$ , there exists a 1-dimensional representation  $\tilde{Q}_s$  defined by

$$\tilde{Q}_s(x, y, w; \theta) = \bar{e}(sw).$$

- For each  $(p, q) \in \mathbb{R}^{2n}$ , there exists a representation  $\tilde{Q}_{p,q}$  on  $L^2(\mathbb{R})$  defined by

$$(\tilde{Q}_{p,q}(x, y, w; \theta)\zeta)(d) = \bar{e}(e^d p \cdot x + e^{-d} q \cdot y) \zeta(d + w).$$

- For  $s \in \mathbb{R}$ , there exists an infinite dimensional irreducible representation  $\tilde{Q}_{r,s}$  on  $L^2(\mathbb{R}^n)$  defined by

$$(\tilde{Q}_{r,s}(x, y, w; \theta)\xi)(u) = \theta \bar{e}(sw) \bar{e}[\eta_\lambda(r)\beta(u, y)] (e^{-\frac{w}{2}})^n \xi(e^{-w}u + e^{-w}x).$$

Vary  $r \in \mathfrak{g}/\mathfrak{z}^\perp$  and check the compatibility condition just like (3.5), to find the appropriate representing pairs. Then the integrated form of these pairs will give us the irreducible representations of  $\tilde{\mathcal{A}}$ , which are stated in the proposition.  $\square$

As in the case of Proposition 3.2 (for the  $C^*$ -algebra  $A$ ), we can see clearly the similarity between the result of Proposition 3.3 and the representation theory of the ordinary group  $C^*$ -algebra  $C^*(\tilde{H})$ . Indeed, except when we study later the notion of “inner tensor product representations” (taking advantage of the Hopf structures of  $A$  and  $\tilde{A}$ ), the representation theories of  $A$  and  $\tilde{A}$  are very similar to those of  $C^*(H)$  and  $C^*(\tilde{H})$ .

In this light, it is interesting to observe that the irreducible representations of  $A$  and  $\tilde{A}$  are in one-to-one correspondence with the dressing orbits (calculated in section 2) in  $G$  and  $\tilde{G}$ , respectively. To emphasize the correspondence, we used the same subscripts for the orbits and the related irreducible representations. In this paper, we will point out this correspondence only. However, it is still true (as in the ordinary Lie group representation theory) that orbit analysis sheds some helpful insight into the study of quantum group representations.

In the following, we give a useful result about the irreducible representations of  $\tilde{A}$  and those of  $A$ . This has been motivated by the orbit analysis, and it is an analog of a similar result for the group representations of  $\tilde{H}$  and  $H$ . See the remark following the proposition.

**Definition 2.** Suppose we are given a representation  $\tilde{\pi}$  of  $\tilde{A}$ . Since it is essentially obtained from a representation  $\tilde{Q}$  of  $\tilde{E}$ , we may consider its restriction  $\tilde{Q}|_E$  to  $E$ . Let us denote by  $\tilde{\pi}|_A$  the representation of  $A$  corresponding to the representation  $\tilde{Q}|_E$  of  $E$ . In this sense, we will call  $\tilde{\pi}|_A$  the *restriction* to  $A$  of the representation  $\tilde{\pi}$ .

**Proposition 3.4.** *Let the notation be as above and consider the restriction to  $A$  of the irreducible representations of  $\tilde{A}$ . We then have:*

$$\tilde{\pi}_{r,s}|_A = \pi_r \quad \text{and} \quad \tilde{\pi}_{p,q}|_A = \int_{\mathbb{R}}^{\oplus} \pi_{e^w p, e^{-w} q} dw.$$

Here  $\int_{\mathbb{R}}^{\oplus}$  denotes the direct integral ([6]) of representations.

PROOF. For any  $\xi \in \tilde{\mathcal{H}}_{r,s} = L^2(\mathbb{R}^n)$ , we have:

$$(\tilde{Q}_{r,s}|_E(x, y; \theta)\xi)(u) = \theta \bar{e}[\eta_\lambda(r)\beta(u, y)]\xi(u + x) = (Q_r(x, y; \theta)\xi)(u).$$

It follows that  $\tilde{\pi}_{r,s}|_A = \pi_r$ . Next, for any  $\zeta \in \tilde{\mathcal{H}}_{p,q} = L^2(\mathbb{R})$ ,

$$(\tilde{Q}_{p,q}|_E(x, y; \theta)\zeta)(w) = \bar{e}(e^w p \cdot x + e^{-w} q \cdot y)\zeta(w) = (Q_{e^w p, e^{-w} q}(x, y; \theta)\zeta)(w).$$

