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Haar measure on a locally compact quantum group*

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Abstract. In the general theory of locally compact quantum groups, the notion of Haar measure (Haar weight) plays the most significant role. The aim of this paper is to carry out a careful analysis regarding Haar weight, in relation to general theory, for the specific non-compact quantum group (A, Δ) constructed earlier by the author. In this way, one can show that (A, Δ) is indeed a "(C^*-algebraic) locally compact quantum group" in the sense of the recently developed definition given by Kustermans and Vaes. Attention will be given to pointing out the relationship between the original construction (obtained by deformation quantization) and the structure maps suggested by general theory.

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0. Introduction

According to the widely accepted paradigm (which goes back to Gelfand and Naimark in the 1940's and has been reaffirmed by Connes and his non-commutative geometry program [5]) that C^*-algebras are quantized/non-commutative locally compact spaces, the C^*-algebra framework is the most natural one in which to formulate a theory of *locally compact quantum groups*. There have been several examples of C^*-algebraic quantum groups constructed, beginning with Woronowicz's (compact) quantum $SU(2)$ group [25]. The examples of non-compact C^*-algebraic quantum groups have been rather scarce, but significant progress has been made over the past decade.

Among the examples of non-compact type is the Hopf C^*-algebra (A, Δ) constructed by the author [8]. The construction is done by the method of deformation quantization, and the approach is a slight generalization of the one

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used in Rieffel's example of a solvable quantum group [19]. In fact, (A, Δ) may be regarded as a "quantized $C^*(H)$" or a "quantized $C_0(G)$", where H is a Heisenberg-type Lie group and G is a certain solvable Lie group carrying a non-linear Poisson structure.

Motivation for choosing suitable comultiplication, counit, antipode (coinverse), and Haar weight on (A, Δ) comes from the information at the level of Poisson–Lie groups. The proofs were given by introducing some tools like the multiplicative unitary operator. In this way, we could argue that (A, Δ), together with its additional structure maps, should be an example of a *non-compact quantum group*.

We further went on to find a "quantum universal R-matrix" type operator related with (A, Δ) and studied its representation theory, indicating that the *-representations of A satisfy an interesting "quasitriangular" type property. See [8] and [9] (More discussion on the representation theory is given in [10].).

However, even with these strong indications suggested by our construction and the representation theoretic applications, we did not quite make it clear whether (A, Δ) actually is a locally compact (C^*-algebraic) quantum group. For instance, in [8], the discussion about the Haar weight on (A, Δ) was rather incomplete, since we restricted our discussion to the level of a dense subalgebra of the C^*-algebra A. Even for the (simpler) example of Rieffel's [19], the full construction of its Haar weight was not carried out. The problem of tying together these loose ends and establishing (A, Δ) as a locally compact quantum group in a suitable sense was postponed to a later occasion.

Part of the reason for the postponement was due to the fact that at the time of writing, the question of the correct definition of a locally compact quantum group had not yet been settled. It was known that simply requiring the existence of a counit and an antipode on the "locally compact quantum semigroup" (A, Δ) is not enough. Some proposals had been made, but they were at a rather primitive stage. Recently, the situation has improved: A new paper by Kustermans and Vaes [14] appeared, in which they give a relatively simple definition of a (reduced) C^*-algebraic quantum group.

In this new definition, the existence of a left invariant (Haar) weight and a right invariant weight plays the central role. In particular, they do not have to include the existence of the antipode and its polar decomposition in their axioms. Unlike the axiom sets of Masuda and Nakagami [15], or those for Kac algebras [6], which are either too complicated or too restrictive, these properties and others can be proved from the defining axioms. We are still far from achieving the goal of formulating a set of axioms in which we do not have to invoke the existence of Haar measure. Considering this, it seems that the definition of Kustermans and Vaes is the most reasonable choice at this moment.

Now that we have an acceptable definition, we are going to return to our example (A, Δ) and verify that (A, Δ) is indeed a locally compact quantum group. Our discussion will begin in Section 1 by describing the definition of Kustermans and Vaes, making precise the notion of a "C^*-algebraic locally compact quantum group".

In section 2, we summarize a few results about our specific example (A, Δ). Instead of repeating our construction carried out in [8], we take a more economical approach of describing results by relying less on the Poisson geometric aspects.

In section 3, which is the main part of this paper, we describe the Haar weight for (A, Δ) and make the notion valid in the C^*-algebra setting. Having the correct left/right invariant weights enables us to conclude that (A, Δ) is indeed a non-compact C^*-algebraic quantum group. We are benefiting a lot from being able to work with our specific example having a tracial weight, but many of the techniques being used here are not necessarily type-specific, and therefore, will be also useful in more general cases: Our discussion on the left invariance of Haar measure is strongly motivated by the new and attractive approach of Van Daele [23], [24].

In Sections 4 and 5, we say a little about the antipode and the modular function of (A, Δ). By comparing our original definitions (motivated by Poisson–Lie group data) with the ones suggested by the general theory, we wish to give some additional perspectives on these maps.

For the discussion to be complete, we need a description of the dual counterpart to (A, Δ). We included a very brief discussion of $(\hat{A}, \hat{\Delta})$ at the end of Section 2, and also added a short Appendix (Section 6). For a more careful discussion on the dual, see [11]. Meanwhile, we plan to pursue in our future papers the discussion on the quantum double, as well as the research on the duality of quantum groups in relation to the Poisson duality at the level of their classical limit.

As a final remark, we point out that while our original construction of (A, Δ) was by deformation quantization of Poisson–Lie groups, it can be also approached algebraically using the recent framework of "twisted bicrossed products" of Vaes and Vainerman [22]. While we do believe in the advantage of the more constructive approach we took in [8] motivated by Poisson–Lie groups (especially in applications involving quantizations or representation theory, as in [9], [10]), complementing it with the more theoretical approach presented here will make our understanding more comprehensive.

1. Definitions, terminologies, and conventions

1.1 Weights on C^*-algebras

We will begin by briefly reviewing the theory of weights on C^*-algebras. The purpose is to make clear the notations used in the main definition of a C^*-

algebraic quantum group (Definition 1.2) and in the proofs in later sections. For a more complete treatment and for standard terminologies on weights, refer to [3]. For instance, recall the standard notations like $\mathfrak{N}_\varphi, \mathfrak{M}_\varphi, {\mathfrak{M}_\varphi}^+, \dots$ associated to a weight φ (on a C^*-algebra A). That is,

- $\mathfrak{N}_\varphi = \{a \in A : \varphi(a^*a) < \infty\}$
- ${\mathfrak{M}_\varphi}^+ = \{a \in A^+ : \varphi(a) < \infty\}$
- $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi$

The weights we will be considering are "proper weights": A proper weight is a non-zero, densely defined weight on a C^*-algebra, which is lower semi-continuous [14].

If we are given a (proper) weight φ on a C^*-algebra, we can define the sets $\mathcal{F}_\varphi$ and $\mathcal{G}_\varphi$ by

$$\mathcal{F}_\varphi = \{\omega \in A_+^* : \omega(x) \le \varphi(x), \forall x \in A^+\}$$
$$\mathcal{G}_\varphi = \{\alpha\omega : \omega \in \mathcal{F}_\varphi, \alpha \in (0, 1)\} \subseteq \mathcal{F}_\varphi.$$

Here A^* denotes the norm dual of A.

These sets have been introduced by Combes, and they play a significant role in the theory of weights. Note that on $\mathcal{F}_\varphi$, one can give a natural order inherited from A_+^*. Meanwhile, $\mathcal{G}_\varphi$ is a directed subset of $\mathcal{F}_\varphi$. That is, for every $\omega_1, \omega_2 \in \mathcal{G}_\varphi$, there exists an element $\omega \in \mathcal{G}_\varphi$ such that $\omega_1, \omega_2 \le \omega$. Because of this, $\mathcal{G}_\varphi$ is often used as an index set (of a net). For a proper weight φ, we would have: $\varphi(x) = \lim(\omega(x))_{\omega\in\mathcal{G}_\varphi}$, for $x \in A^+$.

By standard theory, for a weight φ on a C^*-algebra A, one can associate to it a "GNS-construction" $(\mathcal{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$. Here, $\mathcal{H}_\varphi$ is a Hilbert space, $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathcal{H}_\varphi$ is a linear map such that $\Lambda_\varphi(\mathfrak{N}_\varphi)$ is dense in $\mathcal{H}_\varphi$ and $\langle \Lambda_\varphi(a), \Lambda_\varphi(b)\rangle = \varphi(b^*a)$ for $a, b \in \mathfrak{N}_\varphi$, and π_φ is a representation of A on $\mathcal{H}_\varphi$ defined by $\pi_\varphi(a)\Lambda_\varphi(b) = \Lambda_\varphi(ab)$ for $a \in A$, $b \in \mathfrak{N}_\varphi$. The GNS-construction is unique up to a unitary transformation.

If φ is proper, then $\mathfrak{N}_\varphi$ is dense in A and $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathcal{H}_\varphi$ is a closed map. Also $\pi_\varphi : A \to \mathcal{B}(\mathcal{H}_\varphi)$ is a non-degenerate *-homomorphism. It is not difficult to show that φ has a natural extension to a weight on $M(A)$, which we will still denote by φ. Meanwhile, since we can define for every $\omega \in \mathcal{G}_\varphi$ a unique element $\tilde{\omega} \in \pi_\varphi(A)''_*$ such that $\tilde{\omega} \circ \pi_\varphi = \omega$, we can define a weight $\tilde{\varphi}$ on the von Neumann algebra $\pi_\varphi(A)''$ in the following way: $\tilde{\varphi}(x) = \lim(\tilde{\omega}(x))_{\omega\in\mathcal{G}_\varphi}$, for $x \in (\pi_\varphi(A)'')^+$. Then, by standard terminology [21], $\tilde{\varphi}$ is a "normal", "semi-finite" weight on the von Neumann algebra $\pi_\varphi(A)''$.

Motivated by the properties of normal, semi-finite weights on von Neumann algebras, and to give somewhat of a control over the non-commutativity of A, one introduces the notion of "KMS weights" [12]. The notion as defined below is slightly different from (but equivalent to) the original one given by Combes in [4].

Definition 1.1. *A proper weight φ is called a "KMS weight" if there exists a norm-continuous one-parameter group of automorphisms σ of A such that*

(1) $\varphi \circ \sigma_t = \varphi$, *for all* $t \in \mathbb{R}$.
(2) $\varphi(a^*a) = \varphi(\sigma_{i/2}(a)\sigma_{i/2}(a)^*)$, *for all* $a \in D(\sigma_{i/2})$.

Here $\sigma_{i/2}$ is the analytic extension of the one-parameter group σ_t to $\frac{i}{2}$.

