DEFORMATION QUANTIZATION OF CERTAIN NON-LINEAR POISSON STRUCTURES

BYUNG–JAY KAHNG

ABSTRACT. As a generalization of the linear Poisson bracket on the dual space of a Lie algebra, we introduce certain non-linear Poisson brackets which are "cocycle perturbations" of the linear Poisson bracket. We show that these special Poisson brackets are equivalent to Poisson brackets of central extension type, which resemble the central extensions of an ordinary Lie bracket via Lie algebra cocycles. We are able to formulate (strict) deformation quantizations of these Poisson brackets by means of twisted group C^*–algebras. We also indicate that these deformation quantizations can be used to construct some specific non-compact quantum groups.

INTRODUCTION. Let M be a Poisson manifold. Consider $C^\infty(M)$, the commutative algebra under pointwise multiplication of smooth functions on M. We attempt to deform the pointwise product of smooth functions into a noncommutative product, with respect to a parameter $\hbar$, such that the direction of the deformation is given by the Poisson bracket on M. This problem of finding a *deformation quantization* of M ([33], [1]) is actually a problem dating back to the early days of quantum mechanics [34], [20].

We are particularly interested in the settings where the deformed product of functions is again a function—in contrast to much of the literature on the subject involving formal power series, or the so-called "star products". In this direction, Rieffel has been developing the notion of "strict" deformation quantization, in the C^*–algebra framework [26, 29]. Here, in addition to the requirement that the deformed product of functions is again a function, the deformed algebra is required to have an involution and a C^*–norm. By using the C^*–algebra framework, one gains the advantage of being able to keep the topological and geometric aspects of the given manifold while we perform the quantization.

Let $\mathfrak{h}$ be a (finite dimensional) Lie algebra. It is well-known [35] that the Lie algebra structure on $\mathfrak{h}$ defines a natural Poisson bracket on the

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dual vector space $\mathfrak{h}^*$ of $\mathfrak{h}$, which is called a *linear Poisson bracket*. This Poisson bracket is also called the "Lie–Poisson bracket", to emphasize the fact that it actually was already known to Lie. In [27] Rieffel showed that given the linear Poisson bracket on $\mathfrak{h}^*$, a deformation quantization is provided by the convolution algebra structure on the simply connected Lie group H corresponding to $\mathfrak{h}$. In particular, when $\mathfrak{h}$ is a nilpotent Lie algebra, this is shown to be a strict deformation quantization.

In this paper, we wish to generalize the above situation and to include twisted group convolution algebras into the framework of deformation quantization. We first define a class of Poisson brackets on the dual vector space of a Lie algebra, which contains the linear Poisson bracket as a special case. These Poisson brackets can actually be realized as "central extensions" of the linear Poisson bracket. We then show that twisted group convolution algebras provide deformation quantizations of these Poisson brackets. We obtain strict deformation quantizations when the Lie algebra is nilpotent.

In addition to its interest as a generalization of the deformation quantization problem into the non-linear situation, this result has a nice application to quantum groups. Quantum groups [13], [7] are usually obtained by suitably "deforming" ordinary Lie groups, and as suggested by Drinfeld [13], we expect to obtain quantum groups by deformation quantization of the so-called Poisson–Lie groups [19]. In some cases, the compatible Poisson brackets on the Poisson–Lie groups are shown to be of our special type, in which case we can apply the result of this paper to obtain (strict) deformation quantizations of them.

This enables us to construct some specific non-compact quantum groups. Not only have we actually been able to show [16] that some of the earlier known examples of non-compact quantum groups [28], [31], [11], [18] are obtained in this way, but we also obtain a new class of non-compact quantum groups [15]. Although the method of construction may seem rather naive, our new example is shown to satisfy some interesting properties, including the "quasitriangular" property. We will discuss our construction of quantum groups in a separate paper.

This paper is organized as follows. In the first section, we review the definitions of Poisson brackets and the formulation of (strict) deformation quantization. We also include a discussion on twisted group algebras, which are the main objects of our study. In the second section, we define our special class of non-linear Poisson brackets, as motivated by the central extension of ordinary Lie brackets. We show in the third section that certain twisted group C^*–algebras provide strict deformation quantizations of these special Poisson brackets. We use some

non-trivial results on twisted group C^*–algebras obtained by Packer and Raeburn [21, 22]. We restrict our study to the strict deformation quantization case, but some indications for generalization are also briefly mentioned.

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1. Preliminaries

Let M be a C^∞ manifold, and let $C^\infty(M)$ be the algebra of complex-valued C^∞ functions on M. It is a commutative algebra under pointwise multiplication, and is equipped with an involution given by complex conjugation.

Definition 1.1. By a *Poisson bracket* on M, we mean a skew, bilinear map $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that the following holds:

- $\{\ ,\ \}$ defines a Lie algebra structure on $C^\infty(M)$.
  (i.e. the bracket satisfies the Jacobi identity.)
- (Leibniz rule): $\{f, gh\} = \{f, g\}h + g\{f, h\}$, for $f, g, h \in C^\infty(M)$.

We also require that the Poisson bracket be real, in the sense that $\{f^*, g^*\} = \{f, g\}^*$. A manifold M equipped with such a bracket is called a *Poisson manifold*, and $C^\infty(M)$ is a *Poisson (*–)algebra*.

The deformation quantization will take place in $C^\infty(M)$ (or to allow non-compact M, in $C^\infty_\infty(M)$, which is the space of C^∞ functions vanishing at infinity). The Poisson bracket on M gives the direction of the deformation. Let us formulate the following definition for deformation quantization, which is the "strict" deformation quantization proposed by Rieffel [26]. Depending on the situations, we may also be interested in some other subalgebras of $C^\infty(M)$, on which the deformation takes place. So the definition is formulated at the level of an arbitrary (dense) *–subalgebra $\mathcal{A}$, for example the algebra of Schwartz functions.

Definition 1.2. Let M be a Poisson manifold as above. Let $\mathcal{A}$ be a dense *–subalgebra (with respect to the C^*–norm $\|\ \|_\infty$) of $C_\infty(M)$, on which $\{\ ,\ \}$ is defined with values in $\mathcal{A}$. By a *strict deformation quantization* of $\mathcal{A}$ in the direction of $\{\ ,\ \}$, we mean an open interval I of real numbers containing 0, together with a triple $(\times_\hbar, *^\hbar, \|\ \|_\hbar)$ for each $\hbar \in I$, of an associative product, an involution, and a C^*–norm (for $\times_\hbar$ and $*^\hbar$) on $\mathcal{A}$, such that

(1) For $\hbar = 0$, the operations $\times_\hbar, {}^{*\hbar}, \| \ \|_\hbar$ are the original pointwise product, involution (complex conjugation), and C^*–norm (i. e. sup-norm $\| \ \|_\infty$) on $\mathcal{A}$, respectively.

(2) The completed C^*-algebras $A_\hbar$ form a "continuous field" of C^*–algebras (In particular, the map $\hbar \mapsto \|f\|_\hbar$ is continuous for any $f \in \mathcal{A}$.).

(3) For any $f, g \in \mathcal{A}$,

$$\left\| \frac{f \times_\hbar g - g \times_\hbar f}{i\hbar} - \{f, g\} \right\|_\hbar \to 0$$

as $\hbar \to 0$.

Our main result below will show that certain "twisted group C^*–algebras" provide strict deformation quantizations. Let us briefly discuss about these algebras, mainly to set up our notation. We give here only those results that are needed in later sections. For more discussion on the subject, we refer the reader to the articles by Zeller-Meier [37] (when the group is discrete) and by Busby and Smith [5]. There are also many other recent articles on these algebras including [21, 22, 23], which contain considerably deeper results. To avoid technical pathology, we assume that the groups we consider are discrete or second countable locally compact (e. g. Lie groups), and the C^*–algebras are always separable.

Definition 1.3. ([25]) Let G be a locally compact group with left Haar measure dx and modular function Δ_G. Let G act on a C^*–algebra A and let us denote this action by $\alpha : G \to \mathrm{Aut}(A)$. We assume that α is strongly continuous. Also let $UZM(A)$ be the group of unitary elements in the center of the multiplier algebra, $M(A)$, of A. By a *continuous field* over N of α–*cocycles* of G, where N is a locally compact space, we will mean a function σ on $G \times G \times N$ with values in $UZM(A)$ such that

- If we fix $r \in N$, then σ is a normalized α–cocycle on G. That is,

$$\big(\alpha_x \sigma(y, z; r)\big)\sigma(x, yz; r) = \sigma(x, y; r)\sigma(xy, z; r)$$

and

$$\sigma(x, e; r) = \sigma(e, x; r) = 1$$

for $x, y, z \in G$, where e is the identity element of G.

- If we fix $x, y \in G$, then σ is continuous on N.
- For any $f \in C_\infty(N, A)$ the function

$$(x, y) \mapsto f(\cdot)\sigma(x, y; \cdot)$$

from $G \times G$ to $C_\infty(N, A)$ is Bochner measurable.

For convenience, let us denote by $\sigma^r, r \in N$ the ordinary group cocycle on G defined by $\sigma^r(x, y) = \sigma(x, y; r)$. Corresponding to the continuous field of cocycles $\sigma : r \mapsto \sigma^r$, we can define [25] the twisted convolution and involution on $L^1(G, C_\infty(N, A))$. We define, for $\phi, \psi \in L^1(G, C_\infty(N, A))$:

$$(\phi *_\sigma \psi)(x; r) = \int_G \phi(y; r)\alpha_y(\psi(y^{-1}x; r))\sigma^r(y, y^{-1}x)\, dy$$

and

$$\phi^*(x; r) = \alpha_x(\phi(x^{-1}; r)^*)\sigma^r(x, x^{-1})^*\Delta_G(x^{-1}).$$

We thus obtain a Banach *–algebra. Let us denote this algebra by $L^1(G, N, A, \sigma)$, with the group action α to be understood. We also define $C^*(G, N, A, \sigma)$, the enveloping C^*–algebra of $L^1(G, N, A, \sigma)$. There are also the notions of induced representations and regular representations [5], [37], [25]. So we may as well define the reduced C^*–algebra $C_r^*(G, N, A, \sigma)$. All these are more or less straightforward.