By definition of the direct integrals, we thus obtain:

$$\tilde{Q}_{p,q}|_E = \int_{\mathbb{R}}^{\oplus} Q_{e^w p, e^{-w} q} dw.$$

It follows that:  $\tilde{\pi}_{p,q}|_E = \int_{\mathbb{R}}^{\oplus} \pi_{e^w p, e^{-w} q} dw$ . □

*Remark.* This result has to do with the fact that  $E$  is a normal subgroup of  $\tilde{E}$  with codimension 1. Compare this result with Theorem 6.1 of Kirillov’s fundamental paper [13] or the discussion in section 2.5 of [5], where the analysis of coadjoint orbits was used to obtain a similar result for the representations of ordinary Lie

groups. Although we proved this proposition directly, this strongly indicates the possibility of formulating the proposition via generalized (dressing) orbit theory.

So far we have considered only the representations of  $A$  and  $\tilde{A}$ . In this article, we are not going to discuss the corepresentations of  $(A, \Delta)$  (or of  $(\tilde{A}, \tilde{\Delta})$ ). In fact, it turns out that the corepresentation theory is equivalent to the representation theory of the dual group  $G$ . This is a by-product of the Hopf  $C^*$ -algebra duality between  $(A, \Delta)$  and  $(C^*(G), \hat{\Delta})$ , provided by the “regular” multiplicative unitary operator  $U$  associated with  $(A, \Delta)$  (The definition of  $\hat{\Delta}$  depends on  $U$ ). Since this is the case, the corepresentation theory of  $(A, \Delta)$  is actually simpler.

#### 4. INNER TENSOR PRODUCT OF REPRESENTATIONS

Given any two representations of a Hopf  $C^*$ -algebra  $(B, \Delta)$ , we can define their “(inner) tensor product” [7, §10], [4, §5] as in the below. There is also a corresponding notion for corepresentations. But in the present article, we will not consider this dual notion.

**Definition 3.** Let  $\pi$  and  $\rho$  be representations of a Hopf  $C^*$ -algebra  $(B, \Delta)$ , acting on the Hilbert spaces  $\mathcal{H}_\pi$  and  $\mathcal{H}_\rho$ . Then their *inner tensor product*, denoted by  $\pi \boxtimes \rho$ , is a representation of  $B$  on  $\mathcal{H}_\pi \otimes \mathcal{H}_\rho$  defined by

$$(\pi \boxtimes \rho)(b) = (\pi \otimes \rho)(\Delta(b)), \quad b \in B.$$

Here  $\pi \otimes \rho$  denotes the (outer) tensor product of the representations  $\pi$  and  $\rho$ , which is a representation of  $B \otimes B$  naturally extended to  $M(B \otimes B)$ .

As the name suggests, this notion of inner tensor product is a generalization of the inner tensor product of group representations [8]. For instance, in the case of an ordinary group  $C^*$ -algebra  $C^*(G)$  equipped with its cocommutative (symmetric) comultiplication  $\Delta_0$ , Definition 3 is just the integrated form version of the inner tensor product group representations. In this case, since  $\Delta_0$  is cocommutative, the flip  $\sigma : \mathcal{H}_\pi \otimes \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho \otimes \mathcal{H}_\pi$  provides a natural intertwining operator between  $\pi \boxtimes \rho$  and  $\rho \boxtimes \pi$ .

For general (non-cocommutative) Hopf  $C^*$ -algebras, however, this is not necessarily true. In general,  $\pi \boxtimes \rho$  need not even be equivalent to  $\rho \boxtimes \pi$ . Thus for any Hopf  $C^*$ -algebra (or quantum group), it is an interesting question to ask whether two inner tensor product representations  $\pi \boxtimes \rho$  and  $\rho \boxtimes \pi$  are equivalent and if so, what the intertwining unitary operator between them is.

When a Hopf  $C^*$ -algebra is equipped with a certain “quantum universal  $R$ -matrix” ([7], [4], and Definitions 6.1, 6.2 of [11] for the  $C^*$ -algebra version), we

can give positive answers to these questions. The following result is relatively well known:

**Proposition 4.1.** *Let  $(B, \Delta)$  be a Hopf  $C^*$ -algebra. Suppose that there exists a quantum universal  $R$ -matrix  $R \in M(B \otimes B)$  for  $B$ . Then given any two representations  $\pi$  and  $\rho$  of  $B$  on Hilbert spaces  $\mathcal{H}_\pi$  and  $\mathcal{H}_\rho$ , their inner tensor products  $\pi \boxtimes \rho$  and  $\rho \boxtimes \pi$  are equivalent. The equivalence is established by the (unitary) intertwining operator  $T_{\pi\rho} : \mathcal{H}_\pi \otimes \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho \otimes \mathcal{H}_\pi$ , defined by*

$$T_{\pi\rho} = \sigma \circ (\pi \otimes \rho)(R).$$

Here  $\pi \otimes \rho$  is understood as the natural extension to  $M(B \otimes B)$  of the tensor product  $\pi \otimes \rho : B \otimes B \rightarrow \mathcal{B}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$  and  $\sigma$  is the flip. That is,

$$T_{\pi\rho}((\pi \boxtimes \rho)(b)) = ((\rho \boxtimes \pi)(b))T_{\pi\rho}, \quad b \in B.$$

Furthermore, if the  $R$ -matrix is triangular, then we also have:

$$T_{\rho\pi}T_{\pi\rho} = I_{\mathcal{H}_\pi \otimes \mathcal{H}_\rho} \quad \text{and} \quad T_{\pi\rho}T_{\rho\pi} = I_{\mathcal{H}_\rho \otimes \mathcal{H}_\pi}.$$

PROOF. Let us first calculate how  $T_{\pi\rho}$  acts as an operator. If  $\zeta_1 \in \mathcal{H}_\pi$  and  $\zeta_2 \in \mathcal{H}_\rho$ , we have:

$$T_{\pi\rho}(\zeta_1 \otimes \zeta_2) = \sigma \circ ((\pi \otimes \rho)(R))(\zeta_1 \otimes \zeta_2) = ((\rho \otimes \pi)(R_{21}))(\zeta_2 \otimes \zeta_1).$$

Note that we have  $T_{\pi\rho} = ((\rho \otimes \pi)(R_{21})) \circ \sigma$ , as an operator. Here,  $R_{21} = G \circ R$ . To verify that  $T_{\pi\rho}$  is an intertwining operator between  $\pi \boxtimes \rho$  and  $\rho \boxtimes \pi$ , let us consider an element  $b \in B$ . Then

$$\begin{aligned} T_{\pi\rho}((\pi \boxtimes \rho)(b)) &= T_{\pi\rho}((\pi \otimes \rho)(\Delta b)) = ((\rho \otimes \pi)(R_{21}))((\rho \otimes \pi)(\Delta^{\text{op}}b)) \circ \sigma \\ &= ((\rho \otimes \pi)(R_{21}\Delta^{\text{op}}b)) \circ \sigma. \end{aligned}$$

From  $R\Delta(b)R^{-1} = \Delta^{\text{op}}(b)$ , we have:  $R_{21}\Delta^{\text{op}}(b)R_{21}^{-1} = \Delta(b)$ . It follows that

$$\begin{aligned} T_{\pi\rho}((\pi \boxtimes \rho)(b)) &= ((\rho \otimes \pi)(\Delta(b)R_{21})) \circ \sigma = ((\rho \otimes \pi)(\Delta b))((\rho \otimes \pi)(R_{21})) \circ \sigma \\ &= ((\rho \boxtimes \pi)(b))T_{\pi\rho}. \end{aligned}$$

Furthermore, if  $R$  is triangular (so by definition  $\sigma \circ R = R_{21} = R^{-1}$ ), then,

$$\begin{aligned} T_{\rho\pi}T_{\pi\rho}(\zeta_1 \otimes \zeta_2) &= ((\pi \otimes \rho)(R_{21}))((\pi \otimes \rho)(R))(\zeta_1 \otimes \zeta_2) \\ &= ((\pi \otimes \rho)(R_{21}R))(\zeta_1 \otimes \zeta_2) = (\zeta_1 \otimes \zeta_2). \end{aligned}$$

Since this is true for arbitrary  $\zeta_1 \in \mathcal{H}_\pi$  and  $\zeta_2 \in \mathcal{H}_\rho$ , we have:  $T_{\rho\pi}T_{\pi\rho} = I_{\mathcal{H}_\pi \otimes \mathcal{H}_\rho}$ .  $\square$



In [11, §6], we showed that the “extended” Hopf  $C^*$ -algebra  $(\tilde{A}, \tilde{\Delta})$  has a quasitriangular quantum universal  $R$ -matrix  $R \in M(\tilde{A} \otimes \tilde{A})$ . From this the following Corollary is immediate.