The one-parameter group σ is called the "modular automorphism group" for φ. It is uniquely determined when φ is faithful. Meanwhile, a proper weight φ is said to be "approximately KMS" if the associated (normal, semi-finite) weight $\tilde{\varphi}$ is faithful. A KMS weight is approximately KMS. For more discussion on these classes of weights, including the relationship between the conditions above and the usual KMS condition, see [12]. Finally, note that in the special case when φ is a trace (i. e. $\varphi(a^*a) = \varphi(aa^*)$, for $a \in \mathfrak{N}_\varphi$), it is clear that φ is KMS. The modular automorphism group will be trivial ($\equiv$ Id).

1.2 Definition of a locally compact quantum group

Let A be a C^*-algebra. Suppose $\Delta : A \to M(A \otimes A)$ is a non-degenerate *-homomorphism (Later, Δ will be given certain conditions for it to become a comultiplication.). A proper weight φ on (A, Δ) will be called *left invariant*, if

$$\varphi((\omega \otimes \mathrm{id})(\Delta a)) = \omega(1)\varphi(a), \tag{1.1}$$

for all $a \in \mathfrak{M}_\varphi{}^+$ and $\omega \in A_+^*$. Similarly, φ is called *right invariant*, if

$$\varphi((\mathrm{id} \otimes \omega)(\Delta a)) = \omega(1)\varphi(a). \tag{1.2}$$

By $\omega(1)$, we mean $\|\omega\|$. Note here that we used the extensions of φ to $M(A)$ in the equations, since we only know that $(\omega \otimes \mathrm{id})(\Delta a) \in M(A)^+$. In the definition of locally compact quantum groups (to be given below), the "slices" of Δa will be assumed to be contained in A.

In the definitions above, the left [respectively, right] invariance condition requires the formula (1.1) to hold only for $a \in \mathfrak{M}_\varphi{}^+$. It is a very weak form of left invariance. In the case of locally compact quantum groups, the result can be extended and a much stronger left invariance condition can be proved from it. The proof is non-trivial. It was one of the important contributions made by Kustermans and Vaes.

Next, let us state the definition of a locally compact (C^*-algebraic) quantum group given by Kustermans and Vaes [14]. In the definition, $[X]$ denotes the closed linear span of X.

Definition 1.2. *Consider a C^*-algebra A and a non-degenerate *-homomorphism $\Delta : A \to M(A \otimes A)$ such that*

(1) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$
(2) $[\{(\omega \otimes \mathrm{id})(\Delta a) : \omega \in A^*, a \in A\}] = A$
(3) $[\{(\mathrm{id} \otimes \omega)(\Delta a) : \omega \in A^*, a \in A\}] = A$

Moreover, assume that there exist weights φ and ψ such that

- *φ is a faithful, left invariant approximate KMS weight on (A, Δ).*
- *ψ is a right invariant approximate KMS weight on (A, Δ).*

Then we say that (A, Δ) is a (reduced) C^*-algebraic quantum group.

First condition is the "coassociativity" condition for the "comultiplication" Δ. By the non-degeneracy, it can be naturally extended to $M(A)$ [we can also extend $(\Delta \otimes \mathrm{id})$ and $(\mathrm{id} \otimes \Delta)$], thereby making the expression valid. The two density conditions more or less correspond to the cancellation property in the case of ordinary groups, although they are somewhat weaker. The last axiom corresponds to the existence of Haar measure (The weights φ and ψ actually turn out to be faithful KMS weights.). For more on this definition (e. g. discussions on how one can build other structure maps like the antipode), see [14].

2. The Hopf C^*-algebra (A, Δ)

Our main object of study is the Hopf C^*-algebra (A, Δ) constructed in [8]. As a C^*-algebra, A is isomorphic to a twisted group C^*-algebra $C^*(H/Z, C_0(\mathfrak{g}/\mathfrak{q}), \sigma)$, where H is the $(2n+1)$-dimensional Heisenberg Lie group and Z is the center of H. Whereas, $\mathfrak{g} = \mathfrak{h}^*$ is the dual space of the Lie algebra $\mathfrak{h}$ of H and $\mathfrak{q} = \mathfrak{z}^{\perp}$, for $\mathfrak{z} \subseteq \mathfrak{h}$ corresponding to Z. Since H is a nilpotent Lie group, $H \cong \mathfrak{h}$ and $Z \cong \mathfrak{z}$, as vector spaces. We denoted by σ (not to be confused with the modular automorphism group) the twisting cocycle for the group H/Z. As constructed in [8], σ is a continuous field of cocycles $\mathfrak{g}/\mathfrak{q} \ni r \mapsto \sigma^r$, where

$$\sigma^r((x, y), (x', y')) = \bar{e}[\eta_\lambda(r)\beta(x, y')]. \tag{2.1}$$

Following the notation of the previous paper, we used: $\bar{e}(t) = e^{(-2\pi i)t}$ and $\eta_\lambda(r) = \frac{e^{2\lambda r}-1}{2\lambda}$, where λ is a fixed real constant. We denote by $\beta(\ ,\ )$ the inner product. The elements (x, y), (x', y') are group elements in H/Z.

In [8], we showed that the C^*-algebra A is a deformation quantization (in Rieffel's "strict" sense [18], [20]) of $C_0(G)$, where G is a certain solvable Lie group which is the *dual Poisson–Lie group* of H. The number λ mentioned above determines the group structure of G (When $\lambda = 0$, the group G becomes abelian, which is not very interesting.). See Definition 1.6 of [8] for the precise definition of G. For convenience, we fixed the deformation parameter as $\hbar = 1$. This is the reason why we do not see it in the definition of A. If we wish to illustrate the deformation process, we may just replace β by $\hbar\beta$, and let $\hbar \to 0$. When $\hbar = 0$ (i. e. classical limit), we have $\sigma \equiv 1$, and hence, $A_{\hbar=0} \cong C_0(G)$. Throughout this paper, we will just work with $A = A_{\hbar=1}$.

Let us be a little more specific and recall some of the notations and results obtained in [8], while referring the reader to that paper for more details on the construction of our main example (A, Δ).

We first introduce the subspace $\mathcal{A}$, which is a dense subspace of A consisting of the functions in $S_{3c}(H/Z \times \mathfrak{g}/\mathfrak{q})$, the space of Schwartz functions in the (x, y, r) variables having compact support in the $r (\in \mathfrak{g}/\mathfrak{q})$ variable. On $\mathcal{A}$, we define the (twisted) multiplication and the (twisted) involution as follows:

$$\begin{aligned}(f \times g)(x, y, r) = \int & f(\tilde{x}, \tilde{y}, r) g(x - \tilde{x}, y - \tilde{y}, r) \\ & \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})]\, d\tilde{x} d\tilde{y},\end{aligned} \tag{2.2}$$

and

$$f^*(x, y, r) = \overline{f(-x, -y, r)}\bar{e}[\eta_\lambda(r)\beta(x, y)]. \tag{2.3}$$

It is not difficult to see that $\mathcal{A} = S_{3c}(H/Z \times \mathfrak{g}/\mathfrak{q})$ is closed under the multiplication (2.2) and the involution (2.3). Here, we observe the role being played by the twisting cocycle σ defined in (2.1).

Elements of $\mathcal{A}$ are viewed as operators on the Hilbert space $\mathcal{H} = L^2(H/Z \times \mathfrak{g}/\mathfrak{q})$, via the "regular representation", L, defined by

$$(L_f \xi)(x, y, r) = \int f(\tilde{x}, \tilde{y}, r) \xi(x - \tilde{x}, y - \tilde{y}, r) \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})]\, d\tilde{x} d\tilde{y}. \tag{2.4}$$

For $f \in \mathcal{A}$, define its norm by $\|f\| = \|L_f\|$. Then $(\mathcal{A}, \times, {}^*, \|\ \|)$ as above is a pre-C^*-algebra, whose completion is the C^*-algebra $A \cong C^*(H/Z, C_0(\mathfrak{g}/\mathfrak{q}), \sigma)$.

Remark. To be more precise, the completion of $\mathcal{A}$ with respect to the norm given by the regular representation, L, should be isomorphic to the "reduced" twisted group C^*-algebra $C_r^*(H/Z, C_0(\mathfrak{g}/\mathfrak{q}), \sigma)$. But by using a result of Packer and Raeburn [17], it is rather easy to see that the amenability condition holds in our case, thereby obtaining the isomorphism with the "full" C^*-algebra as above. Meanwhile, we should point out that our definition of $\mathcal{A}$ is slightly different from that of [8]: There, $\mathcal{A}$ is a subspace of $C_0(G)$, while at present we view it as functions contained in $C_0(H/Z \times \mathfrak{g}/\mathfrak{q})$, in the (x, y, r) variables. Nevertheless, they can be regarded as the same since we consider the functions in $\mathcal{A}$ as operators contained in our C^*-algebra A. The identification of the function spaces is given by the (partial) Fourier transform.

The C^*-algebra A becomes a Hopf C^*-algebra, together with its *comultiplication* Δ. In the following proposition, we chose to describe the comultiplication in terms of a certain "multiplicative unitary operator" $U_A \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. See [8] for a discussion on the construction of U_A.

Proposition 2.1. (1) *Let U_A be the operator on $\mathcal{H} \otimes \mathcal{H}$ defined by*

$$U_A\xi(x, y, r, x', y', r') = (e^{-\lambda r'})^n \bar{e}[\eta_\lambda(r')\beta(e^{-\lambda r'}x, y' - e^{-\lambda r'}y)] \\ \xi(e^{-\lambda r'}x, e^{-\lambda r'}y, r+r', x'-e^{-\lambda r'}x, y'-e^{-\lambda r'}y, r').$$

Then U_A is a unitary operator, and is multiplicative. That is,

$$U_{12}U_{13}U_{23} = U_{23}U_{12}.$$

(2) *For $f \in \mathcal{A}$, define Δf by*

$$\Delta f = U_A(f \otimes 1){U_A}^*,$$

where f and Δf are understood as operators L_f and $(L \otimes L)_{\Delta f}$. Then Δ can be extended to a non-degenerate C^*-homomorphism $\Delta : A \to M(A \otimes A)$ satisfying the coassociativity condition:

$$(\Delta \otimes \mathrm{id})(\Delta f) = (\mathrm{id} \otimes \Delta)(\Delta f).$$

Proof. See Proposition 3.1 and Theorem 3.2 of [8], together with the Remark 3.3 following them. □

There is a useful characterization of the C^*-algebra A, via the multiplicative unitary operator U_A. The following result is suggested by the general theory on multiplicative unitaries by Baaj and Skandalis [2].

Proposition 2.2. *Let U_A be as above. Consider the subspace $\mathcal{A}(U_A)$ of $\mathcal{B}(\mathcal{H})$ defined below:*

$$\mathcal{A}(U_A) = \{(\omega \otimes \mathrm{id})(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}.$$

By standard theory, $\mathcal{A}(U_A)$ is a subalgebra of the operator algebra $\mathcal{B}(\mathcal{H})$, and the subspace $\mathcal{A}(U_A)\mathcal{H}$ forms a total set in $\mathcal{H}$.