Compare this definition with the definition given in [5], where the twisted group convolution algebra has been formulated via a single cocycle. Nevertheless, the present definition is no different from the usual one, since we can regard σ also as a single cocycle, taking values in $UZM(C_\infty(N, A))$. The present formulation is useful when we study the continuity problem of the fields of C^*–algebras consisting of twisted group C^*–algebras, by varying the cocycles. Recall that the continuous field property is essential in the definition of the strict deformation quantization (Definition 1.2).

Using the universal property of full C^*–algebras, and also taking advantage of the property of the reduced C^*–algebras that one is able to work with their specific representations, Rieffel in [25] gave an answer to the problem of the continuity of the field of C^*–algebras $\{C^*(G, A, \sigma^r)\}_{r \in N}$, as follows. Here $C^*(G, A, \sigma^r)$ is the twisted group C^*–algebra in the usual sense of [5].

Theorem 1.4. *Let G, A, α be understood as above. Let σ be a continuous field over N of α–cocycles on G. Then*

- *The field $\{C^*(G, A, \sigma^r)\}$ over N is upper semi-continuous.*
- *$\{C_r^*(G, A, \sigma^r)\}$ over N is lower semi-continuous.*
- *Thus, if each (G, A, α, σ^r) satisfies the "amenability condition", i. e. $C_r^*(G, A, \alpha, \sigma^r) = C^*(G, A, \alpha, \sigma^r)$, then it follows that the field of C^*–algebras $\{C^*(G, A, \sigma^r)\}_{r \in N}$ is continuous.*

For the proof of the theorem and the related questions, we will refer the reader to [25], and the references therein. Note that by replacing A with $C_\infty(N, A)$ and by introducing a new base space, we may even

consider a continuous field of the twisted group C^*–algebras given by the cocycles of continuous field type (Definition 1.3).

The C^*–algebra $C^*(G, N, A, \sigma)$ may be regarded as a C^*–algebra of "cross sections" of the continuous field $\{C^*(G, A, \sigma^r)\}_{r \in N}$. It is called the C^*–*algebra of sections of a* C^*–*bundle* by Packer and Raeburn [22] (Compare this terminology with Fell's notion of "C^*–algebraic bundles" [14], which is considerably more general than is needed for our present purposes.). Actually in [22], the continuity problem of twisted group C^*–algebras allowing both the cocycle and the action to vary continuously has been studied in terms of the aforementioned notion of section C^*–algebra of a C^*–bundle. Taking a related viewpoint, Blanchard in [3] has recently developed a framework for a general continuous field of C^*–algebras in terms of "$C_\infty(X)$–algebras", where X in our case is the locally compact base space N. A $C_\infty(X)$–algebra is a certain C^*–algebra having a $C(X)$ module structure. See [3].

We conclude this section by quoting (without proof) a couple of deep theorems of Packer and Raeburn [21, 22] on the structure of twisted group C^*–algebras. We tried to keep Packer and Raeburn's notation and terminology. Although some of them are different from our notation, they are clear enough to understand. For example, $A \times_{\alpha,u} G$ denotes the twisted group C^*–algebra (or "twisted crossed product") $C^*(G, A, \alpha, u)$. All this and more can be found in [21, 22]. These theorems will be used later in the proof of our Theorem 3.4, which is our main result.

Theorem 1.5. *([21]) (Decomposition of twisted crossed products) Suppose that (A, G, α, u) is a separable twisted dynamical system and N is a closed normal subgroup of G. There exists a canonically determined twisted action (β, v) of G/N on $A \times_{\alpha,u} N$ such that:*

$$A \times_{\alpha,u} G \cong (A \times_{\alpha,u} N) \times_{\beta,v} G/N.$$

The next theorem is about the continuity of a field of twisted group C^*–algebras. Compare this with Theorem 1.4, where we considered the continuity problem only when the twisting cocycle is varying. Meanwhile, note in the theorem that G is assumed to be amenable (So by the "stabilization trick" of Packer and Raeburn [21], the amenability condition always holds for any quadruple (G, A, α, u).).

Theorem 1.6. *([22]) Suppose A is the C^*–algebra of sections of a separable C^*–bundle over a locally compact space X, and (α, u) is a twisted action of an amenable locally compact group G on A such that each ideal $I_x = \{a \in A : a(x) = 0\}$ is invariant. Then for each $x \in X$, there is a natural twisted action $(\alpha(x), u(x))$ on the quotient A/I_x, and*

$A \times_{\alpha,u} G$ *is the* C^**–algebra of sections of a* C^**–bundle over* X *with fibers isomorphic to* $(A/I_x) \times_{\alpha(x),u(x)} G$.

2. The non-linear Poisson bracket

Let us begin by trying to characterize the special Poisson brackets that will allow twisted group algebras to be deformation quantizations of them. Recall that the twisting of the convolution algebra structure in a twisted group algebra is given by group (2–)cocycles. Meanwhile any group cocycle for a locally compact group G having values in an abelian group N can be canonically associated with a central extension of G by N, and actually all central extensions are essentially obtained in this way [14].

Since it is known [27] that ordinary group convolution algebras can be regarded as deformation quantizations of linear Poisson brackets, the above observations suggest that twisted group algebras will provide deformation quantizations of certain Poisson brackets which are, in a loose sense, "central extensions" of linear Poisson brackets. Although we have to make clear what we mean by this last statement, this is the main motivation behind the definition of our special type of Poisson bracket formulated below.

Let $\mathfrak{h}$ be a (finite–dimensional) Lie algebra and let us denote by $\mathfrak{g} = \mathfrak{h}^*$ its dual vector space. As usual, we will denote the dual pairing between $\mathfrak{h}$ and $\mathfrak{g}$ by $\langle \; , \; \rangle$. We will have to require later that $\mathfrak{h}$ is a nilpotent or an exponential solvable Lie algebra because of some technical reasons to be discussed below, but for the time being we allow $\mathfrak{h}$ to be a general Lie algebra. Recall [35] that we define the linear Poisson bracket on the dual vector space $\mathfrak{g} = \mathfrak{h}^*$ by

$$(2.1) \qquad \{\phi, \psi\}_{\mathrm{lin}}(\mu) = \big\langle [d\phi(\mu), d\psi(\mu)], \mu \big\rangle$$

where $\phi, \psi \in C^\infty(\mathfrak{g})$ and $\mu \in \mathfrak{g}$. Here $d\phi(\mu)$ and $d\psi(\mu)$ has been naturally realized as elements in $\mathfrak{h}$.

We wish to define a generalization of this Poisson bracket by allowing a suitable "perturbation" of the right-hand side of equation (2.1). This will be done via a certain Lie algebra 2–cocycle on $\mathfrak{h}$, denoted by Ω, having values in $C^\infty(\mathfrak{g})$. That is, we will consider the Poisson brackets of the form:

$$(2.2) \qquad \{\phi, \psi\}(\mu) = \big\langle [d\phi(\mu), d\psi(\mu)], \mu \big\rangle + \Omega\big(d\phi(\mu), d\psi(\mu); \mu\big).$$

As above, we regard $d\phi(\mu)$ and $d\psi(\mu)$ as elements in $\mathfrak{h}$.

Compare equation (2.2) with the definition of the linear Poisson bracket. In the linear Poisson bracket case, the Lie bracket takes values in $\mathfrak{h}$, the elements of which can be regarded as (linear) functions

contained in $C^\infty(\mathfrak{g})$, via the dual pairing. That is, the right-hand side of equation (2.1) can be viewed as the evaluation at $\mu \in \mathfrak{g}$ of a C^∞–function, $[X, Y] \in \mathfrak{h} \subseteq C^\infty(\mathfrak{g})$, where $X = d\phi(\mu)$ and $Y = d\psi(\mu)$. In the "perturbed" case, the right-hand side of equation (2.2) may be viewed as the evaluation at $\mu \in \mathfrak{g}$ of a C^∞–function, $[X, Y] + \Omega(X, Y) \in C^\infty(\mathfrak{g})$, where $X = d\phi(\mu)$ and $Y = d\psi(\mu)$. So to make sense of the Poisson brackets of the type given by equation (2.2), we will first study the "perturbation" of the Lie bracket on $\mathfrak{h}$ by a cocycle. Later, we will find some additional conditions for the cocycle Ω such that the equation (2.2) indeed gives a well-defined Poisson bracket on $\mathfrak{g}$.

Let V be a $U(\mathfrak{h})$–module, possibly infinite dimensional. Consider a *2–cocycle* Ω for $\mathfrak{h}$ having values in V. It is a skew-symmetric, bilinear map from $\mathfrak{h} \times \mathfrak{h}$ into V such that $d\Omega = 0$ (For more discussion on cohomology of Lie algebras, see the standard textbooks on the subject [6, §5], [17].). When V is further viewed as an abelian Lie algebra, the space $\mathfrak{h} \oplus V$ can be given a Lie algebra structure [4], [17] which becomes a *central extension* Lie algebra of $\mathfrak{h}$ by V:

$$\big[(X, v), (Y, w)\big]_{\mathfrak{h} \oplus V} = \big([X, Y], X \cdot w - Y \cdot v + \Omega(X, Y)\big)$$

for $X, Y \in \mathfrak{h}$ and $v, w \in V$. Here the dot denotes the module action. In particular, when V is assumed to be a trivial $U(\mathfrak{h})$–module, we have:

$$(2.3) \qquad \big[(X, v), (Y, w)\big]_{\mathfrak{h} \oplus V} = \big([X, Y], \Omega(X, Y)\big).$$

Let us slightly modify this "central extension" picture as follows, so that we are able to consider a Lie bracket on $\mathfrak{h} + V$, where we now allow $\mathfrak{h} \cap V \neq 0$ in general. Clearly, $\mathfrak{h} \cap V$ is a subspace of $\mathfrak{h}$. However, since $\mathfrak{h}$ is already equipped with its given Lie bracket and since V will be assumed to be an abelian Lie algebra, it is only reasonable to consider the case in which $\mathfrak{h} \cap V$ is an abelian subalgebra of $\mathfrak{h}$. For simplicity, we will further assume that $\mathfrak{h} \cap V$ is a central subalgebra of $\mathfrak{h}$, which means that V is a trivial $U(\mathfrak{h})$–module. Let us denote this central subalgebra by $\mathfrak{z}$. Without loss of generality, we may assume that $\mathfrak{z}$ is the center of $\mathfrak{h}$. In this case, we just replace V by an extended abelian Lie algebra, still denoted by V, satisfying $\mathfrak{h} \cap V = \mathfrak{z}$.