**Corollary 4.2.** *For the Hopf  $C^*$ -algebra  $(\tilde{A}, \tilde{\Delta})$ , any two representations  $\tilde{\pi}$  and  $\tilde{\rho}$  of  $\tilde{A}$  will satisfy:*

$$\tilde{\pi} \boxtimes \tilde{\rho} \cong \tilde{\rho} \boxtimes \tilde{\pi}.$$

By Proposition 4.1, the operator  $T_{\tilde{\pi}\tilde{\rho}} = \sigma \circ (\tilde{\pi} \otimes \tilde{\rho})(R)$  is an intertwining operator for this equivalence.

Unlike  $(\tilde{A}, \tilde{\Delta})$ , however, the Hopf  $C^*$ -algebra  $(A, \Delta)$  does not have its own quantum  $R$ -matrix  $R_A \in M(A \otimes A)$ . Even at the classical, Lie bialgebra level (studied in [11, §1]), we can see that the Poisson structures we consider cannot be obtained from any classical  $r$ -matrix. Because of this, the result like the above Corollary is not automatic for  $(A, \Delta)$ . Even so, we plan to show in the below that the representations of  $(A, \Delta)$  still satisfy the quasitriangular type property.

For this purpose and for possible future use, we are going to calculate here the inner tensor product representations of our Hopf  $C^*$ -algebra  $(A, \Delta)$ . Since it is sufficient to consider the inner tensor products of irreducible representations, let us keep the notation of the previous section and let  $\{\pi_{p,q}\}_{(p,q) \in \mathbb{R}^{2n}}$  and  $\{\pi_r\}_{r \in \mathbb{R}}$  be the irreducible representations of  $A$ . Similarly, let  $\{\tilde{\pi}_s\}_{s \in \mathbb{R}}$ ,  $\{\tilde{\pi}_{p,q}\}_{(p,q) \in \mathbb{R}^{2n}}$ ,  $\{\pi_{r,s}\}_{(r,s) \in \mathbb{R}^2}$  be the irreducible representations of  $\tilde{A}$ . For convenience, we will calculate the inner tensor product representations at the level of our dense subalgebra of functions,  $\mathcal{A}$ .

Let  $f \in \mathcal{A}$  and consider  $\Delta f$ . To carry out our calculations, it is convenient to regard  $\Delta f$  also as a continuous function. By using the definition of  $\Delta$  (given in Theorem 3.2 of [11]) and by using Fourier transform purely formally with the Fourier inversion theorem, it is not difficult to realize  $\Delta f$  as a function in the  $(x, y, r)$  variables:

$$\begin{aligned} \Delta f(x, y, r, x', y', r') &= \int f(x', y', r + r') \bar{e}[p \cdot (e^{\lambda r'} x' - x) + q \cdot (e^{\lambda r'} y' - y)] dpdq. \end{aligned}$$

Note that in the  $(p, q, r) (\in G)$  variables, it is just:

$$\Delta f(p, q, r, p', q', r') = f(e^{\lambda r'} p + p', e^{\lambda r'} q + q', r + r'),$$

which more or less reflects the multiplication law on  $G$ .

We can now explicitly calculate the inner tensor products of irreducible representations of  $\mathcal{A}$ , by  $(\pi \boxtimes \rho)(f) = (\pi \otimes \rho)(\Delta f)$ . We first begin with 1-dimensional representations.

**Proposition 4.3.** *For two 1-dimensional representations  $\pi_{p,q}$  and  $\pi_{p',q'}$  of  $A$ , we have:  $\pi_{p,q} \boxtimes \pi_{p',q'} = \pi_{p+p',q+q'}$ . From this, it follows that:*

$$\pi_{p,q} \boxtimes \pi_{p',q'} = \pi_{p+p',q+q'} = \pi_{p',q'} \boxtimes \pi_{p,q}.$$

PROOF. We have for any  $f \in \mathcal{A}$ ,

$$\begin{aligned} (\pi_{p,q} \boxtimes \pi_{p',q'})(f) &= \int f(x', y', 0) \bar{e}[\tilde{p} \cdot (x' - x) + \tilde{q} \cdot (y' - y)] \\ &\quad \bar{e}[p \cdot x + q \cdot y] \bar{e}[p' \cdot x' + q' \cdot y'] d\tilde{p}d\tilde{q}dx'dy' \\ &= \int f(x, y, 0) \bar{e}[(p + p') \cdot x + (q + q') \cdot y] dx'dy' \\ &= \pi_{p+p',q+q'}(f). \end{aligned}$$

□

For other cases involving infinite dimensional (irreducible) representations, the equivalence between the inner tensor products is not so apparent. However, the inner tensor products of  $\pi_{p,q}$  and  $\pi_r$  has a property of being equivalent to the infinite dimensional representation  $\pi_r$  itself. So in this case, equivalence between the inner tensor products follows rather easily.