We can show that the norm-closure in $\mathcal{B}(\mathcal{H})$ of the algebra $\mathcal{A}(U_A)$ is exactly the C^*-algebra A we are studying. That is,

$$A = \overline{\{(\omega \otimes \mathrm{id})(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\ \|}.$$

Proof. The definition and the properties of $\mathcal{A}(U_A)$ can be found in [2]. We only need to verify the last statement. We will work with the standard notation $\omega_{\xi,\eta}$, where $\xi, \eta \in \mathcal{H}$. It is defined by $\omega_{\xi,\eta}(a) = \langle a\xi, \eta\rangle$, and it is well known that linear combinations of the $\omega_{\xi,\eta}$ are (norm) dense in $\mathcal{B}(\mathcal{H})_*$.

So consider $(\omega_{\xi,\eta} \otimes \mathrm{id})(U_A) \in \mathcal{B}(\mathcal{H})$. We may further assume that ξ and η are continuous functions having compact support. Let $\zeta \in \mathcal{H}$. Then, by using change of variables, we have:

$$\begin{aligned}
&(\omega_{\xi,\eta} \otimes \mathrm{id})(U_A)\zeta(x, y, r) \\
&= \int (U_A(\xi \otimes \zeta))(\tilde{x}, \tilde{y}, \tilde{r}; x, y, r)\overline{\eta(\tilde{x}, \tilde{y}, \tilde{r})}\, d\tilde{x}d\tilde{y}d\tilde{r} \\
&= \int f(\tilde{x}, \tilde{y}, r)\zeta(x - \tilde{x}, y - \tilde{y}, r)\bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})]\, d\tilde{x}d\tilde{y},
\end{aligned}$$

where

$$f(\tilde{x}, \tilde{y}, r) = \int \xi(\tilde{x}, \tilde{y}, \tilde{r} + r)(e^{\lambda r})^n \overline{\eta(e^{\lambda r}\tilde{x}, e^{\lambda r}\tilde{y}, \tilde{r})}\, d\tilde{r}.$$

Since ξ and η are L^2-functions, the integral (and thus f) is well defined. Actually, since f is essentially defined as a convolution product (in r) of two continuous functions having compact support, f will be also continuous with compact support. This means that

$$(\omega_{\xi,\eta} \otimes \mathrm{id})(U_A) = L_f \in A.$$

Meanwhile, since the choice of ξ and η is arbitrary, we can see that the collection of the f will form a total set in the space of continuous functions in the (x, y, r) variables having compact support. It follows from these two conclusions that

$$\overline{\{(\omega \otimes \mathrm{id})(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\ \|} = A.$$

□

Meanwhile, from the proof of Theorem 3.2 of [8], we also have the following result. These are not same as the density conditions of Definition 1.2, but are actually stronger: This is rather well known and can be seen easily by applying linear functionals (Use the fact that any $\omega \in A^*$ has the form $\omega'(\cdot\, b)$, with $\omega' \in A^*$ and $b \in A$.).

Proposition 2.3. *We have:*

(1) $\Delta(A)(1 \otimes A)$ *is dense in* $A \otimes A$.
(2) $\Delta(A)(A \otimes 1)$ *is dense in* $A \otimes A$.

Proof. In the proof of the non-degeneracy of Δ in Theorem 3.2 and Remark 3.3 of [8], we showed that the $(\Delta f)(1 \otimes g)$'s (for $f, g \in \mathcal{A}$) form a total set in the space $S_{3c}(H/Z \times \mathfrak{g}/\mathfrak{q} \times H/Z \times \mathfrak{g}/\mathfrak{q})$, which is in turn shown to be dense in $A \otimes A$: Under the natural injection from $S_{3c}(H/Z \times \mathfrak{g}/\mathfrak{q} \times H/Z \times \mathfrak{g}/\mathfrak{q})$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, the algebraic tensor product $\mathcal{A} \odot \mathcal{A}$ is sent into a dense subset of the algebraic tensor product $A \odot A$. Since elements in $S_{3c}(H/Z \times \mathfrak{g}/\mathfrak{q} \times H/Z \times \mathfrak{g}/\mathfrak{q})$ can be approximated (in the L^1-norm) by elements of $\mathcal{A} \odot \mathcal{A}$, we see that $S_{3c}(H/Z \times \mathfrak{g}/\mathfrak{q} \times H/Z \times \mathfrak{g}/\mathfrak{q})$ is mapped into a dense subset (in the C^* norm) of $A \otimes A$. Thus it follows that $\Delta(A)(1 \otimes A)$ is dense in $A \otimes A$. The second statement can be shown in exactly the same way. □

Turning our attention to the other structures on (A, Δ), we point out that by viewing A as a "quantum $C_0(G)$", we can construct its *counit*, ε, and *antipode*, S. These are described in the following proposition.

Proposition 2.4. (1) *For $f \in \mathcal{A}$, define $\varepsilon : \mathcal{A} \to \mathbb{C}$ by*

$$\varepsilon(f) = \int f(x, y, 0)\, dx dy.$$

Then ε can be extended to a C^*-homomorphism from A to $\mathbb{C}$ satisfying the condition: $(\mathrm{id} \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$.

(2) *Consider a map $S : \mathcal{A} \to \mathcal{A}$ defined by*

$$(S(f))(x, y, r) = (e^{2\lambda r})^n \bar{e}[\eta_\lambda(r)\beta(x, y)] f(-e^{\lambda r}x, -e^{\lambda r}y, -r).$$

Then S can be extended to an anti-automorphism $S : A \to A$, satisfying: $S(S(a)^*)^* = a$ and $(S \otimes S)(\Delta a) = \chi(\Delta(S(a)))$, where χ denotes the flip. Actually, we have: $S^2 = \mathrm{Id}$.

Proof. See Theorem 4.1 and Proposition 4.3 of [8]. We had to use partial Fourier transform to convert these results into the level of functions in the (x, y, r) variables. We also mention here that S is defined by $S(a) = \hat{J} a^* \hat{J}$, where $\hat{J}$ is an involutive operator on $\mathcal{H}$ defined by

$$\hat{J}\xi(x, y, r) = (e^{\lambda r})^n \overline{\xi(e^{\lambda r}x, e^{\lambda r}y, -r)}.$$

Since $\hat{J}$ is an anti-unitary involutive operator, it is easy to see that S is an anti-automorphism such that $S^2 = \mathrm{Id}$. □

Remark. The notation for the operator $\hat{J}$ is motivated by the modular theory and by [15]. Meanwhile, since the square of the antipode is the identity, our example is essentially the *Kac C^*-algebra* (Compare with [7], although our example is actually non-unimodular, unlike in that paper. See also [22], of which our example is a kind of a special case.). We also note that in [8], we used κ to denote the antipode, while we use S here. This is done so that we can match our notation with the preferred notation of Kustermans and Vaes [14].

In general, the counit may as well be unbounded. So Proposition 2.4 implies that what we have is a more restrictive "bounded counit". Having S bounded is also a bonus. Even so, the result of the proposition is not enough to legitimately call S an antipode. To give some support for our choice, we also showed the following, albeit only at the level of the function space $\mathcal{A}$. See section 4 of [8].

Proposition 2.5. *For $f \in \mathcal{A}$, we have:*

$$m((\mathrm{id} \otimes S)(\Delta f)) = m((S \otimes \mathrm{id})(\Delta f)) = \varepsilon(f)1,$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication.

This is the required condition for the antipode in the purely algebraic setting of Hopf algebra theory [16]. In this sense, the proposition gives us a modest justification for our choice of S. However, in the operator algebra setting, this is not the correct way of approach. One of the serious obstacles is that the multiplication m is in general not continuous for the operator norm, thereby giving us trouble extending m to $A \otimes A$ or $M(A \otimes A)$.

Because of this and other reasons (including the obstacles due to possible unboundedness of ε and S), one has to develop a new approach. Motivated by the theory of Kac algebras [6], operator algebraists have been treating the antipode together with the notion of the Haar weight. This is also the approach chosen by Masuda, Nakagami [15] and by Kustermans, Vaes [14]. As we mentioned earlier in this paper, any rigorous discussion about locally compact quantum groups should be built around the notion of Haar weights. In the next section, we will exclusively discuss the Haar weight for our (A, Δ), and establish that (A, Δ) is indeed a "C^*-algebraic locally compact quantum group". We will come back to the discussion of the antipode in section 4.

Before wrapping up this section, let us mention the dual object for our (A, Δ), which would be the "dual locally compact quantum group" $(\hat{A}, \hat{\Delta})$. Our discussion here is kept to a minimum. More careful discussion on $(\hat{A}, \hat{\Delta})$ is presented in a separate paper [11]. Meanwhile, see Appendix (Section 6) for a somewhat different characterization of the dual object.

Proposition 2.6. (1) *Let U_A be as above. Consider the subspace $\hat{\mathcal{A}}(U_A)$ of $\mathcal{B}(\mathcal{H})$ defined by*

$$\hat{\mathcal{A}}(U_A) = \{(\mathrm{id} \otimes \omega)(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}.$$

Then $\hat{\mathcal{A}}(U_A)$ is a subalgebra of the operator algebra $\mathcal{B}(\mathcal{H})$, and the subspace $\hat{\mathcal{A}}(U_A)\mathcal{H}$ forms a total set in $\mathcal{H}$.

(2) *The norm-closure in $\mathcal{B}(\mathcal{H})$ of the algebra $\hat{\mathcal{A}}(U_A)$ is the C^*-algebra $\hat{A}$:*

$$\hat{A} = \overline{\{(\mathrm{id} \otimes \omega)(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\ \|}.$$

(3) *For $b \in \hat{\mathcal{A}}(U_A)$, define $\hat{\Delta} b$ by $\hat{\Delta} b = {U_A}^*(1 \otimes b)U_A$. Then $\hat{\Delta}$ can be extended to the comultiplication $\hat{\Delta} : \hat{A} \to M(\hat{A} \otimes \hat{A})$.*

(4) *The duality exists at the level of $\mathcal{A}(U_A)$ and $\hat{\mathcal{A}}(U_A)$, by the following formula:*

$$\langle L(\omega), \rho(\omega')\rangle = (\omega \otimes \omega')(U_A) = \omega(\rho(\omega')) = \omega'(L(\omega)),$$

where $L(\omega) = (\omega \otimes \mathrm{id})(U_A) \in \mathcal{A}(U_A)$ and $\rho(\omega') = (\mathrm{id} \otimes \omega')(U_A) \in \hat{\mathcal{A}}(U_A)$.