Let us look for a trivial $U(\mathfrak{h})$–module V, which we will view as an abelian Lie algebra, such that $\mathfrak{h} \cap V = \mathfrak{z}$ is the center of $\mathfrak{h}$. Since we eventually want to define a V–valued cocycle for $\mathfrak{h}$, from which we construct a bracket operation on $C^\infty(\mathfrak{g})$, we also require that V is contained in $C^\infty(\mathfrak{g})$. So let us consider the subspace $\mathfrak{q} = \mathfrak{z}^\perp$ of $\mathfrak{g}$, and choose as our V the following:

$$V = C^\infty(\mathfrak{g}/\mathfrak{q}) \subseteq C^\infty(\mathfrak{g}).$$

Here the functions in $V = C^\infty(\mathfrak{g}/\mathfrak{q})$ have been realized as functions in $C^\infty(\mathfrak{g})$, by the "pull-back" using the natural projection $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{q}$.

Since any $X \in \mathfrak{h}$ can be regarded as a linear function contained in $C^\infty(\mathfrak{g})$ via the dual pairing, we can see easily that $X \in \mathfrak{h} \cap V \subseteq C^\infty(\mathfrak{g})$ if and only if $\langle X, \mu + \nu \rangle = \langle X, \mu \rangle$ for all $\mu \in \mathfrak{g}, \nu \in \mathfrak{q}$. It follows immediately that $\mathfrak{h} \cap V = \mathfrak{z}$. On the other hand, consider the representation ad^*. For any $X \in \mathfrak{h}$ and any $\mu \in \mathfrak{g}$, we have $\mathrm{ad}^*_{-X}(\mu) = \nu \in \mathfrak{q}$, since for any $Y \in \mathfrak{z}$, we have $\langle Y, \nu \rangle = \langle Y, \mathrm{ad}^*_{-X}(\mu) \rangle = \langle [X, Y], \mu \rangle = 0$. It follows that

$$\mathrm{ad}^*_X(f)(\mu) = f\big(\mathrm{ad}^*_{-X}(\mu)\big) = f(\nu) = 0,$$

for any $f \in V = C^\infty(\mathfrak{g}/\mathfrak{q})$. By the natural extension of ad^* to $U(\mathfrak{h})$, we can give V the trivial $U(\mathfrak{h})$–module structure.

Remark. When $\mathfrak{h}$ has a trivial center, the space V will be just $\{0\}$. To avoid this problem, we could have considered $C^\infty(\mathfrak{g})^H$, the space of $\mathrm{Ad}^* H$–invariant C^∞ functions on $\mathfrak{g}$. It is always nonempty (It contains the so-called Casimir elements [32].). It also satisfies $\mathfrak{h} \cap C^\infty(\mathfrak{g})^H = \mathfrak{z}$ and can be given the trivial $U(\mathfrak{h})$–module structure. However, it does not satisfy the following property (Lemma 2.1), which we need later when we define our Poisson bracket. For this reason, we choose our V as it is defined above. At least for nilpotent $\mathfrak{h}$, which is the case we are going to study most of the time, this is less of a problem since $\mathfrak{h}$ has a non-trivial center.

Lemma 2.1. *Let $V \subseteq C^\infty(\mathfrak{g})$ be defined as above. Then for any function $\chi \in V$ and for any $\mu \in \mathfrak{g}$, we have: $d\chi(\mu) \in \mathfrak{z}$.*

Proof. Since $V \subseteq C^\infty(\mathfrak{g})$, it follows that $d\chi(\mu) \in \mathfrak{h}$. Recall that $d\chi(\mu)$ defines a linear functional on $\mathfrak{g} = \mathfrak{h}^*$ by

$$\big\langle d\chi(\mu), \nu \big\rangle = \left(\frac{d}{dt} \right)\bigg|_{t=0} \chi(\mu + t\nu).$$

To see if $d\chi$ is contained in $\mathfrak{z}$, suppose $\nu \in \mathfrak{q} = \mathfrak{z}^\perp$. Since $\chi \in V = C^\infty(\mathfrak{g}/\mathfrak{q})$, we know that $\chi(\mu + \nu) = \chi(\mu)$, for all $\nu \in \mathfrak{q}$. Therefore, the above expression becomes:

$$\big\langle d\chi(\mu), \nu \big\rangle = \left(\frac{d}{dt} \right)\bigg|_{t=0} \chi(\mu) = 0.$$

Since $\nu \in \mathfrak{q}$ is arbitrary, we thus have: $d\chi(\mu) \in \mathfrak{z}$. $\qquad\square$

Let us now turn to the discussion of defining a (perturbed) bracket operation on $\mathfrak{h}+V$, which will enable us to formulate our special Poisson bracket on $C^\infty(\mathfrak{g})$. Since $\mathfrak{h}$ and V are subspaces of $\mathfrak{h} + V$, there exists a (linear) surjective map, $\mathfrak{h} \to (\mathfrak{h} + V)/V$, whose kernel is $\mathfrak{h} \cap V = \mathfrak{z}$. We

thus obtain a vector space isomorphism, in a canonical way, between $(\mathfrak{h} + V)/V$ and $\mathfrak{h}/\mathfrak{z}$. The map from $\mathfrak{h} + V$ onto $\mathfrak{h}/\mathfrak{z}$ is a canonical one, which extends the canonical projection of $\mathfrak{h}$ onto $\mathfrak{h}/\mathfrak{z}$. Therefore, it is reasonable to consider a cocycle for $\mathfrak{h}/\mathfrak{z}$ having values in V (viewed as a trivial $U(\mathfrak{h}/\mathfrak{z})$–module) and use it to define a bracket operation on $\mathfrak{h} + V$. Let Ω be such a cocycle for $\mathfrak{h}/\mathfrak{z}$.

Remark. Note that in this setting, the cocycle Ω can naturally be identified with a cocycle $\tilde{\Omega}$ for $\mathfrak{h}$ having values in V (considered as a trivial $U(\mathfrak{h})$–module), satisfying the following "centrality condition":

$$(2.4) \qquad\qquad \tilde{\Omega}(Z, Y) = \tilde{\Omega}(Y, Z) = 0,$$

for any $Z \in \mathfrak{z}$ and any $Y \in \mathfrak{h}$. In fact, we may define $\tilde{\Omega}$ as $\tilde{\Omega}(X, Y) = \Omega(\dot{X}, \dot{Y})$, where $\dot{X}$ denotes the image in $\mathfrak{h}/\mathfrak{z}$ of X under the canonical projection. For this reason, we will from time to time use the same notation, Ω, to denote both Ω and $\tilde{\Omega}$.

By viewing Ω as a cocycle for $\mathfrak{h}$, we can define, as in equation (2.3), a Lie bracket on $\mathfrak{h} \oplus V$:

$$\big[(X, v), (Y, w)\big]_{\mathfrak{h}\oplus V} = \big([X, Y], \Omega(X, Y)\big).$$

To define a bracket operation on $\mathfrak{h} + V$, consider the natural surjective map from $\mathfrak{h} \oplus V$ onto $\mathfrak{h} + V$, whose kernel is:

$$\delta = \{(Z, -Z) : Z \in \mathfrak{z}\} \subseteq \mathfrak{h} \oplus V.$$

Since δ is clearly central with respect to the Lie bracket $[\ ,\ ]_{\mathfrak{h}\oplus V}$ given above, it is an ideal. Therefore, $\mathfrak{h} + V = (\mathfrak{h} \oplus V)/\delta$ is a Lie algebra. The Lie bracket on it is given by:

$$(2.5)\ \ [X + v, Y + w]_{\mathfrak{h}+V} = [X, Y] + \Omega(X, Y), \qquad X, Y \in \mathfrak{h}, \quad v, w \in V$$

which is the given Lie bracket on $\mathfrak{h}$ plus a cocycle term. In this sense, equation (2.5) may be considered as a "perturbed Lie bracket" of the given Lie bracket on $\mathfrak{h}$. Compare this with equation (2.2), where the linear Poisson bracket (given by the Lie bracket on $\mathfrak{h}$) is "perturbed" by a certain cocycle Ω.

Using the observation given above as motivation, let us define more rigorously our Poisson bracket on $\mathfrak{g} = \mathfrak{h}^*$. This is, in fact, a "cocycle perturbation" of $\{\ ,\ \}_{\mathrm{lin}}$ on $\mathfrak{g}$.

Theorem 2.2. *Let $\mathfrak{h}$ be a Lie algebra with center $\mathfrak{z}$ and let $\mathfrak{g} = \mathfrak{h}^*$ be the dual vector space of $\mathfrak{h}$. Consider the vector space $V = C^\infty(\mathfrak{g}/\mathfrak{q}) \subseteq C^\infty(\mathfrak{g})$ as above, where $\mathfrak{q} = \mathfrak{z}^\perp$. Let us give V the trivial $U(\mathfrak{h})$–module structure. Let Ω be a Lie algebra 2–cocycle for $\mathfrak{h}$ having values in V,*

satisfying the centrality condition. That is, Ω is a skew-symmetric, bilinear map from $\mathfrak{h} \times \mathfrak{h}$ into V such that:

$$\Omega\big(X, [Y, Z]\big) + \Omega\big(Y, [Z, X]\big) + \Omega\big(Z, [X, Y]\big) = 0, \qquad X, Y, Z \in \mathfrak{h}$$

satisfying: $\Omega(Z, Y) = \Omega(Y, Z) = 0$ for $Z \in \mathfrak{z}$ and any $Y \in \mathfrak{h}$. Then the bracket operation $\{\ ,\ \}_\Omega : C^\infty(\mathfrak{g}) \times C^\infty(\mathfrak{g}) \to C^\infty(\mathfrak{g})$ defined by

$$\{\phi, \psi\}_\Omega(\mu) = \big\langle [d\phi(\mu), d\psi(\mu)], \mu \big\rangle + \Omega\big(d\phi(\mu), d\psi(\mu); \mu\big)$$

is a Poisson bracket on $\mathfrak{g}$.

Remark. If we denote $d\phi(\mu)$ and $d\psi(\mu)$ by X and Y, as elements in $\mathfrak{h}$, the right-hand side of the definition of the Poisson bracket may be viewed as the evaluation at $\mu \in \mathfrak{g}$ of a C^∞–function, $[X, Y] + \Omega(X, Y) \in \mathfrak{h} + V \in C^\infty(\mathfrak{g})$. Note that this expression is just the Lie bracket on $\mathfrak{h} + V$ defined earlier by equation (2.5).