**Proposition 4.4.** *Consider a 1-dimensional representation  $\pi_{p,q}$  and an infinite dimensional representation  $\pi_r$  of  $A$ . Their inner tensor product is equivalent to the irreducible representation  $\pi_r$ . We thus have:*

$$\pi_r \boxtimes \pi_{p,q} \cong \pi_r \cong \pi_{p,q} \boxtimes \pi_r.$$

PROOF. For  $\xi \in \mathcal{H}_r \otimes \mathcal{H}_{p,q} \cong L^2(\mathbb{R}^n) \otimes \mathbb{C} \cong L^2(\mathbb{R}^n)$  and for  $f \in \mathcal{A}$ , we have:

$$\begin{aligned} ((\pi_r \boxtimes \pi_{p,q})(f)\xi)(u) &= \int f(x', y', r) \bar{e}[\tilde{p} \cdot (x' - x) + \tilde{q} \cdot (y' - y)] \\ &\quad \bar{e}[\eta_\lambda(r)\beta(u, y)] \bar{e}[p \cdot x' + q \cdot y'] \xi(u + x) d\tilde{p}d\tilde{q}dx'dy' \\ &= \int f(x, y, r) \bar{e}[p \cdot x + q \cdot y] \bar{e}[\eta_\lambda(r)\beta(u, y)] \xi(u + x) dx'dy'. \end{aligned}$$

Similarly for  $\xi \in \mathcal{H}_{p,q} \otimes \mathcal{H}_r \cong L^2(\mathbb{R}^n)$ ,

$$\begin{aligned} ((\pi_{p,q} \boxtimes \pi_r)(f)\xi)(u) &= \int f(x', y', r) \bar{e}[\tilde{p} \cdot (e^{\lambda r} x' - x) + \tilde{q} \cdot (e^{\lambda r} y' - y)] \\ &\quad \bar{e}[p \cdot x + q \cdot y] \bar{e}[\eta_\lambda(r)\beta(u, y')] \xi(u + x') d\tilde{p}d\tilde{q}dx'dy' \\ &= \int f(x, y, r) \bar{e}[e^{\lambda r} p \cdot x + e^{\lambda r} q \cdot y] \bar{e}[\eta_\lambda(r)\beta(u, y)] \xi(u + x) dx dy. \end{aligned}$$

Before proving  $\pi_r \boxtimes \pi_{p,q} \cong \pi_{p,q} \boxtimes \pi_r$ , let us show the equivalence  $\pi_r \otimes \pi_{p,q} \cong \pi_r$ . This equivalence is suggested by the corresponding result at the level of Heisenberg Lie group representation theory, which is obtained by using the standard analysis via “characters” [14], [5]. In our case, the equivalence is established by the intertwining operator  $S : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  defined by:  $S\xi(u) = \bar{e}(p \cdot u)\xi\left(u - \frac{q}{\eta_\lambda(r)}\right)$ . Indeed for  $f \in \mathcal{A}$ ,

$$\begin{aligned} S((\pi_r \boxtimes \pi_{p,q})(f))\xi(u) &= \int \bar{e}(p \cdot u) f(x, y, r) \bar{e}[p \cdot x + q \cdot y] \\ &\quad \bar{e}\left[\eta_\lambda(r)\beta\left(u - \frac{q}{\eta_\lambda(r)}, y\right)\right] \xi\left(u - \frac{q}{\eta_\lambda(r)} + x\right) dx dy, \end{aligned}$$

and

$$(\pi_r(f))S\xi(u) = \int f(x, y, r) \bar{e}[\eta_\lambda(r)\beta(u, y)] \bar{e}[p \cdot (u + x)] \xi\left(u + x - \frac{q}{\eta_\lambda(r)}\right) dx dy.$$

We thus have:  $S((\pi_r \boxtimes \pi_{p,q})(f)) = (\pi_r(f))S$ , proving the equivalence:  $\pi_r \boxtimes \pi_{p,q} \cong \pi_r$ . It is easy to check that  $S^{-1}$  gives the intertwining operator for the equivalence:  $\pi_r \cong \pi_r \boxtimes \pi_{p,q}$ .