Proof. Since our multiplicative unitary operator U_A is known to be regular, we can just follow the standard theory of multiplicative unitary operators [2]. □

At this moment, $(\hat{A}, \hat{\Delta})$ is just a quantum semigroup, having only the comultiplication. However, once we establish in section 3 the proof that our (A, Δ) is indeed a (C^*-algebraic) locally compact quantum group, we can apply the general theory [14], and show that $(\hat{A}, \hat{\Delta})$ is also a locally compact quantum group. Meanwhile, we can give a more specific description of the C^*-algebra $\hat{A}$, as presented below.

Proposition 2.7. *Let $\hat{\mathcal{A}}$ be the space of Schwartz functions in the (x, y, r) variables having compact support in the r variable. For $f \in \hat{\mathcal{A}}$, define the operator $\rho_f \in \mathcal{B}(\mathcal{H})$ by*

$$(\rho_f \zeta)(x, y, r) = \int (e^{\lambda\tilde{r}})^n f(x, y, \tilde{r})\zeta(e^{\lambda\tilde{r}}x, e^{\lambda\tilde{r}}y, r - \tilde{r})\, d\tilde{r}. \tag{2.5}$$

Then the C^*-algebra $\hat{A}$ is generated by the operators ρ_f.

Proof. As in the proof of Proposition 2.2, consider the operators $(\mathrm{id} \otimes \omega_{\xi,\eta})(U_A)$ in $\hat{\mathcal{A}}(U_A)$. Without loss of generality, we can assume that ξ and η are continuous functions having compact support. Let $\zeta \in \mathcal{H}$. Then we have:

$$\begin{aligned}
&(\mathrm{id} \otimes \omega_{\xi,\eta})(U_A)\zeta(x, y, r)\\
&= \int (U_A(\zeta \otimes \xi))(x, y, r; \tilde{x}, \tilde{y}, \tilde{r})\overline{\eta(\tilde{x}, \tilde{y}, \tilde{r})}\, d\tilde{x}d\tilde{y}d\tilde{r}\\
&= \int (e^{\lambda\tilde{r}})^n f(x, y, \tilde{r})\zeta(e^{\lambda\tilde{r}}x, e^{\lambda\tilde{r}}y, r - \tilde{r})\, d\tilde{r},
\end{aligned}$$

where

$$f(x, y, \tilde{r}) = \int \bar{e}[\eta_\lambda(\tilde{r})\beta(x, y - e^{-\lambda\tilde{r}}\tilde{y})] \\ \xi(\tilde{x} - e^{\lambda\tilde{r}}x, \tilde{y} - e^{\lambda\tilde{r}}y, -\tilde{r})\overline{\eta(\tilde{x}, \tilde{y}, -\tilde{r})}\, d\tilde{x}d\tilde{y}.$$

We see that $(\mathrm{id} \otimes \omega_{\xi,\eta})(U_A) = \rho_f$, and f is continuous with compact support. By the same argument we used in the proof of Proposition 2.2, we conclude that the C^*-algebra generated by the operators ρ_f, $f \in \hat{\mathcal{A}}$, coincides with the C^*-algebra $\hat{A}$. □

The above characterization of $\hat{A}$ is useful when we wish to regard $\hat{A}$ as a deformation quantization of a Poisson–Lie group. Actually, by using partial Fourier transform, we can show without difficulty that $\hat{A} \cong \rho(C^*(G))$, where ρ is the right regular representation of $C^*(G)$. In a future paper, we will have an occasion to discuss the duality between (A, Δ) and $(\hat{A}, \hat{\Delta})$ in relation to the Poisson–Lie group duality between G and H.

3. Haar weight

We have been arguing that (A, Δ) *is* a "quantized $C_0(G)$". This viewpoint has been helpful in our construction of its comultiplication Δ, counit ε, and antipode S.

To discuss the (left invariant) Haar weight on (A, Δ), we pull this viewpoint once more. Recall that the group structure on G has been specifically chosen (Definition 1.6 of [8]) so that the Lebesgue measure on G becomes its left invariant Haar measure. This suggests us to build the Haar weight on (A, Δ) from the Lebesgue measure on G. At the level of functions in $\mathcal{A}$, this suggestion takes the following form:

Definition 3.1. *On $\mathcal{A}$, define a linear functional φ by*

$$\varphi(f) = \int f(0, 0, r)\, dr.$$

In section 5 of [8], we obtained some results (including the "left invariance" property) indicating that our choice of φ is a correct one. However, the discussion was limited to the level of functions in $\mathcal{A}$, and thus not very satisfactory.

Jumping up from the function level to the operator level can be quite technical, and it is not necessarily an easy task (For example, see [1], [23].). Whereas, if one wants to rigorously formulate the construction of a locally compact quantum group in the operator algebra setting, this step of "jumping up" (extending φ to a weight) is very crucial.

Fortunately in our case, the discussion will be much simpler than some of the difficult examples, since we can show that φ is tracial. Note the following:

Proposition 3.2. *Let φ be defined on $\mathcal{A}$ as in Definition 3.1. Then for $f \in \mathcal{A}$, we have:*

$$\varphi(f^* \times f) = \varphi(f \times f^*) = \|f\|_2^2,$$

where f^* is the C^*-involution of f, as given in (2.3).

Proof. By equations (2.2) and (2.3), we have:

$$\begin{aligned}
(f^* \times f)(x, y, r) &= \int f^*(\tilde{x}, \tilde{y}, r) f(x - \tilde{x}, y - \tilde{y}, r) \\
&\qquad \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})]\, d\tilde{x} d\tilde{y} \\
&= \int \overline{f(-\tilde{x}, -\tilde{y}, r)} \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, \tilde{y})] f(x - \tilde{x}, y - \tilde{y}, r) \\
&\qquad \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})]\, d\tilde{x} d\tilde{y} \\
&= \int \overline{f(-\tilde{x}, -\tilde{y}, r)} f(x - \tilde{x}, y - \tilde{y}, r) \\
&\qquad \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y)]\, d\tilde{x} d\tilde{y}.
\end{aligned}$$

It follows that:

$$\begin{aligned}
\varphi(f^* \times f) &= \int (f^* \times f)(0, 0, r)\, dr \\
&= \int \overline{f(-\tilde{x}, -\tilde{y}, r)} f(-\tilde{x}, -\tilde{y}, r)\, d\tilde{x} d\tilde{y} dr \\
&= \int \overline{f(\tilde{x}, \tilde{y}, r)} f(\tilde{x}, \tilde{y}, r)\, d\tilde{x} d\tilde{y} dr = \|f\|_2^2.
\end{aligned}$$

The identity $\varphi(f \times f^*) = \|f\|_2^2$ can be proved similarly. □

Corollary. *By Proposition 3.2, we see that φ is a faithful, positive linear functional which is a trace.*

Now, let us begin the discussion of constructing a weight on (A, Δ) extending φ. As a first step, let us consider the associated GNS construction for φ. We can see below that the "regular representation" L on $\mathcal{H}$ we defined earlier is the GNS representation for φ.

Proposition 3.3. *Consider the Hilbert space $\mathcal{H} = L^2(H/Z \times \mathfrak{g}/\mathfrak{q})$, and let $\Lambda : \mathcal{A} \hookrightarrow \mathcal{H}$ be the inclusion map. Then for $f, g \in \mathcal{A}$, we have:*

$$\langle \Lambda(f), \Lambda(g) \rangle = \varphi(g^* \times f).$$

Here $\langle\ ,\ \rangle$ is the inner product on $\mathcal{H}$, conjugate in the second place. Meanwhile, left multiplication gives a non-degenerate *-representation, $\pi_\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, which coincides with the "regular representation" L.

By essential uniqueness of the GNS construction, we conclude that $(\mathcal{H}, \Lambda, \pi_\varphi)$ is the GNS triple associated with φ.

Proof. Since $\mathcal{A} = S_{3c}(H/Z \times \mathfrak{g}/\mathfrak{q})$, clearly $\mathcal{A}$ is a dense subspace of $\mathcal{H}$. The inclusion map (i. e. $\Lambda(f) = f$) carries $\mathcal{A}$ into $\mathcal{H}$. Now for $f, g \in \mathcal{A}$,

$$\begin{aligned}\varphi(g^* \times f) &= \int g^*(\tilde{x}, \tilde{y}, r) f(0 - \tilde{x}, 0 - \tilde{y}, r) \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, 0 - \tilde{y})]\, d\tilde{x} d\tilde{y} dr \\ &= \int \overline{g(-\tilde{x}, -\tilde{y}, r)} \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, \tilde{y})] f(-\tilde{x}, -\tilde{y}, r) \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, -\tilde{y})]\, d\tilde{x} d\tilde{y} dr \\ &= \int \overline{g(\tilde{x}, \tilde{y}, r)} f(\tilde{x}, \tilde{y}, r)\, d\tilde{x} d\tilde{y} dr = \langle f, g \rangle = \langle \Lambda(f), \Lambda(g) \rangle.\end{aligned}$$

Consider now the left-multiplication representation π_φ. Then for $f, \xi \in \mathcal{A}$,

$$\begin{aligned}(\pi_\varphi(f))(\Lambda(\xi))(x, y, r) &:= (\Lambda(f \times \xi))(x, y, r) = (f \times \xi)(x, y, r) \\ &= \int f(\tilde{x}, \tilde{y}, r) \xi(x - \tilde{x}, y - \tilde{y}, r) \\ &\quad \bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})]\, d\tilde{x} d\tilde{y} \\ &= L_f \xi(x, y, r).\end{aligned}$$

This shows that π_φ is just the *-representation L of equation (2.4). Since $\mathcal{A} = \Lambda(\mathcal{A})$ is dense in $\mathcal{H}$, it is also clear that $\overline{\pi_\varphi(\mathcal{A})\mathcal{H}}^{\|\ \|_2} = \mathcal{H}$, which means that $\pi_\varphi (= L)$ is non-degenerate. □

We are ready to show that $\mathcal{A} (\subseteq \mathcal{H})$ is a "left Hilbert algebra" (See definition below.).

Definition 3.4. *([4], [21]) By a* left Hilbert algebra, *we mean an involutive algebra $\mathcal{U}$ equipped with a scalar product such that the involution is an anti-linear preclosed mapping in the associated Hilbert space $\mathcal{H}$ and such that the left-multiplication representation π of $\mathcal{U}$ is non-degenerate, bounded, and involutive.*

Proposition 3.5. *The algebra $\mathcal{A}$, together with its inner product inherited from that of $\mathcal{H}$, is a left Hilbert algebra.*

Proof. We view $\mathcal{A} = \Lambda(\mathcal{A}) \subseteq \mathcal{H}$. It is an involutive algebra equipped with the inner product inherited from that of $\mathcal{H}$. Since φ is a trace, the map $f \mapsto f^*$ is not just closable, but it is actually isometry and hence bounded. Note that for every $f, g \in \mathcal{A}$, we have:

$$\langle f^*, g \rangle = \varphi(g^* \times f^*) = \varphi(f^* \times g^*) = \langle g^*, f \rangle,$$

where we used the property that φ is a trace. The remaining conditions for $\mathcal{A}$ being a left Hilbert algebra are immediate consequences of the previous proposition. □

Remark. The closure of the involution on $\mathcal{A}$ is often denoted by T. The map T is a closed, anti-linear map on $\mathcal{H}$, and $\mathcal{A}$ is a core for T (Actually, T is bounded.). Define $\nabla = T^*T$. Clearly, $\mathcal{A} \subseteq D(\nabla)$ and $\nabla(f) = f$ for $f \in \mathcal{A}$. In other words, $\nabla = \mathrm{Id}$. The polar decomposition of T is given by $T = J\nabla^{\frac{1}{2}}$, where ∇ is as above and J is an anti-unitary operator. Obviously in our case, $T = J$. The "modular operator" ∇ plays an important role in the formulation of the KMS property. But as we see here, we can ignore ∇ from now on, all due to the property that φ is a trace.