Proof. Since $d\phi(\mu)$ and $d\psi(\mu)$ can be naturally viewed as elements in $\mathfrak{h}$, it is easy to see that $\{\ ,\ \}_\Omega$ is indeed a map from $C^\infty(\mathfrak{g}) \times C^\infty(\mathfrak{g})$ into $C^\infty(\mathfrak{g})$. The skew-symmetry and bilinearity are clear.

To verify the Jacobi identity, consider the functions ϕ_1, ϕ_2, ϕ_3 in $C^\infty(\mathfrak{g})$. We may write $\{\phi_2, \phi_3\}_\Omega$ as:

$$\{\phi_2, \phi_3\}_\Omega(\mu) = \{\phi_2, \phi_3\}_{\mathrm{lin}}(\mu) + \chi(\mu),$$

where χ is a function in V. We therefore have:

$$d\big(\{\phi_2, \phi_3\}_\Omega\big)(\mu) = \big[d\phi_2(\mu), d\phi_3(\mu)\big] + d\chi(\mu).$$

The first term in the right hand side is the differential of the linear Poisson bracket, which is rather well known [35]. Moreover, since $\chi \in V$, it follows from Lemma 2.1 that $d\chi(\mu) \in \mathfrak{z}$, which is "central" with respect to both $[\ ,\ ]$ and Ω. We thus have:

$$\big\{\phi_1, \{\phi_2, \phi_3\}_\Omega\big\}_\Omega(\mu)$$
$$= \big\langle \big[d\phi_1(\mu), d(\{\phi_2, \phi_3\}_\Omega)(\mu)\big], \mu \big\rangle + \Omega\big(d\phi_1(\mu), d(\{\phi_2, \phi_3\}_\Omega)(\mu); \mu\big)$$
$$= \big\langle \big[d\phi_1(\mu), [d\phi_2(\mu), d\phi_3(\mu)]\big], \mu \big\rangle + \Omega\big(d\phi_1(\mu), [d\phi_2(\mu), d\phi_3(\mu)]; \mu\big),$$

and similarly for $\big\{\phi_2, \{\phi_3, \phi_1\}_\Omega\big\}_\Omega$ and $\big\{\phi_3, \{\phi_1, \phi_2\}_\Omega\big\}_\Omega$. So the Jacobi identity for $\{\ ,\ \}_\Omega$ follows from that of the Lie bracket $[\ ,\ ]$ and the cocycle identity for Ω. That is,

$$\big\{\phi_1, \{\phi_2, \phi_3\}_\Omega\big\}_\Omega(\mu) + \big\{\phi_2, \{\phi_3, \phi_1\}_\Omega\big\}_\Omega(\mu) + \big\{\phi_3, \{\phi_1, \phi_2\}_\Omega\big\}_\Omega(\mu) = 0.$$

Finally, since $d(\phi\psi) = (d\phi)\psi + \phi(d\psi)$ for any $\phi, \psi \in C^\infty(\mathfrak{g})$, the Leibniz rule for the bracket is also clear. $\qquad\square$

Remark. This is the special type of Poisson bracket we will work with from now on. The linear Poisson bracket $\{\ ,\ \}_{\mathrm{lin}}$ on $\mathfrak{g} = \mathfrak{h}^*$ is clearly of this type, since it corresponds to the case when the cocycle Ω is trivial. Meanwhile when Ω is a scalar-valued cocycle, we obtain the so-called *affine Poisson bracket* [2], [30]. Affine Poisson structures occur naturally in the study of symplectic actions of Lie groups with general (not necessarily equivariant for the coadjoint action) moment mappings. The notion of affine Poisson brackets has generalization also to the groupoid level. See [12] or [36].

Suppose we are given a Poisson bracket of our special type, $\{\ ,\ \}_{\Omega}$, on $\mathfrak{g} = \mathfrak{h}^*$. To discuss its (strict) deformation quantization, it is useful to observe that $\{\ ,\ \}_{\Omega}$ can be viewed as a "central extension" of the linear Poisson bracket on the dual vector space of the Lie algebra $\mathfrak{h}/\mathfrak{z}$. This actually follows from the fact that the Lie bracket $[\ ,\ ]_{\mathfrak{h}+V}$ given by equation (2.5) can be transferred to a Lie bracket on $\mathfrak{h}/\mathfrak{z} \oplus V$, which turns out to be a central extension of the Lie bracket on $\mathfrak{h}/\mathfrak{z}$. Let us make this observation more precise.

Consider the exact sequence of Lie algebras,

$$0 \to \mathfrak{z} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\rho} \mathfrak{h}/\mathfrak{z} \to 0$$

such that ι and ρ are the injection and the quotient map, respectively. Let us fix a linear map $\tau : \mathfrak{h}/\mathfrak{z} \to \mathfrak{h}$ such that $\rho\tau = \mathrm{id}$. In this case, the exactness implies that the map

$$(2.6) \qquad \omega_0 : (x, y) \mapsto \iota^{-1}\big([\tau(x), \tau(y)] - \tau([x, y]_{\mathfrak{h}/\mathfrak{z}})\big)$$

is well-defined from $\mathfrak{h}/\mathfrak{z} \times \mathfrak{h}/\mathfrak{z}$ into $\mathfrak{z}$, and it is actually a Lie algebra cocycle for $\mathfrak{h}/\mathfrak{z}$ having values in $\mathfrak{z}$. See [4], [17]. Then the Lie bracket on $\mathfrak{h}$ can be written as follows:

$$(2.7) \quad [X, Y] = \tau\big([\rho(X), \rho(Y)]_{\mathfrak{h}/\mathfrak{z}}\big) + \iota\big(\omega_0(\rho(X), \rho(Y))\big), \qquad X, Y \in \mathfrak{h}.$$

Since we have been regarding the center $\mathfrak{z}$ as a subalgebra of $\mathfrak{h}$ such that $\mathfrak{h} \cap V = \mathfrak{z}$, we may ignore the map ι and view $\mathfrak{z}$ and its image in $\mathfrak{h}$ or V as the same. Then τ is actually the map that determines the vector space isomorphism between $\mathfrak{h}/\mathfrak{z} \oplus V$ and $\mathfrak{h} + V$. Under this isomorphism and by using equation (2.7), the Lie bracket $[\ ,\ ]_{\mathfrak{h}+V}$ of equation (2.5) is transferred to a Lie bracket on $\mathfrak{h}/\mathfrak{z} \oplus V$ defined by:
$$(2.8)$$
$$\big[(x, v), (y, w)\big]_{\mathfrak{h}/\mathfrak{z} \oplus V} = [x, y]_{\mathfrak{h}/\mathfrak{z}} + \omega_0(x, y) + \Omega(x, y) = [x, y]_{\mathfrak{h}/\mathfrak{z}} + \omega(x, y).$$

Here ω_0 is the cocycle defined in equation (2.6) and we regarded Ω as a cocycle for $\mathfrak{h}/\mathfrak{z}$, as assured by an earlier remark. For convenience, we introduced a new (V–valued) cocycle ω for $\mathfrak{h}/\mathfrak{z}$, as a sum of the two cocycles ω_0 and Ω. Then it is clear that equation (2.8) defines a central

extension of the Lie bracket $[\,,\,]_{\mathfrak{h}/\mathfrak{z}}$, where the extension is given by the cocycle ω. We can now define a Poisson bracket on $\mathfrak{g}$ modeled after this central extension type Lie bracket.

Theorem 2.3. *Let $\mathfrak{h}$ be a Lie algebra with center $\mathfrak{z}$ and let us fix the maps ρ and τ given above. Let $\mathfrak{g} = \mathfrak{h}^*$. Consider the vector space $V \subseteq C^\infty(\mathfrak{g})$ defined above and let us give V the trivial $U(\mathfrak{h}/\mathfrak{z})$–module structure. Suppose ω is a Lie algebra cocycle for $\mathfrak{h}/\mathfrak{z}$ having values in V. Then the bracket operation $\{\,,\,\}_\omega : C^\infty(\mathfrak{g}) \times C^\infty(\mathfrak{g}) \to C^\infty(\mathfrak{g})$ defined by*

$$\{\phi, \psi\}_\omega(\mu) = \left\langle \tau([d\dot{\phi}(\mu), d\dot{\psi}(\mu)]_{\mathfrak{h}/\mathfrak{z}}), \mu \right\rangle + \omega\left(d\dot{\phi}(\mu), d\dot{\psi}(\mu); \mu\right)$$

is a Poisson bracket on $\mathfrak{g}$. Here $\dot{X}$ denotes the image of X under the canonical projection ρ of $\mathfrak{h}$ onto $\mathfrak{h}/\mathfrak{z}$.

Proof. Define $\Omega : \mathfrak{h}/\mathfrak{z} \times \mathfrak{h}/\mathfrak{z} \to V$ by

(2.9) $$\Omega(x, y) = \omega(x, y) - \omega_0(x, y),$$

where ω_0 is the cocycle for $\mathfrak{h}/\mathfrak{z}$ defined in equation (2.6). Then the discussion in the previous paragraph implies that the bracket $\{\,,\,\}_\omega$ is equivalent to the Poisson bracket $\{\,,\,\}_\Omega$ given in Theorem 2.2. Therefore, it is clearly a Poisson bracket on $\mathfrak{g}$. $\qquad\square$

Although the present formulation depends on the choice of the map τ and hence is not canonical, this Poisson bracket is, by construction, equivalent to the canonical Poisson bracket given in Theorem 2.2. The relationship between them is given by equation (2.9). In particular, if we consider the cocycle ω_0 of equation (2.6) in place of ω, so that Ω is trivial, we obtain the linear Poisson bracket $\{\,,\,\}_{\text{lin}}$ on $\mathfrak{g}$. Therefore, to find a (strict) deformation quantization of our Poisson bracket $\{\,,\,\}_\Omega$ of Theorem 2.2, we may as well try to find a (strict) deformation quantization of the central extension type Poisson bracket $\{\,,\,\}_\omega$. This change in our point of view is useful when we work with specific examples, where the choice of coordinates are usually apparent.

3. Twisted group C^*–algebras as deformation quantizations

As we mentioned earlier, we expect that twisted group C^*–algebras will be deformation quantizations of the Poisson brackets of "central extension" type. These are in fact the special type of Poisson brackets we defined in the previous section. A more canonical description has been given in Theorem 2.2, while an equivalent, "central extension" type description has been given in Theorem 2.3.