Meanwhile, from the explicit calculations given at the beginning of the proof, it is apparent that we have:  $\pi_{p,q} \boxtimes \pi_r = \pi_r \boxtimes \pi_{e^{\lambda r} p, e^{\lambda r} q}$ . We thus obtain the equivalence:  $\pi_{p,q} \boxtimes \pi_r \cong \pi_r$ , via the intertwining operator similar to the above  $S$ , replacing  $p$  and  $q$  with  $e^{\lambda r} p$  and  $e^{\lambda r} q$ . Combining these results, we can find the intertwining operator  $T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  between  $\pi_r \boxtimes \pi_{p,q}$  and  $\pi_{p,q} \boxtimes \pi_r$ , obtained by multiplying the respective intertwining operators for the equivalences  $\pi_r \boxtimes \pi_{p,q} \cong \pi_r$  and  $\pi_r \cong \pi_{p,q} \boxtimes \pi_r$ . By straightforward calculation, we have the following expression for  $T$ :

$$T\xi(u) = \bar{e}\left[p \cdot u - e^{\lambda r} p \cdot \left(u - \frac{q}{\eta_\lambda(r)} + \frac{e^{\lambda r} q}{\eta_\lambda(r)}\right)\right] \xi\left(u - \frac{q}{\eta_\lambda(r)} + \frac{e^{\lambda r} q}{\eta_\lambda(r)}\right).$$

It is clear that  $T^{-1}$  gives the intertwining operator between  $\pi_{p,q} \boxtimes \pi_r$  and  $\pi_r \boxtimes \pi_{p,q}$ . □

So far nothing very interesting has happened, in the sense that the results are similar to those of the Heisenberg group representation theory. However, a breakdown of this analogy occurs when we consider inner tensor products of two infinite dimensional representations  $\pi_r$  and  $\pi_{r'}$ . Let us first prove the equivalence of  $\pi_r \boxtimes \pi_{r'}$  and  $\pi_{r'} \boxtimes \pi_r$ .

**Proposition 4.5.** *Consider a pair  $(\pi_r, \pi_{r'})$  of two infinite dimensional irreducible representations of  $A$ . Then we have:*

$$\pi_r \boxtimes \pi_{r'} \cong \pi_{r'} \boxtimes \pi_r,$$

where the equivalence between them is given by the intertwining operator  $T_{\pi_r, \pi_{r'}} : L^2(\mathbb{R}^{2n}) \rightarrow L^2(\mathbb{R}^{2n})$  defined by

$$T_{\pi_r, \pi_{r'}} \xi(v, u) = (e^{-\frac{\lambda r}{2}})^n (e^{-\frac{\lambda r'}{2}})^n \xi(e^{-\lambda r'} u + (e^{\lambda r'} - e^{-\lambda r'}) e^{-\lambda r} v, e^{-\lambda r} v).$$

PROOF. Let  $f \in \mathcal{A}$ . For  $\xi \in \mathcal{H}_r \otimes \mathcal{H}_{r'} \cong L^2(\mathbb{R}^{2n})$ , we have:

$$\begin{aligned} & ((\pi_r \boxtimes \pi_{r'})(f)\xi)(u, v) \\ &= \int f(x', y', r + r') \bar{e}[\tilde{p} \cdot (e^{\lambda r'} x' - x) + \tilde{q} \cdot (e^{\lambda r'} y' - y)] \\ & \quad \bar{e}[\eta_\lambda(r)\beta(u, y)] \bar{e}[\eta_\lambda(r')\beta(v, y')] \xi(u + x, v + x') d\tilde{p}d\tilde{q}dx'dy' \\ &= \int f(x, y, r + r') \bar{e}[\eta_\lambda(r)\beta(u, e^{\lambda r'} y)] \bar{e}[\eta_\lambda(r')\beta(v, y)] \xi(u + e^{\lambda r'} x, v + x) dx dy. \end{aligned}$$

Similarly for  $\xi \in \mathcal{H}_{r'} \otimes \mathcal{H}_r \cong L^2(\mathbb{R}^{2n})$ , by interchanging the roles of  $r$  and  $r'$ ,

$$\begin{aligned} ((\pi_{r'} \boxtimes \pi_r)(f)\xi)(v, u) &= \int f(x, y, r + r') \bar{e}[\eta_\lambda(r')\beta(v, e^{\lambda r} y)] \bar{e}[\eta_\lambda(r)\beta(u, y)] \\ & \quad \xi(v + e^{\lambda r} x, u + x) dx dy. \end{aligned}$$