Since we have a left Hilbert algebra structure on $\mathcal{A}$, we can apply the result of Combes ([4], [21]) to obtain a weight extending φ. Although it is true that we do not necessarily have to rely a lot on the theory of weights on C^*-algebras (Since φ is a trace in our case, we could use even earlier results of Dixmier), we nevertheless choose here the more general approach. The advantage is that the process will remain essentially the same even in more difficult examples where we may encounter non-tracial weights.

Theorem 3.6. *There is a faithful, lower semi-continuous weight on the C^*-algebra A extending the linear functional φ. We will use the notation φ_A to denote this weight.*

Proof. The representation $\pi_\varphi(= L)$ generates the von Neumann algebra $L(\mathcal{A})''$ on the Hilbert space $\mathcal{H}$. It would be actually the von Neumann algebra M_A generated by A. On this von Neumann algebra, we can define as in the below a faithful, semi-finite, normal weight $\tilde{\varphi}$ (See Theorem 2.11 of [4].):

For $p \in L(\mathcal{A})''$ and $p \geq 0$, define $\tilde{\varphi}(p)$ by

$$\tilde{\varphi}(p) = \begin{cases} \|\xi\|^2 = \langle \xi, \xi \rangle & \text{if } \exists \xi \in \mathcal{A}'' \text{ such that } p^{1/2} = \pi_\varphi(\xi) \\ +\infty & \text{otherwise} \end{cases}$$

Here $\mathcal{A}''$ denotes the set of "left bounded elements" [4, §2].

We restrict this normal weight to the C^*-algebra $\overline{L(\mathcal{A})}^{\|\ \|}$ (norm-closure). Then the restriction is a faithful, lower semi-continuous weight. Since $\pi_\varphi(= L)$ extends from $\mathcal{A}$ to an isomorphism $A \cong \overline{L(\mathcal{A})}^{\|\ \|}$, we can use this isomorphism to obtain the faithful, lower semi-continuous weight (to be denoted by φ_A) on A.

It is clear from the construction that φ_A extends the linear functional φ on $\mathcal{A}$. To see this explicitly, suppose $f \in \mathcal{A}$. Then $\pi_\varphi(f)^*\pi_\varphi(f) \in L(\mathcal{A})''$. According to the theory of left Hilbert algebras, we then have $\pi_\varphi(f)^*\pi_\varphi(f) \in \mathfrak{M}_{\tilde{\varphi}}{}^+$ and

$$\tilde{\varphi}(\pi_\varphi(f)^*\pi_\varphi(f)) = \langle \Lambda(f), \Lambda(f) \rangle = \langle f, f \rangle = \varphi(f^*f).$$

But $\pi_\varphi(f)^*\pi_\varphi(f) = \pi_\varphi(f^*f) \in \overline{L(\mathcal{A})}^{\|\ \|} \cong A$, and since φ_A is the restriction of $\tilde{\varphi}$ to A, it follows that $\pi_\varphi(f^*f) \in \mathfrak{M}_{\varphi_A}$, and $\varphi_A(\pi_\varphi(f^*f)) = \varphi(f^*f)$. By using polarization, we conclude that in general, $L(\mathcal{A}) \subseteq \mathfrak{M}_{\varphi_A}$ and

$$\varphi_A(\pi_\varphi(f)) = \varphi(f), \qquad \forall f \in \mathcal{A}.$$

$\square$

Remark. From the proof of the proposition, we can see that φ_A is densely defined (note that we have $L(\mathcal{A}) \subseteq \mathfrak{M}_{\varphi_A}$). It is a faithful weight since the linear functional φ is faithful on $\mathcal{A}$. Since φ_A is obtained by restricting the normal weight $\tilde{\varphi}$ on the von Neumann algebra level, it follows that it is also KMS (We will not give proof of this here, since φ being a trace makes this last statement redundant: See comment after Definition 1.1.). In the terminology of the first section, φ_A is a "proper" weight, which is "faithful" and "KMS" (actually a trace).

From now on, let us turn our attention to the weight φ_A. Consider the GNS triple associated with φ_A, given by the following ingredients:

- $\mathcal{H}_{\varphi_A} = \mathcal{H}$
- $\Lambda_{\varphi_A} : \mathfrak{N}_{\varphi_A} \to \mathcal{H}$. The proof of the previous theorem suggests that for $a \in \mathfrak{N}_{\varphi_A}$, there exists a unique "left bounded" element $v \in \mathcal{H}$. We define $\Lambda_{\varphi_A}(a) = v$.
- $\pi_{\varphi_A} : A \to \mathcal{B}(\mathcal{H})$ is the inclusion map.

Note that for $f \in \mathcal{A}$, we have: $\Lambda_{\varphi_A}(\pi_\varphi(f)) = \Lambda(f)$. So we know that Λ_{φ_A} has a dense range in $\mathcal{H}$.

Define Λ_0 as the closure of the mapping $L(\mathcal{A}) \to \mathcal{H} : \pi_\varphi(f) \mapsto \Lambda(f)$. Let us denote by $\mathcal{A}_0$ the domain of Λ_0. Clearly, Λ_0 is a restriction of Λ_{φ_A}. By using the properties of φ_A, including its lower semi-continuity and the "left invariance" at the level of the *-algebra $\mathcal{A}$, one can improve the left invariance up to the level of $\mathcal{A}_0$. One can also show that $\mathcal{A}_0 = \mathfrak{N}_{\varphi_A}$ and that $L(\mathcal{A})$ is a core for Λ_{φ_A} (We can more or less follow the discussion in section 6 of [13].). From these results, the left invariance of φ_A can be proved at the C^*-algebra level (A similar result can be found in Corollary 6.14 of [13].).

However, we plan to present a somewhat different proof of the left invariance, which is in the spirit of Van Daele's recently developed method [23], [24]. The main strategy is to show that there exists a faithful, semi-finite, normal weight μ on $\mathcal{B}(\mathcal{H})$ such that at least formally, $\mu(ba) = \varphi_B(b)\varphi_A(a)$ for $b \in B$, $a \in A$. [See Appendix (Section 6) for the definition of the "dual" B and of the weight φ_B.]

Proposition 3.7. *Let γ be the unbounded operator on $\mathcal{H}$ having $\Lambda(\mathcal{A})$ as a core and is defined by*

$$\gamma\Lambda(f) = \Lambda(\gamma f), \qquad f \in \mathcal{A},$$

where $\gamma f \in \mathcal{A}$ is such that $\gamma f(x, y, r) = (e^{2\lambda r})^n f(x, y, r)$.

Now on $\mathcal{B}(\mathcal{H})$, we define a linear functional μ by

$$\mu := \operatorname{Tr}(\gamma\ \cdot).$$

Then μ is a faithful, semi-finite, normal weight on $\mathcal{B}(\mathcal{H})$ such that for $b \in \mathfrak{N}_{\varphi_B}$ and $a \in \mathfrak{N}_{\varphi_A}$,

$$\mu(b^*a^*ab) = \varphi_B(b^*b)\varphi_A(a^*a).$$

Proof. The operator γ is very much related with the "modular function" operator, $\tilde{\delta}$, discussed in section 5 (In our case, $\gamma = \tilde{\delta}^{-1}$.). For more precise definition of γ, see Definition 2.6 of [24].

Let us verify the last statement, at the dense function algebra level of $b \in \hat{\mathcal{A}}(\subseteq M_B)$ and $a \in \mathcal{A}(\subseteq M_A)$. For this, note that by Lemma 6.5 of Appendix and Proposition 3.2, we have:

$$\varphi_B(b^*b) = \|b\|_2^2, \qquad \text{and} \qquad \varphi_A(a^*a) = \|a\|_2^2.$$

Meanwhile, by equations (6.1) and (2.4), we have:

$$\begin{aligned}(b^*a^*ab)\xi(x,y,r) = \int & \overline{b(e^{\lambda r}x, e^{\lambda r}y, \tilde{r}-r)}\,\overline{a(\tilde{x},\tilde{y},\tilde{r})}e[\eta_\lambda(\tilde{r})\beta(\tilde{x},y)] \\ & a(\hat{x},\hat{y},\tilde{r})\bar{e}[\eta_\lambda(\tilde{r})\beta(\hat{x},y+\tilde{y}-\hat{y})] \\ & b(e^{\lambda\hat{r}}(x+\tilde{x}-\hat{x}), e^{\lambda\hat{r}}(y+\tilde{y}-\hat{y}), \tilde{r}-\hat{r}) \\ & \xi(x+\tilde{x}-\hat{x}, y+\tilde{y}-\hat{y}, \hat{r})\, d\tilde{r}d\tilde{x}d\tilde{y}d\hat{x}d\hat{y}d\hat{r}.\end{aligned}$$

If we let (ξ_l) be an orthonormal basis in $\mathcal{H}$, we then have:

$$\begin{aligned}\mu(b^*a^*ab) &= \operatorname{Tr}(\gamma b^*a^*ab) = \sum_l \langle(\gamma b^*a^*ab)\xi_l, \xi_l\rangle \\ &= \sum_l \left(\int (e^{2\lambda r})^n (b^*a^*ab)\xi_l(x,y,r)\overline{\xi_l(x,y,r)}\, dxdydr\right) \\ &= \int (e^{2\lambda r})^n \overline{b(e^{\lambda r}x, e^{\lambda r}y, \tilde{r}-r)}\,\overline{a(\tilde{x},\tilde{y},\tilde{r})}e[\eta_\lambda(\tilde{r})\beta(\tilde{x},y)] \\ &\quad a(\tilde{x},\tilde{y},\tilde{r})\bar{e}[\eta_\lambda(\tilde{r})\beta(\tilde{x},y)]b(e^{\lambda r}x, e^{\lambda r}y, \tilde{r}-r)\, dxdydrd\tilde{x}d\tilde{y}d\tilde{r} \\ &= \int \overline{b(x,y,r)}\,\overline{a(\tilde{x},\tilde{y},\tilde{r})}a(\tilde{x},\tilde{y},\tilde{r})b(x,y,r)\, dxdydrd\tilde{x}d\tilde{y}d\tilde{r} \\ &= \|b\|_2^2\|a\|_2^2 = \varphi_B(b^*b)\varphi_A(a^*a).\end{aligned}$$

We used the change of variables.