Let us from now on consider the Poisson bracket $\{\ ,\ \}_\omega$ on $\mathfrak{g} = \mathfrak{h}^*$, as defined in Theorem 2.3. For convenience, we will fix the map $\tau : \mathfrak{h}/\mathfrak{z} \to \mathfrak{h}$ and identify $\mathfrak{h}/\mathfrak{z}$ with its image $\tau(\mathfrak{h}/\mathfrak{z}) \subseteq \mathfrak{h}$ under τ. To find a deformation quantization of $\{\ ,\ \}_\omega$, we will look for a group (2–)cocycle, σ, for the Lie group H/Z of $\mathfrak{h}/\mathfrak{z}$, corresponding to the Lie algebra cocycle ω. Then we will form a twisted group C^*–algebra of H/Z with σ, which we will show below will give us a strict deformation quantization of $C^\infty(\mathfrak{g})$ in the direction of $\{\ ,\ \}_\omega$. By the equivalence of the Poisson brackets $\{\ ,\ \}_\omega$ and $\{\ ,\ \}_\Omega$, this may also be interpreted as giving a strict deformation quantization of $C^\infty(\mathfrak{g})$ in the direction of $\{\ ,\ \}_\Omega$. This result will be a generalization of the result by Rieffel [27] saying that an ordinary group C^*–algebra $C^*(H)$ provides a deformation quantization of $C^\infty(\mathfrak{h}^*)$ in the direction of the linear Poisson bracket on $\mathfrak{h}^*$.

Recall that the cocycle ω provides a Lie bracket on the space $\mathfrak{h}/\mathfrak{z} \oplus V$. If we restrict this Lie bracket to $\mathfrak{h}/\mathfrak{z}$, we obtain the map $[\ ,\ ]_\omega : \mathfrak{h}/\mathfrak{z} \times \mathfrak{h}/\mathfrak{z} \to \mathfrak{h}/\mathfrak{z} \oplus V$ defined by:

$$(3.1) \qquad [x, y]_\omega = [x, y]_{\mathfrak{h}/\mathfrak{z}} + \omega(x, y).$$

For the time being, to make our book keeping simpler, let us denote by $\mathfrak{k}$ and K the Lie algebra $\mathfrak{h}/\mathfrak{z}$ and its Lie group H/Z. We now try to construct a group-like structure corresponding to $[\ ,\ ]_\omega$. From equation (3.1), we expect to obtain a cocycle extension of the Lie group $K = H/Z$ via a certain group cocycle corresponding to ω. As a first step, let us consider the following Baker–Campbell–Hausdorff series for $\mathfrak{k} \oplus V$, ignoring the convergence problem for the moment. Define

$$S(X,Y) = X+Y+\frac{1}{2}[X,Y]_{\mathfrak{k}\oplus V}+\frac{1}{12}[X,[X,Y]_{\mathfrak{k}\oplus V}]_{\mathfrak{k}\oplus V}+\frac{1}{12}[Y,[Y,X]_{\mathfrak{k}\oplus V}]_{\mathfrak{k}\oplus V}+\dots$$

for $X, Y \in \mathfrak{k} \oplus V$. Let us also define $S_\hbar$ by $S_\hbar(X,Y) = \frac{1}{\hbar}S(\hbar X, \hbar Y)$ for $\hbar \neq 0$ in $\mathbb{R}$. For $\hbar = 0$, we let $S_0(X,Y) = X + Y$.

Lemma 3.1. *Let $\hbar \in \mathbb{R}$ be fixed and let $S_\hbar$ be as above. Then we have, at least formally (ignoring the convergence problem),*

$$S_\hbar\big(X, S_\hbar(Y, Z)\big) = S_\hbar\big(S_\hbar(X, Y), Z\big)$$
$$S_\hbar(X, -X) = 0, \quad S_\hbar(X, 0) = S_\hbar(0, X) = X$$

for $X, Y, Z \in \mathfrak{k} \oplus V$.

Proof. Since $(X,Y) \mapsto \frac{1}{\hbar}[\hbar X, \hbar Y]_{\mathfrak{k}\oplus V} = \hbar[X,Y]_{\mathfrak{k}\oplus V}$ is a Lie bracket, the above property of the Baker–Campbell–Hausdorff series is a standard result in Lie algebra theory [4], [32]. $\square$

Note that when the cocycle ω is trivial, the map

$$S_\hbar(x, y) = x *_\hbar y, \qquad x, y \in \mathfrak{k}$$

is an associative multiplication defined locally in a neighborhood of $(0, 0)$, on which the series converges [32]. This becomes a globally well-defined group multiplication on $\mathfrak{k}$ when $\mathfrak{k}$ is an exponential solvable Lie algebra. In general when the cocycle ω is nontrivial, the convergence problem of the series $S_\hbar$ is not as simple because we are allowing V to be an infinite dimensional vector space. Unless we have more knowledge about the Lie algebra and the cocycle, we cannot avoid this rather serious problem. But let us postpone the discussion of the convergence problem a while longer and consider, purely formally, a restriction of the series $S_\hbar$ to $\mathfrak{k} \times \mathfrak{k}$. Let us write:

(3.2) $$S_\hbar(x, y) = x *_\hbar y + R_\hbar(x, y)$$

where $x, y \in \mathfrak{k}$ and $x *_\hbar y$ is the "group multiplication" on $\mathfrak{k}$ defined as above. Since $[x, y] \in \mathfrak{k}$ and since $[x, y]_\omega - [x, y] = \omega(x, y)$ lies in V which is central, it is clear that $R_\hbar(x, y) \in V$, if it converges, and this is the term carrying all the information on the twisting of the Lie bracket. It turns out that $R_\hbar(\ ,\ )$ is a "group cocycle" for $(\mathfrak{k}, *_\hbar)$ having values in V.

Proposition 3.2. *Let the notation be as above. Then the map $R_\hbar$ is a "group cocycle" for $(\mathfrak{k}, *_\hbar)$ having values in the additive abelian group V. That is, it satisfies, at least formally, the following conditions for a normalized group cocycle:*

$$R_\hbar(y, z) + R_\hbar(x, y *_\hbar z) = R_\hbar(x, y) + R_\hbar(x *_\hbar y, z)$$
$$R_\hbar(x, -x) = 0, \quad R_\hbar(x, 0) = R_\hbar(0, x) = 0$$

for $x, y, z \in \mathfrak{k}$.

Proof. From Lemma 3.1, we know that

$$S_\hbar\big(x, S_\hbar(y, z)\big) = S_\hbar\big(S_\hbar(x, y), z\big), \qquad x, y, z \in \mathfrak{k}.$$

If we rewrite both sides, since $R_\hbar(\ ,\ )$ is central, we have:

$(\text{LHS}) = S_\hbar\big(x, y *_\hbar z + R_\hbar(y, z)\big) = x *_\hbar (y *_\hbar z) + R_\hbar(y, z) + R_\hbar(x, y *_\hbar z),$

$(\text{RHS}) = S_\hbar\big(x *_\hbar y + R_\hbar(x, y), z\big) = (x *_\hbar y) *_\hbar z + R_\hbar(x, y) + R_\hbar(x *_\hbar y, z).$

So we have the following equation:

$$R_\hbar(y, z) + R_\hbar(x, y *_\hbar z) = R_\hbar(x, y) + R_\hbar(x *_\hbar y, z).$$

Also from the lemma, we have:

$$R_\hbar(x, -x) = 0, \quad R_\hbar(x, 0) = R_\hbar(0, x) = 0.$$

$\square$

As we mentioned above, this proposition does not make much sense unless we clear up the convergence problem of the series $S_\hbar$. Note that the "multiplication", $*_\hbar$, on $\mathfrak{k}$ is only locally defined. The definition of $R_\hbar(\ ,\ )$ is even more difficult because of the fact that it takes values in an infinite dimensional vector space V. Fortunately, in some special cases, these convergence problems do become simpler. Let us mention a few here.

When the cocycle is known to take values in a finite dimensional subspace W of V, the space $\mathfrak{k} + W$ becomes a finite dimensional Lie algebra such that the convergence of $S_\hbar$ is obtained at least locally in a neighborhood of 0. But this case is rather uninteresting, because the cocycle extension becomes just another (finite dimensional) Lie algebra. The corresponding Poisson bracket obtained as in Theorem 2.3 is just the linear Poisson bracket on the dual space of this extended Lie algebra. In particular, when $\omega = \omega_0$, the series $S_\hbar$ becomes just the Baker–Campbell–Hausdorff series for the Lie algebra $\mathfrak{h}/\mathfrak{z} + \mathfrak{z} = \mathfrak{h}$ and the corresponding Poisson bracket is the linear Poisson bracket on $\mathfrak{h}^*$.

Meanwhile, when the Lie algebra $\mathfrak{k}$ is nilpotent, the series $S_\hbar$ becomes a finite series and hence always converges, whether or not the cocycle ω takes values in an infinite dimensional vector space. In particular, $x *_\hbar y$ and $R_\hbar(x,y)$ can be defined for any $x, y \in \mathfrak{k}$.

Despite this attention to detail which we have to make, there are still possibilities for generalization. There are some special cases of exponential solvable Lie algebras that do not fall into one of these cases but whose convergence problem (at least locally) can still be managed. Meanwhile in a formal power series setting, since the convergence problem becomes less crucial, the result of the above proposition is still valid for any Lie algebra. But in these general settings, we no longer expect to obtain "strict deformation quantizations" as we do below. For this, some generalized notion of a group cocycle, in a "local" sense, needs to be developed. This search for a correct, weaker notion of deformation quantizations, will be postponed as a future project.

Since we wish to establish a "strict" deformation quantization (in the sense of Definition 1.2) of our special type of a Poisson bracket, we will consider the case when the Lie algebra $\mathfrak{k}$ is nilpotent. So from now on, let us assume that the Lie algebra $\mathfrak{h}$ is nilpotent. Then $\mathfrak{k} = \mathfrak{h}/\mathfrak{z}$ also becomes nilpotent. Let us choose and fix a basis for $\mathfrak{h}$ (for example, we can take the "Malcev basis" [10]) such that elements of $\mathfrak{z}$ can be written as $z = (0, z)$ and elements of $\mathfrak{k} = \mathfrak{h}/\mathfrak{z}$ can be written as $x = (x, 0)$.

Recall that we have defined our V as the space $C^\infty(\mathfrak{g}/\mathfrak{q})$, where $\mathfrak{q}$ is the subspace $\mathfrak{q} = \mathfrak{z}^\perp$ of $\mathfrak{g}$.