To prove the equivalence between  $\pi_r \boxtimes \pi_{r'}$  and  $\pi_{r'} \boxtimes \pi_r$ , it is useful to recall the fact that any two representations  $\tilde{\pi}$  and  $\tilde{\rho}$  of the “extended” Hopf  $C^*$ -algebra  $(\tilde{A}, \tilde{\Delta})$  satisfy:  $\tilde{\pi} \boxtimes \tilde{\rho} \cong \tilde{\rho} \boxtimes \tilde{\pi}$ . In particular, we would have:  $\tilde{\pi}_{r,0} \boxtimes \tilde{\pi}_{r',0} \cong \tilde{\pi}_{r',0} \boxtimes \tilde{\pi}_{r,0}$ . Its intertwining operator is:  $T_{\tilde{\pi}_{r,0}, \tilde{\pi}_{r',0}} = \sigma \circ ((\tilde{\pi}_{r,0} \otimes \tilde{\pi}_{r',0})(R))$ , by Corollary 4.2 of Proposition 4.1. By restriction to  $A$  (in the sense of Definition 2), we obtain:

$$(\tilde{\pi}_{r,0}|_A) \boxtimes (\tilde{\pi}_{r',0}|_A) \cong (\tilde{\pi}_{r',0}|_A) \boxtimes (\tilde{\pi}_{r,0}|_A),$$

which, by Proposition 3.4, is just:  $\pi_r \boxtimes \pi_{r'} \cong \pi_{r'} \boxtimes \pi_r$ .

To find the intertwining operator for this equivalence, let us find an explicit expression for the operator  $T_{\tilde{\pi}_{r,0}, \tilde{\pi}_{r',0}}$ . Recall first that by equation (6.3) and Definition 6.3 of [11], the quantum  $R$ -matrix for  $(\tilde{A}, \tilde{\Delta})$  is considered as a continuous

function defined by:

$$R(p, q, r, s, p', q', r', s') = \Phi(p, q, r, s, p', q', r', s')\Phi'(p, q, r, s, p', q', r', s') \\ = \bar{e}[\lambda(rs' + r's)]\bar{e}[2\lambda(e^{-\lambda r'})p \cdot q'].$$

We then calculate  $(\tilde{\pi}_{r,0} \otimes \tilde{\pi}_{r',0})(R)$  as an operator on  $\tilde{\mathcal{H}}_{r,0} \otimes \tilde{\mathcal{H}}_{r',0} \cong L^2(\mathbb{R}^{2n})$ . By a straightforward calculation, we obtain for  $\xi \in L^2(\mathbb{R}^{2n})$ ,

$$((\tilde{\pi}_{r,0} \otimes \tilde{\pi}_{r',0})(\Phi)\xi)(u, v) \\ = \int \bar{e}[\lambda(rs' + r's)]e[p \cdot x + q \cdot y + sw + p' \cdot x' + q' \cdot y' + s'w'] \\ (e^{-\frac{w}{2}})^n (e^{-\frac{w'}{2}})^n \xi(e^{-w}u + e^{-w}x, e^{-w'}v + e^{-w'}x') \\ \bar{e}[\eta_\lambda(r)\beta(u, y)]\bar{e}[\eta_\lambda(r')\beta(v, y')] dpdqdsdp'dq'ds'dxdydw dx'dy'dw' \\ = (e^{-\frac{\lambda r}{2}})^n (e^{-\frac{\lambda r'}{2}})^n \xi(e^{-\lambda r'}u, e^{-\lambda r}v).$$

Similarly,

$$((\tilde{\pi}_{r,0} \otimes \tilde{\pi}_{r',0})(\Psi)\xi)(u, v) = \xi(u + 2\lambda e^{-\lambda r'}\eta_\lambda(r')v, v).$$

Since  $\eta_\lambda(r') = \frac{e^{2\lambda r'} - 1}{2\lambda}$ , we thus have:

$$((\tilde{\pi}_{r,0} \otimes \tilde{\pi}_{r',0})(R)\xi)(u, v) = ((\tilde{\pi}_{r,0} \otimes \tilde{\pi}_{r',0})(\Phi))((\tilde{\pi}_{r,0} \otimes \tilde{\pi}_{r',0})(\Psi))\xi(u, v) \\ = (e^{-\frac{\lambda r}{2}})^n (e^{-\frac{\lambda r'}{2}})^n \xi(e^{-\lambda r'}u + (e^{\lambda r'} - e^{-\lambda r'})e^{-\lambda r}v, e^{-\lambda r}v).$$

By applying the flip  $\sigma$ , we therefore obtain:

$$T_{\tilde{\pi}_{r,0}\tilde{\pi}_{r',0}}\xi(v, u) = \sigma \circ ((\tilde{\pi}_{r,0} \otimes \tilde{\pi}_{r',0})(R))\xi(v, u) \\ = (e^{-\frac{\lambda r}{2}})^n (e^{-\frac{\lambda r'}{2}})^n \xi(e^{-\lambda r'}u + (e^{\lambda r'} - e^{-\lambda r'})e^{-\lambda r}v, e^{-\lambda r}v).$$