Since $\hat{\mathcal{A}}$ and $\mathcal{A}$ generate the von Neumann algebras M_B and M_A, while M_BM_A is σ-strongly dense in $\mathcal{B}(\mathcal{H})$ (see Lemma 6.4 of Appendix), this will characterize μ. The properties of μ being faithful, semi-finite, and normal follow from those of φ_A and φ_B, as well as the fact that μ is a trace. $\square$

The significance of the above proposition is that for a certain well-chosen element $b \in B$, we know that $A \ni a \mapsto \mu(b^*ab)$ is a scalar multiple of φ_A. This observation is useful in our proof of the left invariance of φ_A. Before we present our main theorem, let us first introduce a lemma on the linear forms $\omega_{\xi,\eta}$.

Lemma 3.8. *Let $\xi, \eta \in \mathcal{H}$ and consider $\omega_{\xi,\eta}$, as defined earlier. If (ξ_k) forms an orthonormal basis of $\mathcal{H}$, we have:*

$$\omega_{\xi,\eta}(ab) = \sum_k \omega_{\xi_k,\eta}(a)\omega_{\xi,\xi_k}(b), \qquad a, b \in \mathcal{B}(\mathcal{H}).$$

Proof. We have:

$$\begin{aligned}
\omega_{\xi,\eta}(ab) &= \langle ab\xi, \eta\rangle = \langle b\xi, a^*\eta\rangle \\
&= \sum_k \langle b\xi, \xi_k\rangle\langle \xi_k, a^*\eta\rangle = \sum_k \langle b\xi, \xi_k\rangle\langle a\xi_k, \eta\rangle \\
&= \sum_k \omega_{\xi,\xi_k}(b)\omega_{\xi_k,\eta}(a).
\end{aligned}$$

□

The following theorem shows the left invariance of φ_A, as defined by equation (1.1).

Theorem 3.9. *For any positive element $a \in A$ such that $\varphi_A(a) < \infty$, and for $\omega \in A_+^*$, we have:*

$$\varphi_A((\omega \otimes \mathrm{id})(\Delta a)) = \omega(1)\varphi_A(a).$$

Proof. As stated above, let $a \in \mathfrak{M}_{\varphi_A}{}^+$ and let $\omega \in A_+^*$. Without loss of generality, we can assume that ω is a (positive) vector state. That is, we can assume that there is a vector $\zeta \in \mathcal{H}$ such that $\omega = \omega_{\zeta,\zeta}$.

Now consider $(\omega \otimes \mathrm{id})(\Delta a) = (\omega_{\zeta,\zeta} \otimes \mathrm{id})(\Delta a)$. For our purposes, it is more convenient to express Δa in terms of the "dual" multiplicative unitary operator defined in Lemma 6.1 in Appendix: From Proposition 6.3, we know that $\Delta a = \widehat{U_A}^*(1 \otimes a)\widehat{U_A}$. If we let (ξ_k) be an orthonormal basis in $\mathcal{H}$, we would then have:

$$\begin{aligned}
(\omega \otimes \mathrm{id})(\Delta a) &= (\omega_{\zeta,\zeta} \otimes \mathrm{id})(\widehat{U_A}^*(1 \otimes a)\widehat{U_A}) \\
&= \sum_k [(\omega_{\xi_k,\zeta} \otimes \mathrm{id})(\widehat{U_A}^*)]a[(\omega_{\zeta,\xi_k} \otimes \mathrm{id})(\widehat{U_A})] \\
&= \sum_k {v_k}^* a^{\frac{1}{2}} a^{\frac{1}{2}} v_k.
\end{aligned}$$

The sum is convergent in the σ-weak topology on the von Neumann algebra M_A (Use Lemma 3.8.). For convenience, we let $v_k = (\omega_{\zeta,\xi_k} \otimes \mathrm{id})(\widehat{U_A}) (\in \mathcal{B}(\mathcal{H}))$. Note that since $\widehat{U_A}$ is unitary, the operators v_k have the property that for the orthonormal basis (ξ_l) of $\mathcal{H}$, we have:

$$\begin{aligned}\sum_k \langle v_k\xi_l, v_k\xi_j\rangle &= \langle \widehat{U_A}(\zeta \otimes \xi_l), \widehat{U_A}(\zeta \otimes \xi_j)\rangle \\ &= \langle \zeta \otimes \xi_l, \zeta \otimes \xi_j\rangle = \langle \zeta, \zeta\rangle\langle \xi_l, \xi_j\rangle. \qquad (3.1)\end{aligned}$$

Next, suggested by Proposition 3.7 and the comments following it, let us choose a fixed element $b \in \hat{\mathcal{A}} (\subseteq \mathfrak{N}_{\varphi_B})$, so that we have:

$$\varphi_A(a) = \left(\frac{1}{\|b\|_2^2}\right) \mu(b^*ab), \qquad \text{for } a \in \mathfrak{M}_{\varphi_A}.$$

Then combining these observations, we have the following:

$$\begin{aligned}\varphi_A((\omega_{\zeta,\zeta} \otimes \mathrm{id})(\Delta a)) &= \varphi_A\left(\sum_k {v_k}^* a^{\frac{1}{2}} a^{\frac{1}{2}} v_k\right) = \sum_k \varphi_A({v_k}^* a^{\frac{1}{2}} a^{\frac{1}{2}} v_k) \\ &= \frac{1}{\|b\|_2^2} \sum_k \mu(b^* {v_k}^* a^{\frac{1}{2}} a^{\frac{1}{2}} v_k b) = \frac{1}{\|b\|_2^2} \sum_{k,l} \mathrm{Tr}(\gamma b^* {v_k}^* a^{\frac{1}{2}} a^{\frac{1}{2}} v_k b\xi_l, \xi_l) \\ &= \frac{1}{\|b\|_2^2} \sum_{k,l} \langle v_k \gamma^{\frac{1}{2}} a^{\frac{1}{2}} b\xi_l, v_k \gamma^{\frac{1}{2}} a^{\frac{1}{2}} b\xi_l\rangle \\ &= \frac{1}{\|b\|_2^2} \sum_l \langle \zeta, \zeta\rangle \langle \gamma^{\frac{1}{2}} a^{\frac{1}{2}} b\xi_l, \gamma^{\frac{1}{2}} a^{\frac{1}{2}} b\xi_l\rangle \qquad \text{by equation (3.1)} \\ &= \frac{1}{\|b\|_2^2} \langle \zeta, \zeta\rangle \mu(b^*ab) = \langle \zeta, \zeta\rangle \varphi_A(a) = \|\omega\| \varphi_A(a) = \omega(1)\varphi_A(a).\end{aligned}$$

□

As we remarked in section 1, proving this "weak" version of the left invariance is enough. In this way, we have shown that φ_A is a proper, faithful, KMS (tracial) weight on (A, Δ), which is left invariant. This satisfies the requirement of Definition 1.2.

We now need to talk about the right invariant weight on (A, Δ). Again by viewing (A, Δ) as a "quantized $C_0(G)$", we try to build the weight from the right Haar measure of G (The group structure of G as defined in Definition 1.6 of [8] immediately gives us the natural choice for its right Haar measure.). Just as we did at the beginning of this section, this suggestion lets us to consider the linear functional ψ on $\mathcal{A}$, as described below.

Definition 3.10. *On $\mathcal{A}$, define a linear functional ψ by*

$$\psi(f) = \int f(0, 0, r)(e^{-2\lambda r})^n \, dr.$$

It is helpful to realize that at the level of the *-algebra $\mathcal{A}$, we have: $\psi = \varphi \circ S$, where S is the antipodal map we defined in Proposition 2.4. Indeed, for $f \in \mathcal{A}$, we have:

$$\begin{aligned}\varphi(S(f)) &= \int (S(f))(0,0,r)\,dr = \int (e^{2\lambda r})^n \bar{e}[\eta_\lambda(r)\beta(0,0)] f(0,0,-r)\,dr \\ &= \int f(0,0,r)(e^{-2\lambda r})^n\,dr = \psi(f).\end{aligned}$$

Therefore, to extend ψ to the C^*-algebra level, we may consider $\psi_A = \varphi_A \circ S$, where S is now regarded as an antiautomorphism on A.

Remark. Defining $\psi_A = \varphi_A \circ S$ is not entirely correct: In general, the "antipode" S may not be defined everywhere and can be unbounded. However, even in the general case, the antipode can be always written in the form $S = R\tau_{-\frac{i}{2}}$ ("polar decomposition" of S), where τ is the so-called "scaling group" and R is the "unitary antipode". In our case, $\tau \equiv \mathrm{Id}$ and $R = S$ (See section 4.). The correct way of defining ψ_A would be: $\psi_A = \varphi_A \circ R$, which is true in general.

Since R is an (anti-)automorphism on A, it follows that $\psi_A = \varphi_A \circ R$ is clearly a faithful, lower semi-continuous, densely defined KMS weight on A, extending ψ. Checking the "right invariance" is straightforward, if we use the property of R.

Theorem 3.11. *Let $\psi_A = \varphi_A \circ R$. It is a proper, faithful, KMS (and tracial) weight on A. It is also "right invariant". That is, for $a \in \mathfrak{M}_{\psi_A}{}^+$ and for $\omega \in A_+^*$, we have:*

$$\psi_A((\mathrm{id} \otimes \omega)(\Delta a)) = \omega(1)\psi_A(a).$$

Proof. Recall from Proposition 2.4 that $R(= S)$ satisfies $(R \otimes R)(\Delta a) = \chi(\Delta(R(a)))$, where χ denotes the flip. We thus have:

$$\begin{aligned}\psi_A((\mathrm{id} \otimes \omega)(\Delta a)) &= \varphi_A(R((\mathrm{id} \otimes \omega)(\Delta a))) \\ &= \varphi_A((\mathrm{id} \otimes \omega)((R \otimes R)(\Delta a))) \\ &= \varphi_A((\mathrm{id} \otimes \omega)(\chi(\Delta(R(a))))) = \varphi_A((\omega \otimes \mathrm{id})(\Delta(R(a)))) \\ &= \omega(1)\varphi_A(R(a)) \qquad\qquad \varphi_A\text{: left invariant} \\ &= \omega(1)\psi_A(a).\end{aligned}$$

□

We thus have the weight ψ_A on (A, Δ), satisfying the requirement of Definition 1.2. For another characterization of the right invariant weight, see section 5.

Finally, we are now able to say that (A, Δ) is indeed a *(C^*-algebraic) locally compact quantum group*, in the sense of [14].