By Proposition 3.2, we obtain a group cocyle $R_\hbar$ for the nilpotent Lie group $K_\hbar = (\mathfrak{h}/\mathfrak{z}, *_\hbar)$. Here and from now on, if $\mathfrak{g}$ is a Lie algebra with the corresponding (simply connected) Lie group G, we will denote by $G_\hbar$ for the (simply connected) Lie group corresponding to $\mathfrak{g}$ whose Lie bracket is given by $\hbar[\, , \,]$. Since $R_\hbar$ is a continuous function–valued cocycle having values in $V = C^\infty(\mathfrak{g}/\mathfrak{q})$, it is more convenient to introduce instead the following (continuous) family of ordinary $\mathbb{T}$–valued cocycles, $r \mapsto \sigma_\hbar^r$. Below and through out the rest of the paper, $e(t)$ denotes the function $\exp\big[(2\pi i)t\big]$. Also $\bar{e}(t) = \exp\big[(-2\pi i)t\big]$. The proof of the following proposition is immediate from Proposition 3.2.

Proposition 3.3. *For a fixed $r \in \mathfrak{g}/\mathfrak{q}$, define the map $\sigma_\hbar^r : \mathfrak{h}/\mathfrak{z} \times \mathfrak{h}/\mathfrak{z} \to \mathbb{T}$ by*

$$\sigma_\hbar^r(x, y) = \bar{e}\big[R_\hbar(x, y; r)\big] = \exp\big[(-2\pi i) R_\hbar(x, y; r)\big]$$

where $R_\hbar(x, y; r)$ is the evaluation at r of $R_\hbar(x, y) \in C^\infty(\mathfrak{g}/\mathfrak{q})$. Then $\sigma_\hbar^r$ is a smooth normalized group cocycle for $K_\hbar$ having values in $\mathbb{T}$. That is,

$$\sigma_\hbar^\mu(y, z)\sigma_\hbar^\mu(x, y *_\hbar z) = \sigma_\hbar^\mu(x, y)\sigma_\hbar^\mu(x *_\hbar y, z)$$
$$\sigma_\hbar^\mu(x, 0) = \sigma_\hbar^\mu(0, x) = 1$$

for $x, y, z \in K_\hbar = (\mathfrak{h}/\mathfrak{z}, *_\hbar)$. Moreover, if we fix $x, y \in K_\hbar$, then $\mu \mapsto \sigma_\hbar^\mu(x, y)$ is a C^∞–function from $\mathfrak{g}/\mathfrak{q}$ into $\mathbb{T}$.

Remark. Since the functions in $V = C^\infty(\mathfrak{g}/\mathfrak{q})$ are invariant under the coadjoint action of H, by an observation made earlier, they are also invariant under the coadjoint action of $K_\hbar$. Thus the cocycle condition of the proposition can also be interpreted as the condition for a *normalized α–cocycle* in Definition 1.3, where α in this case is the coadjoint action of $K_\hbar$. Although this interpretation is not directly needed in our discussion below, this still suggests a possibility of future generalization.

Since $\sigma_\hbar : r \to \sigma_\hbar^r$ is a continuous field of normalized $\mathbb{T}$–cocycles (Definition 1.3) for the Lie group $K_\hbar$, it follows that we can, as in section 1, define a twisted convolution algebra $L^1\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q})\big)$. For f, g in $L^1\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q})\big)$, we have:

$$(3.3) \qquad (f *_{\sigma_\hbar} g)(y; r) = \int_{\mathfrak{h}/\mathfrak{z}} f(x; r)g(x^{-1} *_\hbar y; r)\sigma_\hbar^r(x, x^{-1} *_\hbar y) \, dx.$$

Here dx denotes the left Haar measure for the (nilpotent) Lie group $K_\hbar = (\mathfrak{h}/\mathfrak{z}, *_\hbar)$, which is just a fixed Lebesgue measure for the underlying vector space $\mathfrak{k} = \mathfrak{h}/\mathfrak{z}$. We will show below that as $\hbar$ approaches

0, the family of twisted convolution algebras $\left\{L^1\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q})\big)\right\}_{\hbar\in\mathbb{R}}$ provides a deformation quantization of our Poisson bracket $\{\ ,\ \}_\omega$ on $\mathfrak{g}$. Since we prefer to find a deformation at the level of continuous functions on $\mathfrak{g}$, we need to develop suitable machinary.

Choose a Lebesgue measure, dX, on $\mathfrak{h}$ and Lebesgue measures, dx and dz, on $\mathfrak{h}/\mathfrak{z}$ and $\mathfrak{z}$ respectively, such that we have: $dX = dxdz$. These Lebesgue measures will be Haar measures for the (nilpotent) Lie groups corresponding to the Lie algebras $\mathfrak{h}$, $\mathfrak{h}/\mathfrak{z}$, and $\mathfrak{z}$. In particular, Haar measure for $K_\hbar = (\mathfrak{h}/\mathfrak{z}, *_\hbar)$ is a Lebesgue measure dx for $\mathfrak{h}/\mathfrak{z}$. Meanwhile, we may write the dual vector space $\mathfrak{g} = \mathfrak{h}^*$ as $\mathfrak{g} = \mathfrak{q} \oplus (\mathfrak{g}/\mathfrak{q})$, a direct product of subspaces, where $\mathfrak{q} = \mathfrak{z}^\perp$. By elementary linear algebra, we can realize $\mathfrak{q}$ and $\mathfrak{g}/\mathfrak{q}$ as dual vector spaces of $\mathfrak{k} = \mathfrak{h}/\mathfrak{z}$ and $\mathfrak{z}$, respectively. Therefore, we are able to choose dual (Plancherel) measures, dq and dr, for $\mathfrak{q}$ and $\mathfrak{g}/\mathfrak{q}$ such that $d\mu = dqdr$ becomes a Plancherel measure for $\mathfrak{g} = \mathfrak{h}^*$. All these measures are essentially Lebesgue measures.

We then define the Fourier transform, $\mathcal{F}$, between the spaces of Schwartz functions $S(\mathfrak{h})$ and $S(\mathfrak{g})$ by

$$(\mathcal{F}f)(\mu) = \int_{\mathfrak{h}} f(X)\bar{e}\big[\langle X, \mu\rangle\big]\,dX, \qquad f \in S(\mathfrak{h})$$

and the inverse Fourier transform, $\mathcal{F}^{-1}$, from $S(\mathfrak{g})$ to $S(\mathfrak{h})$ by

$$(\mathcal{F}^{-1}\phi)(X) = \int_{\mathfrak{g}} \phi(\mu)e\big[\langle X, \mu\rangle\big]\,d\mu, \qquad \phi \in S(\mathfrak{g}).$$

Here again, $e(t) = \exp\big[(2\pi i)t\big]$ and $\bar{e}(t) = \exp\big[(-2\pi i)t\big]$. Our choice of the Plancherel measure means that we have $\mathcal{F}^{-1}(\mathcal{F}f) = f$ for all $f \in S(\mathfrak{h})$ and $\mathcal{F}(\mathcal{F}^{-1}\phi) = \phi$ for all $\phi \in S(\mathfrak{g})$. This is the Fourier inversion theorem. Let us also define the partial Fourier transform, $^\wedge$, from $S(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ to $S(\mathfrak{g}) = S(\mathfrak{q} \times \mathfrak{g}/\mathfrak{q})$ by

$$f^\wedge(q; r) = \int_{\mathfrak{h}/\mathfrak{z}} f(x; r)\bar{e}\big[\langle x, q\rangle\big]\,dx.$$

Its inverse Fourier transform $^\vee$ is similarly defined from $S(\mathfrak{g})$ to $S(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ by replacing $\bar{e}$ with e in the definition. Again, we have the Fourier inversion theorem: $(f^\wedge)^\vee = f$ for all $f \in S(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ and $(\phi^\vee)^\wedge = \phi$ for all $\phi \in S(\mathfrak{g})$.

We are now ready to state and prove our main theorem. We show that given our Poisson bracket $\{\ ,\ \}_\omega$ on $\mathfrak{g}$, its strict deformation quantization is essentially given by a family of twisted group C^*–algebras.

Theorem 3.4. *Let $\mathfrak{h}$ be a nilpotent Lie algebra. Let the notation be as above and let ω be a Lie algebra cocycle for $\mathfrak{h}/\mathfrak{z}$ having values in*

$V = C^\infty(\mathfrak{g}/\mathfrak{q})$. *Suppose that $\mathfrak{g} = \mathfrak{h}^*$ is equipped with a Poisson bracket of our special type, $\{\ ,\ \}_\omega$, as in Theorem 2.3. Then there exists a dense (with respect to the usual $\|\ \|_\infty$ norm) subspace, $\mathcal{A} \subseteq S(\mathfrak{g})$ of $C_\infty(\mathfrak{g})$ such that for a fixed $\hbar \in \mathbb{R}$, the following operation, $\times_\hbar$, defined by:*

$$(3.4) \qquad (\phi \times_\hbar \psi) = (\phi^\vee *_{\sigma_\hbar} \psi^\vee)^\wedge, \qquad \phi, \psi \in \mathcal{A}$$

is a well-defined multiplication on $\mathcal{A}$. Here $*_{\sigma_\hbar}$ is the twisted convolution defined in equation (3.3). Moreover, the following properties hold:

- *We can define a suitable involution, $*_\hbar$, and a C^*–norm, $\|\ \|_\hbar$, on $\mathcal{A}$ such that the C^*–completion of $(\mathcal{A}, \times_\hbar, *_\hbar)$ with respect to $\|\ \|_\hbar$ defines a C^*–algebra $A_\hbar$.*
- *For $\hbar \in \mathbb{R}$, the C^*–algebras $A_\hbar$ form a continuous field of C^*–algebras.*
- *$(\mathcal{A}, \times_\hbar, *_\hbar, \|\ \|_\hbar)_{\hbar \in \mathbb{R}}$ is a strict deformation quantization of $\mathcal{A} \subseteq C^\infty(\mathfrak{g})$ in the direction of the Poisson bracket $(1/2\pi)\{\ ,\ \}_\omega$ on $\mathfrak{g}$. In particular, we have:*

$$(3.5) \qquad \left\| \frac{\phi \times_\hbar \psi - \psi \times_\hbar \phi}{\hbar} - \frac{i}{2\pi}\{\phi, \psi\}_\omega \right\|_\hbar \to 0$$

as $\hbar \to 0$.