Define  $T_{\pi_r\pi_{r'}}$  by  $T_{\pi_r\pi_{r'}} = T_{\tilde{\pi}_{r,0}\tilde{\pi}_{r',0}}$ . Then it is a straightforward calculation to show that  $T_{\pi_r\pi_{r'}}$  is an intertwining operator between  $\pi_r \boxtimes \pi_{r'}$  and  $\pi_{r'} \boxtimes \pi_r$ . For  $f \in \mathcal{A}$  and  $\xi \in L^2(\mathbb{R}^{2n})$ , we have:

$$T_{\pi_r\pi_{r'}}((\pi_r \boxtimes \pi_{r'})(f))\xi(v, u) \\ = \int f(x, y, r + r')\bar{e}[\eta_\lambda(r)\beta(e^{-\lambda r'}u + (e^{\lambda r'} - e^{-\lambda r'})e^{-\lambda r}v, e^{\lambda r'}y)]\bar{e}[\eta_\lambda(r')\beta(e^{-\lambda r}v, y)] \\ (e^{-\frac{\lambda r}{2}})^n (e^{-\frac{\lambda r'}{2}})^n \xi(e^{-\lambda r'}u + (e^{\lambda r'} - e^{-\lambda r'})e^{-\lambda r}v + e^{\lambda r'}x, e^{-\lambda r}v + x) dx dy \\ = ((\pi_{r'} \boxtimes \pi_r)(f))T_{\pi_r\pi_{r'}}\xi(v, u).$$

Since  $\xi$  is arbitrary, it follows that:

$$T_{\pi_r\pi_{r'}}((\pi_r \boxtimes \pi_{r'})(f)) = ((\pi_{r'} \boxtimes \pi_r)(f))T_{\pi_r\pi_{r'}}.$$

□

Observe that by interchanging  $r$  and  $r'$ , we are able to find the expression for the intertwining operator  $T_{\pi_{r'}\pi_r}$ :

$$T_{\pi_{r'}\pi_r}\xi(u, v) = (e^{-\frac{\lambda r}{2}})^n (e^{-\frac{\lambda r'}{2}})^n \xi(e^{-\lambda r}v + (e^{\lambda r} - e^{-\lambda r})e^{-\lambda r'}u, e^{-\lambda r'}u).$$

For  $f \in \mathcal{A}$ , we will have:  $T_{\pi_{r'}\pi_r}((\pi_{r'} \boxtimes \pi_r)(f)) = ((\pi_r \boxtimes \pi_{r'})(f))T_{\pi_{r'}\pi_r}$ .

However, note that unlike in the ordinary group representation theory or in the cases equipped with “triangular” quantum  $R$ -matrices, we no longer have:  $T_{\pi_{r'}\pi_r}T_{\pi_r\pi_{r'}} = I$ . We instead have:

$$T_{\pi_{r'}\pi_r}T_{\pi_r\pi_{r'}}\xi(u, v) = (e^{-\lambda r})^n (e^{\lambda r'})^n \xi(u - (1 - e^{-2\lambda r'})e^{-2\lambda r}u + (e^{\lambda r'} - e^{-\lambda r'})e^{-2\lambda r}v, e^{-2\lambda r}v + (1 - e^{-2\lambda r})e^{-\lambda r'}u),$$

which is clearly not the identity operator. Let us summarize our results in the following theorem:

**Theorem 4.6.** *Given any two representations  $\pi$  and  $\rho$  (acting on the Hilbert spaces  $\mathcal{H}_\pi$  and  $\mathcal{H}_\rho$ ) of the Hopf  $C^*$ -algebra  $(A, \Delta)$ , their inner tensor products “commutes” (i. e.  $\pi \boxtimes \rho$  and  $\rho \boxtimes \pi$  are equivalent). However, the intertwining operators  $T_{\pi\rho}$  and  $T_{\rho\pi}$  are in general not inverses of each other (i. e.  $T_{\rho\pi}T_{\pi\rho} \neq I_{\mathcal{H}_\pi \otimes \mathcal{H}_\rho}$ ).*

This theorem means that the category of representations of  $(A, \Delta)$  is essentially a “quasitriangular monoidal category” (See [4, §15] or [2]). In our case, it is interesting to point out that  $(A, \Delta)$  possesses the quasitriangular type property without the existence of its own quantum  $R$ -matrix  $R_A$ . Meanwhile, since there have been only a handful of examples so far of non-compact,  $C^*$ -algebraic quantum groups possessing the property of quasitriangularity, having these examples  $(A, \Delta)$  and  $(\tilde{A}, \tilde{\Delta})$  would benefit the study of non-compact quantum groups and its development.

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