Theorem 3.12. *The pair (A, Δ), together with the weights φ_A and ψ_A on it, is a C^*-algebraic locally compact quantum group.*

Proof. Combine the results of Proposition 2.1 and Proposition 2.3 on the comultiplication Δ. Theorem 3.6 and Theorem 3.9 gives the left invariant weight ϕ_A, while Theorem 3.11 gives the right invariant weight ψ_A. By Definition 1.2, we conclude that (A, Δ) is a *(reduced) C^*-algebraic quantum group*. $\square$

4. Antipode

According to the general theory (by Kustermans and Vaes [14]), the result of Theorem 3.12 is enough to establish our main goal of showing that (A, Δ) is indeed a C^*-algebraic locally compact quantum group (satisfying Definition 1.2).

Assuming both the left invariant and the right invariant weights in the definition may look somewhat peculiar, while there is no mention on the antipode. However, using these rather simple set of axioms, Kustermans and Vaes could prove additional properties for (A, Δ), so that it can be legitimately called a locally compact quantum group. They first construct a manageable multiplicative unitary operator (in the sense of [2] and [27]) associated with (A, Δ). [In our case, this unitary operator W coincides with our $\widehat{U_A}$ defined in Appendix.] More significantly, they then construct the antipode and its polar decomposition. The uniqueness (up to scalar multiplication) of the Haar weight is also obtained.

An aspect of note through all this is that in this new definition, the "left (or right) invariance" of a weight has been formulated without invoking the antipode, while a characterization of the antipode is given without explicitly referring to any invariant weights. This is much simpler and is a fundamental improvement over earlier frameworks, where one usually requires certain conditions of the type:

$$(\mathrm{id} \otimes \varphi)((1 \otimes a)(\Delta b)) = S((\mathrm{id} \otimes \varphi)((\Delta a)(1 \otimes b))).$$

It is also more natural. Note that in the cases of ordinary locally compact groups or Hopf algebras in the purely algebraic setting, the axioms of the antipode do not have to require any relationships to invariant measures.

For details on the general theory, we will refer the reader to [14]. What we plan to do in this section is to match the general theory with our specific example. Let us see if we can re-construct S from (A, Δ).

By general theory ([27], [14]), the antipode, S, can be characterized such that $\{(\omega \otimes \mathrm{id})(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}$ is a core for S and

$$S((\omega \otimes \mathrm{id})(U_A)) = (\omega \otimes \mathrm{id})({U_A}^*), \qquad \omega \in \mathcal{B}(\mathcal{H})_*. \tag{4.1}$$

It is a closed linear operator on A. The domain $D(S)$ is a subalgebra of A and S is antimultiplicative: i. e. $S(ab) = S(b)S(a)$, for any $a, b \in D(S)$. The image $S(D(S))$ coincides with $D(S)^*$ and $S(S(a)^*)^* = a$ for any $a \in D(S)$. The operator S admits the (unique) "polar decomposition": $S = R\tau_{-\frac{i}{2}}$, where R is the "unitary antipode" and $\tau_{-\frac{i}{2}}$ is the analytic generator of a certain one parameter group $(\tau_t)_{t\in\mathbb{R}}$ of *-automorphisms of A (called the "scaling group").

Remark. In [14], the scaling group and the unitary antipode are constructed first (using only the multiplicative unitary operator and the invariant weights), from which they define the antipode via $S = R\tau_{-\frac{i}{2}}$. The characterization given above is due to Woronowicz [27].

To compare S given by equation (4.1) with our own S defined in Proposition 2.4, let us again consider $\omega_{\xi,\eta}$. From the proof of Proposition 2.2, we know that

$$(\omega_{\xi,\eta} \otimes \mathrm{id})(U_A) = L_f,$$

where

$$f(\tilde{x}, \tilde{y}, r) = \int \xi(\tilde{x}, \tilde{y}, \tilde{r} + r)(e^{\lambda r})^n \overline{\eta(e^{\lambda r}\tilde{x}, e^{\lambda r}\tilde{y}, \tilde{r})}\, d\tilde{r}.$$

We can carry out a similar computation for $(\omega_{\xi,\eta} \otimes \mathrm{id})(U_A{}^*)$. For $\zeta \in \mathcal{H}$, we would have (again using change of variables):

$$\begin{aligned}
(S(L_f))\zeta(x, y, r) &= S((\omega_{\xi,\eta} \otimes \mathrm{id})(U_A))\zeta(x, y, r) \\
&= (\omega_{\xi,\eta} \otimes \mathrm{id})(U_A{}^*)\zeta(x, y, r) \\
&= \int (U_A{}^*(\xi \otimes \zeta))(\tilde{x}, \tilde{y}, \tilde{r}; x, y, r)\overline{\eta(\tilde{x}, \tilde{y}, \tilde{r})}\, d\tilde{x} d\tilde{y} d\tilde{r} \\
&= \int g(\tilde{x}, \tilde{y}, r)\zeta(x-\tilde{x}, y-\tilde{y}, r)\bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y-\tilde{y})]\, d\tilde{x} d\tilde{y} \\
&= L_g\zeta(x, y, r),
\end{aligned}$$

where

$$\begin{aligned}
g(\tilde{x}, \tilde{y}, r) = \int &(e^{\lambda r})^n \xi(-e^{\lambda r}\tilde{x}, -e^{\lambda r}\tilde{y}, \tilde{r} - r)\overline{\eta(-\tilde{x}, -\tilde{y}, \tilde{r})} \\
&\bar{e}[\eta_\lambda(r)\beta(\tilde{x}, \tilde{y})]\, d\tilde{r}.
\end{aligned}$$

This means that $S(f) = g$. Comparing with f, we see that

$$(S(f))(x, y, r) = g(x, y, r) = (e^{2\lambda r})^n f(-e^{\lambda r}\tilde{x}, -e^{\lambda r}\tilde{y}, -r)\bar{e}[\eta_\lambda(r)\beta(x, y)].$$

This is exactly the expression we gave in Proposition 2.4, verifying that our situation agrees perfectly with the general theory.

Since we have already seen that $S : A \to A$ is an antiautomorphism (defined everywhere on A), the uniqueness of the polar decomposition implies that $R = S$ and $\tau \equiv \mathrm{Id}$.

As a final comment on the general theory, we point out that after one defines the antipode as $S = R\tau_{-\frac{1}{2}}$, one proves that for $a, b \in \mathfrak{N}_{\psi_A}$,

$$S((\psi_A \otimes \mathrm{id})((a^* \otimes 1)(\Delta b))) = (\psi_A \otimes \mathrm{id})(\Delta(a^*)(b \otimes 1)).$$

In this way, one can "define" S, as well as give a stronger version of the invariance of ψ_A. The fact that this result could be obtained from the defining axioms (as opposed to being one of the axioms itself) was the significant achievement of [14].

5. Modular function

To motivate the modular function of (A, Δ), let us re-visit our right invariant weight ψ_A. We will keep the notation of Section 3. Recall that at the level of the dense *-algebra $\mathcal{A}$, the right invariant weight is given by the linear functional ψ:

$$\psi(f) = \int f(0, 0, r)(e^{-2\lambda r})^n \, dr.$$

Let us consider the Hilbert space $\mathcal{H}_R$, which will be the GNS Hilbert space for ψ. It is defined such that $\mathcal{H}_R = \mathcal{H}$ as a space and the inner product on it is defined by

$$\langle f, g \rangle_R = \int f(x, y, r)\overline{g(x, y, r)}(e^{-2\lambda r})^n \, dr.$$

Let Λ_R be the inclusion map $\Lambda_R : \mathcal{A} \hookrightarrow \mathcal{H}_R$. We can see easily that for $f, g \in \mathcal{A}$,

$$\langle \Lambda_R(f), \Lambda_R(g) \rangle_R = \langle \Lambda(f), \Lambda(\delta g) \rangle.$$

Here $\delta g \in \mathcal{A}$ defined by $\delta g(x, y, r) = (e^{-2\lambda r})^n g(x, y, r)$.

For motivational purposes, let us be less rigorous for the time being. Observe that working purely formally, we can regard δg as follows:

$$\begin{aligned}
\delta g(x, y, r) &= (e^{-2\lambda r})^n g(x, y, r) \\
&= \int \delta(\tilde{x}, \tilde{y}, r) g(x - \tilde{x}, y - \tilde{y}, r)\bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})] \, d\tilde{x} d\tilde{y} \\
&= (\delta \times g)(x, y, r),
\end{aligned}$$

where δ is considered as a (Dirac delta type) "function" in the (x, y, r) variables such that

$$\delta(x, y, r) = 0, \qquad \text{(if } x \neq 0 \text{ or } y \neq 0\text{)}$$
$$\int \delta(x, y, r)\, dxdy = (e^{-2\lambda r})^n.$$

At the level of the functions in the (x, y, z) variables, δ corresponds to the following "function" (we may use partial Fourier transform, again purely formally):

$$\delta(x, y, z) = \int e[(-e^{-\lambda r} p) \cdot x + (-e^{-\lambda r} q) \cdot y + (-r)z]\, dpdqdr.$$

In this formulation, we see an indication of the inverse operation on G (Note that in G, we have $(p, q, r)^{-1} = (-e^{-\lambda r} p, -e^{-\lambda r} q, -r)$.).

These remarks modestly justifies our intention to call δ a "modular function". Certainly, we see that δ plays an important role relating $\langle\ ,\ \rangle_R$ and $\langle\ ,\ \rangle$, or in other words, relating ψ and φ. A word of caution is that δ is not bounded and not exactly a function. What we plan to do here is to make this notion precise in the C^*-algebra setting.

Note that since $\mathcal{A}$ is a dense subspace of $\mathcal{H}$, we may already regard the map $\tilde{\delta} : \mathcal{A} \ni g \mapsto \delta g \in \mathcal{A}$ as an operator on $\mathcal{H}$. It would be an (unbounded) operator affiliated with the von Neumann algebra $L(A)'' = M_A$, since for an arbitrary element $b \in L(A)'$ and $g \in \mathcal{A} (\subseteq A)$, we would have:

$$b\tilde{\delta}g = \tilde{\delta}gb = \tilde{\delta}bg.$$

By viewing $\mathcal{A}$ as a dense subspace of $\mathcal{H}$, we conclude that $\tilde{\delta}$ commutes with $b \in L(A)'$, proving our claim that $\tilde{\delta}$ is affiliated with M_A.