Proof. (Step 1). The twisted convolution, equation (3.3), has been defined between functions on $\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}$. To define a multiplication between functions on $\mathfrak{g}$, we use the partial Fourier transform to transfer the twisted convolution to $S(\mathfrak{g})$.

Although $S(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \subseteq L^1(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}))$, it is in general not true that $S(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is an algebra under the twisted convolution $*_{\sigma_\hbar}$, unless the cocycle is trivial. Still, at least on $C_c^\infty(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$, the C^∞–functions on $\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}$ with compact support, the twisted convolution is closed. This result actually corresponds to a similar result in the trivial cocycle case (i.e. the crossed products [24]), and the proof is also done by straightforward calculation. So for the purpose of proving the theorem, we may take $C_c^\infty(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ as the subspace on which the twisted convolution is closed.

We will let $\mathcal{A}$ be the image of $C_c^\infty(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ in $S(\mathfrak{g})$ under the partial Fourier transform, $\wedge$. By the inverse partial Fourier transform, $\vee$, the subspace $\mathcal{A}$ is carried back onto $C_c^\infty(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$. Therefore, it follows immediately that equation (3.4) defines a closed multiplication on $\mathcal{A} \subseteq S(\mathfrak{g})$. Since $C_c^\infty(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q})$ is dense in $S(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \subseteq L^1(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}))$ with respect to the L^1–norm, it is clear that $\mathcal{A}$ is dense in $S(\mathfrak{g}) \subseteq C_\infty(\mathfrak{g}/\mathfrak{q})$ with respect to the $\|\ \|_\infty$ norm.

Since we have defined our deformed multiplication on $\mathcal{A}$ to be isomorphic to the twisted convolution on $C_c^\infty(\mathfrak{h}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{q}) \subseteq L^1\big(\mathfrak{h}/\mathfrak{z}, C_\infty(\mathfrak{g}/\mathfrak{q})\big)$, we may also transfer other structures on the twisted convolution algebra to $\mathcal{A}$ via partial Fourier transform. On the twisted convolution algebra, the involution is given by the following formula:

$$(3.6) \qquad f^*(x;r) = \overline{f(x^{-1};r)\sigma_\hbar^r(x,x^{-1})}\Delta_{K_\hbar}(x^{-1}).$$

Here $\Delta_{\mathfrak{h}/\mathfrak{z}} \equiv 1$, since $K_\hbar = (\mathfrak{h}/\mathfrak{z}, *_\hbar)$ is a nilpotent Lie group. There also exists a canonical C^*–norm, which is dominated by the L^1–norm of the twisted group algebra, such that the completion with respect to the C^*–norm gives rise to the enveloping C^*–algebra $C^*\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big)$. Via partial Fourier transform, we transfer these structures to $\mathcal{A}$ to define its involution, $^{*_\hbar}$, and the C^*–norm, $\| \ \|_\hbar$. Let us denote the C^*–completion of $\big(\mathcal{A}, \times_\hbar, {}^{*_\hbar}, \| \ \|_\hbar\big)$ by $A_\hbar$. This proves the first assertion of the theorem. We have $A_\hbar \cong C^*\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big)$. Moreover, since the group $K_\hbar$ is amenable (nilpotent), the "amenability condition" always holds for the twisted convolution algebra, that is, $C^*\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big) = C_r^*\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big)$.

(Step 2: Continuity of the field of C^*–algebras $\{A_\hbar\}_{\hbar\in\mathbb{R}}$). For $\hbar \neq 0$, there exists a group isomorphism between $K_\hbar = (\mathfrak{h}/\mathfrak{z}, *_\hbar)$ and $K = (\mathfrak{h}/\mathfrak{z}, *)$ given by $x \mapsto \hbar x$. For convenience, let us use the same notation, $\sigma_\hbar$, for the group cocycle for K transferred by the isomorphism from the cocycle $\sigma_\hbar$ for $K_\hbar$. Then we have $A_\hbar \cong C^*\big(K, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big)$, with the cocycle $\sigma_\hbar$ now viewed as the cocycle for K. Moreover, it is not difficult to see that $\hbar \to \sigma_\hbar$ forms a continuous field of cocycles. Since each $A_\hbar$ is a twisted group C^*–algebra satisfying the amenability condition, we conclude by Theorem 1.4 that $\{A_\hbar\}_{\hbar\in\mathbb{R},\hbar\neq 0}$ forms a continuous field of C^*–algebras.

When $h = 0$, we no longer have the isomorphism between K_0 and K in general. Therefore, A_0 cannot be regarded as a twisted group algebra of K. However, note that the problem will go away when $\mathfrak{h}/\mathfrak{z}$ is abelian. In this case, each Lie group $K_\hbar$, including $\hbar = 0$, is just the additive Lie group K_0. That is, $x *_\hbar y = x + y$ for every $x, y \in K_\hbar$. So each C^*–algebra $A_\hbar$ becomes $A_\hbar \cong C^*\big(K_0, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big)$. Here we have used the notation K_0 instead of $K_\hbar$ to emphasize the fact that it is the same additive abelian Lie group for every $\hbar \in \mathbb{R}$. But note that the cocycle still depends on $\hbar$. Since $\hbar \to \sigma_\hbar$ can be viewed as a continuous field of cocycles for K_0 and since the amenability condition holds, it follows that $\{A_\hbar\}_{\hbar\in\mathbb{R}}$ is a continuous field of C^*–algebras.

In general, $\mathfrak{h}/\mathfrak{z}$ is not abelian and this argument is no longer valid. In this case, we may break down the nilpotent Lie algebra $\mathfrak{h}/\mathfrak{z}$ into its center and the corresponding quotient algebra. If the resulting quotient

algebra is not abelian, we again break it into its center and the quotient. Since $\mathfrak{h}/\mathfrak{z}$ is a finite dimensional nilpotent Lie algebra, it is clear that this process will end with our $\mathfrak{h}/\mathfrak{z}$ broken into several abelian Lie algebras. By using the nontrivial structural theorems by Packer and Raeburn (Theorems 1.5 and 1.6), we can now prove the continuity of the field of C^*–algebras $\{A_\hbar\}_{\hbar \in \mathbb{R}}$.

Recall that for a given $\hbar \in \mathbb{R}$,

$$A_\hbar \cong C^*\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big)$$

which we will write as $A_\hbar = B_\hbar^0 \times_{\alpha^0(\hbar), \sigma^0(\hbar)} (N^0)_\hbar$. That is, $B_\hbar^0 = C_\infty(\mathfrak{g}/\mathfrak{q})$ is the C^*–algebra, $N^0 = K$ is the nilpotent Lie group, and $\sigma^0(\hbar) = \sigma_\hbar$ is the cocycle for $(N^0)_\hbar = K_\hbar$. For the moment, $\alpha^0(\hbar)$ is the trivial action. When N^0 is an abelian Lie group so that $(N^0)_\hbar = N^0$ for all $\hbar$, we have already shown that the field of C^*–algebras $\{A_\hbar\}_{\hbar \in \mathbb{R}}$ is continuous. We have to prove the result for nonabelian N^0.

Since N^0 is a nilpotent Lie group, it has a nontrivial center $Z^0 \subseteq N^0$ as a normal subgroup. Denote by N^1 the quotient Lie group N^0/Z^0. It is clear that $(N^1)_\hbar = (N^0)_\hbar/(Z^0)_\hbar = (N^0)_\hbar/Z^0$, since Z^0 is abelian. Since N^1 is also nilpotent, we can similarly define Z^1 and N^2. We continue this (finite) process until we have obtained an abelian group N^k. Meanwhile by Theorem 1.5, the C^*–algebra $A_\hbar$ can be written as

$$A_\hbar = B_\hbar^0 \times_{\alpha^0(\hbar), \sigma^0(\hbar)} (N^0)_\hbar = \big(B_\hbar^0 \times_{\alpha^0(\hbar), \sigma^0(\hbar)} Z^0\big) \times_{\alpha^1(\hbar), \sigma^1(\hbar)} (N^1)_\hbar$$

which we may denote by $A_\hbar = B_\hbar^1 \times_{\alpha^1(\hbar), \sigma^1(\hbar)} (N^1)_\hbar$. If we define $B_\hbar^2, B_\hbar^3, \ldots$ in a similar manner, since N^k is assumed to be abelian, we obtain: $A_\hbar = B_\hbar^k \times_{\alpha^k(\hbar), \sigma^k(\hbar)} N^k$.

Let us now apply Theorem 1.6. We will prove the continuity of $\{A_\hbar\}_{\hbar \in \mathbb{R}}$ by induction on k. When $k = 0$, note that $(N^0)_\hbar = N^0$ since it is abelian. This is just the case we have proved earlier. As an induction hypothesis, suppose that the result holds for all positive integer less then k. Since by definition $B_\hbar^k = B_\hbar^{k-1} \times_{\alpha^{k-1}(\hbar), \sigma^{k-1}(\hbar)} Z^{k-1}$, where Z^{k-1} is an abelian Lie group, it follows that the field $\{B_\hbar^k\}_{\hbar \in \mathbb{R}}$ is continuous. Let us denote by B the corresponding C^*–algebra of sections. Note that $\big(\alpha^k(\hbar), \sigma^k(\hbar)\big)$ is a twisted action of N^k on each fibre $B_\hbar^k$ while the continuity of $\hbar \to \big(\alpha^k(\hbar), \sigma^k(\hbar)\big)$ is obvious from the construction, which we will regard as a continuous field of twisted action (α, σ) on B. Therefore, we conclude from Theorem 1.6 that the C^*–algebra $B \times_{\alpha, \sigma} N^k$ is the algebra of sections of a C^*–bundle over $\mathbb{R}$ with fibres isomorphic to $B_\hbar^k \times_{\alpha^k(\hbar), \sigma^k(\hbar)} N^k$. This means that the field of C^*–algebras $\hbar \to A_\hbar = B_\hbar^k \times_{\alpha^k(\hbar), \sigma^k(\hbar)} N^k$ is continuous.

(Step 3: Proof of the deformation property). On $\mathcal{A}$, we will form the expression, $(\phi \times_\hbar \psi - \psi \times_\hbar \phi)/\hbar$, and compare this with our Poisson bracket on $\mathfrak{g}$ defined in Theorem 2.3.