We may pull down the operator $\tilde{\delta}$ to the C^*-algebra level, and obtain an operator affiliated with A, in the C^*-algebra setting (c. f. in the sense of Woronowicz [26]). So define first a closed linear (unbounded) operator, N, from $\mathcal{H}$ into $\mathcal{H}_R$ such that $\Lambda(\mathcal{A})$ is a core for N and

$$N\Lambda(f) := \Lambda_R(f), \qquad f \in \mathcal{A}.$$

Then N is a densely defined, injective operator with dense range. Note also that $\langle N\Lambda(f), \Lambda_R(g)\rangle_R = \langle \Lambda(f), \Lambda(\delta g)\rangle$. So we have $\Lambda_R(\mathcal{A}) \subseteq D(N^*)$, and

$$N^*\Lambda_R(g) = \Lambda(\delta g), \qquad g \in \mathcal{A}.$$

Consider the following operator (which will be the "modular function"). Clearly, $\Lambda(\mathcal{A}) \subseteq D(\delta_A)$ and $\delta_A \Lambda(f) = \Lambda(\delta f)$.

Definition 5.1. *Define $\delta_A = N^*N$. It is an injective, positive (unbounded) operator on $\mathcal{H}$.*

By general theory, we can say that δ_A is the appropriate definition of the "modular function" in the C^*-algebra setting.

Theorem 5.2. *Let δ_A be defined as above. Then the following properties hold.*

(1) *δ_A is an operator affiliated with the C^*-algebra A.*
(2) $\Delta(\delta_A) = \delta_A \otimes \delta_A$.
(3) $\tau_t(\delta_A) = \delta_A$ *and* $R(\delta_A) = \delta_A^{-1}$.
(4) $\psi(a) = \varphi(\delta_A^{\frac{1}{2}} a \delta_A^{\frac{1}{2}})$, *for* $a \in \mathcal{A}$.

Proof. It is not difficult to see that δ_A is cut down from the operator $\tilde{\delta}$. For proof of the statements, see [14, §7] or see [13, §8]. There are also important relations relating the modular automorphism groups corresponding to φ_A and ψ_A, but in our case they become trivial. □

6. Appendix: An alternative formulation of the dual

The aim of this Appendix is to present a dual counterpart to our locally compact quantum group (A, Δ), which is slightly different (though isomorphic) from $(\hat{A}, \hat{\Delta})$ defined in Section 2. It would be actually the Hopf C^*-algebra having the opposite multiplication and the opposite comultiplication to $(\hat{A}, \hat{\Delta})$. To avoid a lengthy discussion, we plan to give only a brief treatment. But we will include results that are relevant to our main theorem in Section 3.

Let us define (B, Δ_B), by again using the language of multiplicative unitary operators. We begin with a lemma, which is motivated by the general theory of multiplicative unitary operators [2].

Lemma 6.1. *Let $j \in \mathcal{B}(\mathcal{H})$ be defined by*

$$j\xi(x, y, r) = (e^{\lambda r})^n \bar{e}[\eta_\lambda(r)\beta(x, y)]\xi(-e^{\lambda r}x, -e^{\lambda r}y, -r).$$

Then j is a unitary operator such that $j^2 = 1$. Moreover, the operator $\widehat{U_A}$ defined by

$$\widehat{U_A} = \Sigma(j \otimes 1)U_A(j \otimes 1)\Sigma, \qquad \Sigma \text{ denotes the flip}$$

is multiplicative unitary and is regular. For $\xi \in \mathcal{H}$, we specifically have:

$$\widehat{U_A}\xi(x, y, r, x', y', r') = e[\eta_\lambda(r)\beta(e^{\lambda(r'-r)}x', y)] \\ \xi(x + e^{\lambda(r'-r)}x', y + e^{\lambda(r'-r)}y', r; x', y', r' - r).$$

Remark. The proof is straightforward. What is really going on is that the triple $(\mathcal{H}, U_A, j)$ forms a *Kac system*, in the terminology of Baaj and Skandalis (See section 6 of [2].). The operator j may be written as $j = \hat{J}J = J\hat{J}$, where $\hat{J}$

is the anti-unitary operator defined in the proof of Proposition 2.4, while J is the anti-unitary operator determining the *-operation of A as mentioned in the remark following Proposition 3.5. We will have an occasion to say more about these operators in our future paper.

Definition 6.2. *Let $\widehat{U_A}$ be the multiplicative unitary operator obtained above. Define (B, Δ_B) as follows:*

(1) *Let $\mathcal{A}(\widehat{U_A})$ be the subspace of $\mathcal{B}(\mathcal{H})$ defined by*

$$\mathcal{A}(\widehat{U_A}) = \{(\omega \otimes \mathrm{id})(\hat{U}_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}.$$

Then $\mathcal{A}(\widehat{U_A})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, and the subspace $\mathcal{A}(\widehat{U_A})\mathcal{H}$ forms a total set in $\mathcal{H}$.

(2) *The norm-closure in $\mathcal{B}(\mathcal{H})$ of the algebra $\mathcal{A}(\widehat{U_A})$ is the C^*-algebra B. The σ-strong closure of $\mathcal{A}(\widehat{U_A})$ in $\mathcal{B}(\mathcal{H})$ will be the von Neumann algebra M_B.*

(3) *For $b \in \mathcal{A}(\widehat{U_A})$, define $\Delta_B(b)$ by $\Delta_B(b) = \widehat{U_A}(b \otimes 1)\widehat{U_A}^*$. Then Δ_B can be extended to the comultiplication on B, and also to the level of the von Neumann algebra M_B.*

We do not give the proof here, since it is essentially the same as in Propositions 2.2 and 2.6. Let us just add a brief clarification: By a comultiplication on B, we mean a non-degenerate C^*-homomorphism $\Delta_B : B \to M(B \otimes B)$ satisfying the coassociativity; whereas by a comultiplication on M_B, we mean a unital normal *-homomorphism $\Delta_B : M_B \to M_B \otimes M_B$ satisfying the coassociativity. In the following proposition, we give a more specific description of the C^*-algebra B.

Proposition 6.3. (1) *For $\omega \in \mathcal{B}(\mathcal{H})_*$, we let $\lambda(\omega) = (\omega \otimes \mathrm{id})(\widehat{U_A})$. Then we have: $\lambda(\omega) = j\rho(\omega)j$, where $\rho(\omega) = (\mathrm{id} \otimes \omega)(U_A) \in \hat{\mathcal{A}}(U_A)$ as defined in equation* (2.5).

(2) *Let $\hat{\mathcal{A}}$ be the space of Schwartz functions in the (x, y, r) variables having compact support in the r variable. For $f \in \hat{\mathcal{A}}$, define the operator $\lambda_f \in \mathcal{B}(\mathcal{H})$ by*

$$(\lambda_f \zeta)(x, y, r) = \int f(e^{\lambda\tilde{r}}x, e^{\lambda\tilde{r}}y, r - \tilde{r})\zeta(x, y, \tilde{r})\, d\tilde{r}. \tag{6.1}$$

Then the C^*-algebra B is generated by the operators λ_f. By partial Fourier transform, we can also show that $B \cong \lambda(C^*(G))$, where λ is the left regular representation of $C^*(G)$.

(3) *For any $f, g \in \hat{\mathcal{A}}$, we have: $[\rho_f, \lambda_g] = 0$. Actually, $M_B = M_{\hat{A}}'$, where $M_{\hat{A}}$ is the von Neumann algebra generated by $\hat{A}$.*

(4) *For $b \in (\hat{A}, \hat{\Delta})$, we have: $(\lambda \otimes \lambda)(\hat{\Delta}b) = \widehat{U_A}(\lambda(b) \otimes 1)\widehat{U_A}^*$.*

(5) *Dually, there exists an alternative characterization of the C^*-algebra A:*

$$A = \overline{\{(\mathrm{id}\otimes\omega)(\widehat{U_A}) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\ \|}.$$

And for $a \in A$, we have: $(L \otimes L)(\Delta a) = \widehat{U_A}^*(1 \otimes L(a))\widehat{U_A}$, where L is the regular representation of A defined in Section 2.

Proof. See Proposition 6.8 of [2]. For instance, for the first statement, note that:

$$\lambda(\omega) = (\mathrm{id}\otimes\omega)((j \otimes 1)U_A(j \otimes 1)) = j\rho(\omega)j.$$

The second statement is a consequence of this result. We can also give a direct proof, just as in Propositions 2.2 and 2.6. Actually, we have:

$$j\rho_f j = \lambda_{\tilde{f}}, \qquad f \in \hat{\mathcal{A}},$$

where $\tilde{f}(x, y, r) = \bar{e}[\eta_\lambda(r)\beta(x, y)]f(-e^{\lambda r}x, -e^{\lambda r}y, -r)$.

For the third and fourth statements, we can again refer to general theory (Proposition 6.8 of [2]), or we can just give a direct proof. Since we see (up to partial Fourier transform in the x and y variables) that ρ_f and λ_f are essentially the right and left regular representations of $C^*(G)$, the result follows easily. The last statement is also straightforward (similar to Proposition 2.2). □

Remark. The above proposition implies that at least at the level of the dense subalgebra of functions, B has an opposite algebra structure to that of $\hat{A}$. Meanwhile, (4) above implies that $(\hat{A}, \hat{\Delta}) \cong (B, \Delta_B)$ as Hopf C^*-algebras.

It turns out that working with (B, Δ_B) and $M_B = M_{\hat{A}}{}'$ is more convenient in our proof of Theorem 3.9. Here are a couple of lemmas that are useful in Section 3. Similar results exist for $\hat{A}$ and $M_{\hat{A}}$. We took light versions of the proofs.

Lemma 6.4. *Let M_A and M_B be the enveloping von Neumann algebras of A and B. We have:*

(1) $\widehat{U_A} \in M_A \otimes M_B$.
(2) $M_B \cap M_A = \mathbb{C}1$.
(3) *The linear space $M_B M_A$ is σ-strongly dense in $\mathcal{B}(\mathcal{H})$.*

Proof. The first statement is immediate from general theory, once we realize (see previous proposition) that $\widehat{U_A}$ determines B and A (as well as M_B and M_A). We also have: $\widehat{U_A} \in M(A \otimes B)$. The remaining two results also follow from the same realization. For instance, we could modify the proof of Proposition 2.5 of [24]. □

Lemma 6.5. *On $\hat{\mathcal{A}} \subseteq B$, consider a linear functional φ_B defined by*

$$\varphi_B(\lambda_f) = \int f(x, y, 0)\, dx dy.$$

It can be extended to a faithful, semi-finite, normal weight $\tilde{\varphi}_B$ on M_B.

Remark. The idea for proof of this lemma is pretty much the same as the early part of section 3 (but φ_B is no longer a trace). It turns out that φ_B will be an (invariant) Haar weight for (B, Δ_B), although for our current purposes, this result is not immediately necessary. We will make all these clear in our future paper. Meanwhile, by a straightforward calculation using equation (6.1), we see easily that:

$$\varphi_B({\lambda_f}^* \lambda_f) = \|f\|_2^2.$$

This last result will be useful in the proof of Proposition 3.7.

Using multiplicative unitary operators, we can also give notions that are analogous to "opposite dual" or "co-opposite dual" [11]. For a more careful discussion on the duality, as well as on the notion of quantum double, refer to our future paper.

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