Recall that a given function $\phi \in S(\mathfrak{g})$ can be written as

$$\phi(\mu) = \int (\mathcal{F}^{-1}\phi)(X)\bar{e}\big[\langle X, \mu \rangle\big]\, dX$$

by the Fourier inversion theorem. So we have

$$d\phi(\mu) = (-2\pi i) \int (\mathcal{F}^{-1}\phi)(X)\bar{e}\big[\langle X, \mu \rangle\big] X\, dX.$$

Therefore the Poisson bracket $\{\ ,\ \}_\omega$ becomes, for $\phi, \psi \in \mathcal{A} \subseteq S(\mathfrak{g})$,

$$\{\phi, \psi\}_\omega(\mu) = \big\langle [d\dot\phi(\mu), d\dot\psi(\mu)], \mu \big\rangle + \omega\big(d\dot\phi(\mu), d\dot\psi(\mu); \mu\big)$$

$$= (-4\pi^2) \int (\mathcal{F}^{-1}\phi)(X)(\mathcal{F}^{-1}\psi)(Y)\bar{e}\big[\langle X + Y, \mu \rangle\big]$$

$$\big(\langle [\dot X, \dot Y], \mu \rangle + \omega(\dot X, \dot Y; \mu)\big)\, dX dY.$$

If we write an element $\mu \in \mathfrak{g} = \mathfrak{q} \oplus (\mathfrak{g}/\mathfrak{q})$ as $\mu = (q, r)$ and similarly, elements $X, Y \in \mathfrak{h} = \mathfrak{h}/\mathfrak{z} \oplus \mathfrak{z}$ as $X = (x, z)$, $Y = (y, z')$, then we obtain:

$$\{\phi, \psi\}_\omega(\mu) = (-4\pi^2) \int (\mathcal{F}^{-1}\phi)(x, z)(\mathcal{F}^{-1}\psi)(y, z')\bar{e}\big[\langle (x + y, z + z'), (q, r) \rangle\big]$$

$$\big(\langle [x, y], q \rangle + \omega(x, y; r)\big)\, dx dz dy dz'.$$

Meanwhile, deformed multiplication on $\mathcal{A}$ can be written as follows:

$$(\phi \times_\hbar \psi)(q, r) = \int \phi^\vee(x, r)\psi^\vee(x^{-1} *_\hbar y, r)\sigma_\hbar^r(x, x^{-1} *_\hbar y)\bar{e}\big[\langle y, q \rangle\big]\, dx dy$$

$$= \int (\mathcal{F}^{-1}\phi)(x, z)(\mathcal{F}^{-1}\psi)(y, z')\bar{e}\big[\langle z + z', r \rangle\big]$$

$$\bar{e}\big[R_\hbar(x, y; r)\big]\bar{e}\big[\langle x *_\hbar y, q \rangle\big]\, dz dz' dx dy.$$

To prove the deformation property, we must now show that the expression $(\phi \times_\hbar \psi - \psi \times_\hbar \phi)/\hbar$ approaches $(i/2\pi)\{\phi, \psi\}_\omega$, in the sense of equation (3.5), as $\hbar \to 0$.

Since each $A_\hbar$ (for $\hbar \neq 0$) is isomorphic to the twisted group C^*–algebra $C^*\big(K, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big)$, the C^*–norm $\|\ \|_\hbar$ is dominated by the "L^1–norm" on $L^1\big(K, C_\infty(\mathfrak{g}/\mathfrak{q})\big)$, which is actually equivalent (via the partial Fourier transform in $r \in \mathfrak{g}/\mathfrak{q}$ variable) to the L^1–norm on $L^1\big(\mathfrak{h}/\mathfrak{z} \times \mathfrak{z}\big) = L^1(\mathfrak{h})$. It is also clear that even for $\hbar = 0$, the C^*–norm $\|\ \|_{\hbar=0}$ (which is just the sup norm on $C_\infty(\mathfrak{g})$) is dominated by the L^1–norm on $L^1(\mathfrak{h})$, similarly by the Fourier transform. Therefore to prove equation (3.5), it is sufficient to show the convergence with respect to the L^1–norm on $L^1(\mathfrak{h})$. But first, let us show that we have

at least the pointwise convergence valid in $\mathcal{A}$, thereby giving us a mild justification to our situation. Since

$$\bar{e}\big[\langle x *_\hbar y, q\rangle + R_\hbar(x, y; r)\big]$$

$$= \bar{e}\left[\langle x + y, q\rangle + \frac{\hbar}{2}\langle [x, y]_{\mathfrak{h}/\mathfrak{z}}, q\rangle + \frac{\hbar}{2}\omega(x, y; r) + O(\hbar^2)\right]$$

$$= \bar{e}\big[\langle x + y, q\rangle\big] + (-2\pi i)\frac{\hbar}{2}\bar{e}\big[\langle x + y, q\rangle\big]\big(\langle [x, y], q\rangle + \omega(x, y; r)\big) + O(\hbar^2),$$

we have:

$$\lim_{\hbar \to 0}\left(\frac{\bar{e}\big[\langle x *_\hbar y, q\rangle + R_\hbar(x, y; r)\big] - \bar{e}\big[\langle y *_\hbar x, q\rangle + R_\hbar(y, x; r)\big]}{\hbar}\right)$$

$$= (-2\pi i)\bar{e}\big[\langle x + y, q\rangle\big]\big(\langle [x, y], q\rangle + \omega(x, y; r)\big).$$

From this, the pointwise convergence follows. That is, for $\mu = (q, r)$ in $\mathfrak{g}$, we have:

$$\left(\frac{\phi \times_\hbar \psi - \psi \times_\hbar \phi}{\hbar}\right)(\mu) - \frac{i}{2\pi}\{\phi, \psi\}_\omega(\mu) \longrightarrow 0$$

as $\hbar \to 0$. In a formal power series setting (for example, at the QUE algebra level), this kind of proof (showing pointwise convergence) is usually sufficient.

Let us now consider the convergence problem with respect to the L^1–norm on $L^1(\mathfrak{h})$, transferred to the $\mathcal{A} \subseteq S(\mathfrak{g})$ level via Fourier transform. For the linear Poisson bracket on $\mathfrak{g}$, Rieffel [27] has shown the L^1–convergence in $S(\mathfrak{h})$, and hence in $S(\mathfrak{g})$. The idea is to give a bound for the L^1–norm for the expression, $(\phi \times_\hbar \psi - \psi \times_\hbar \phi)/\hbar - (i/2\pi)\{\phi, \psi\}_\omega$. Then by Lebesgue's dominated convergence theorem, the result follows. In our case however, since the cocycles $R(\ ,\ )$ and $\omega(\ ,\ )$ have values in $V = C_\infty(\mathfrak{g}/\mathfrak{q})$ and since we allowed them to be possibly non-polynomial functions, we do not in general expect the convergence to take place in $S(\mathfrak{g})$. Actually, even our deformed multiplication, $\times_\hbar$, had to be defined on a subspace $\mathcal{A} \subseteq S(\mathfrak{g})$.

At least in $\mathcal{A}$, we are able to find a suitable L^1–bound for the above expression, since the convergence involving the cocycle terms can now be controlled in a compact set such that on this compact set, we have a uniform convergence. So the dominated convergence theorem again can be applied to assure L^1–convergence. Thus our proof is complete. $\square$

Remark. Sometimes, we are able to find a bigger space $\mathcal{A}$ in which the deformed multiplication is closed and the deformation property of equation (3.5) is satisfied. Since elements of the subalgebra $\mathcal{A}$ are

in a certain sense "smooth functions", it is always desirable, for non-commutative geometry purposes [8, 9], to have as big an $\mathcal{A}$ as possible.

Remark. When $\{\ ,\ \}_\omega$ is the linear Poisson bracket on $\mathfrak{g} = \mathfrak{h}^*$ (when $\omega = \omega_0$), then we can show that:

$$A_\hbar \cong C^*\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big) \cong C^*\big(K_\hbar, C^*(Z), \sigma_\hbar\big) \cong C^*(H_\hbar).$$

The twisted convolution (3.4) on $L^1\big(K_\hbar, C_\infty(\mathfrak{g}/\mathfrak{q}), \sigma_\hbar\big)$ is just the ordinary convolution on $L^1(H_\hbar)$. So in this case, the theorem implies that a strict deformation quantization of the linear Poisson bracket on the dual vector space of a nilpotent Lie algebra $\mathfrak{h}$ is provided by a family of ordinary group C^*–algebras $\big\{C^*(H_\hbar)\big\}_{\hbar \in \mathbb{R}}$. This is the result obtained in [27]. The dense subalgebra $\mathcal{A}$ on which the deformation takes place is the space of Schwartz functions.

Although we have obtained the theorem assuming $\mathfrak{h}$ to be nilpotent, much of the argument will work if $\mathfrak{h}$ is at least an exponential solvable Lie algebra. Some technical problems come up. First, we have to use the Haar measure for the group $(\mathfrak{h}/\mathfrak{z}, *_\hbar)$, which would no longer coincide with the Lebesgue measure on $\mathfrak{h}/\mathfrak{z}$, to define the twisted convolution product. So some modifications to the definitions of dual Plancherel measure and Fourier transform are necessary. We have already mentioned the serious problems of defining the group cocycles and of correctly formulating the context of strict deformation quantization.

At least in the linear Poisson bracket case, Rieffel in [27] has studied these problems when $\mathfrak{h}$ is exponential, and even more general cases including the Lie groups which are only locally diffeomorphic to vector spaces, by relaxing the conditions for the deformation quantization. It will be an interesting future project to find a correct formulation of the definitions such that the twisted group C^*–algebras arising from non-nilpotent Lie algebras (at least those corresponding to exponential Lie algebras) fit into the framework of deformation quantization, probably by allowing some mild relaxation on the strictness condition.

In another direction, there is a natural next step of our theorem to study more general types of twisted group C^*–algebras ("twisted crossed products" by Packer and Raeburn), with nontrivial actions as well as nontrivial cocycles, as possible deformation quantizations of Poisson brackets. At present, we do not have a genuine example that does not degenerate into either crossed products (only actions are nontrivial) or twisted group C^*–algebras with only cocycles nontrivial. Still, there are some positive indications that these C^*–algebras would

provide a right framework for the setting mentioned above—involving an exponential solvable Lie algebra and its dual vector space.

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Department of Mathematics, University of California at Berkeley, Berkeley, CA 94704

E-mail address: bjkahng@math.berkeley.edu