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QUANTUM DOUBLE CONSTRUCTION IN THE C^*-ALGEBRA SETTING OF CERTAIN HEISENBERG-TYPE QUANTUM GROUPS

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Abstract. We carry out the *quantum double construction* of the specific quantum groups we constructed earlier, namely, the "quantum Heisenberg group algebra" (A, Δ) and its dual $(\hat{A}, \hat{\Delta})$. Our approach is by constructing a suitable multiplicative unitary operator, retaining the C^*-algebra framework of locally compact quantum groups. We also discuss the dual of the quantum double and the Haar weights on both of these double objects. Towards the end, a construction of a (quasitriangular) "quantum universal R-matrix" is given.

Introduction. The quantum double construction, which was originally introduced by Drinfeld in the mid-80's for (finite-dimensional) Hopf algebras [7], is among the most celebrated methods of constructing non-commutative and non-cocommutative Hopf algebras. Even in the case of an ordinary group, equivalently for the algebra of (continuous) functions $C(G)$, the quantum double construction leads to an interesting crossed product algebra $C(G) \rtimes_\alpha G$, where α is the conjugation [20], [17].

We wish to carry out a similar construction in the framework of (C^*-algebraic) locally compact quantum groups. This is not totally a new endeavor: As early as in [23], Podles and Woronowicz has constructed their example of a quantum Lorentz group, by considering the quantum double of the compact quantum group $SU_\mu(2)$; Baaj and Skandalis [1] have a version in the context of the multiplicative

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unitary operators; And more recently, Yamanouchi [28] has made this more systematic while Baaj and Vaes [2] consider a more generalized framework of double crossed products.

On the other hand, as is the case for a lot of going-ons in the study of locally compact quantum groups (especially for the non-compact ones), there have been only a handful of work done on actual examples. In this paper, we make a modest contribution by considering the case of the "quantum Heisenberg group algebra" (A, Δ) and the "quantum Heisenberg group" $(\hat{A}, \hat{\Delta})$, which are the specific, mutually dual non-compact quantum groups we obtained previously (See [9], [12], [11].). As we pointed out in a separate paper [14], the examples (A, Δ) and $(\hat{A}, \hat{\Delta})$ are similar, but actually different from (and more general than) the earlier known examples by Rieffel [24] and by Van Daele [26]. We will obtain here the quantum double object of (A, Δ) and $(\hat{A}, \hat{\Delta})$, and show that the quantum double is also a valid locally compact quantum group.

In addition to finding a new example of a quantum group and enriching the duality picture between (A, Δ) and $(\hat{A}, \hat{\Delta})$, there are other merits of studying the quantum double. An interesting point is that the quantum double is not just an algebraic object, but also a nice non-commutative geometric object of study: Note that since the quantum double is obtained as a generalized crossed product, it can be considered as a kind of a "quantized space", while being a quantum group means it is also "group-like". It will be an interesting future research project to further explore how these two different flavors arise together in our example.

At present, the goal of this paper is to give an actual construction of the quantum double, give a concrete realization as an operator algebra on a specific Hilbert space, establish it as a C^*-algebraic, locally compact quantum group in the sense of Kustermans, Vaes [18], or of Masuda, Nakagami, Woronowicz [21]. Also constructed here are the dual object of the quantum double (again a locally compact quantum group), and the "quantum universal R-matrix" type operator for the quantum double.

Our construction method and techniques are strongly motivated by and are based on the fundamental paper by Baaj and Skandalis [1]. Therefore, many of the proofs are not genuinely new.

On the other hand, we note that the presentation given in [1], as well as the ones in [28], [2], are somewhat less suitable for developing a rich connection with the Poisson–Lie group theory. Our presentation is made hoping to improve this situation. By explicitly working with a dense subalgebra of functions contained in a C^*-algebra, we make it much easier to establish a link between the quantum (C^*-algebra) setting and the classical (Poisson–Lie group) level.

The example we are studying may be considered a simple one from quantum group theory point of view, but it is actually not so dull from quantization aspects (Note that it comes from a certain non-linear Poisson bracket: See [9], [14].). The results developed here will be helpful in our future work, where we plan to explore the properties of the quantum double in relation to the Poisson geometric objects like Lie bialgebras and dressing actions.

Here is how the paper is organized: In Section 1, we briefly summarize the quantum double construction in the (finite-dimensional) Hopf algebra setting. We generally follow Majid [20]. We will use this section as a guide for our main construction in the C^*-algebra framework.

In Section 2, we describe the specific quantum groups (A, Δ) and $(\hat{A}, \hat{\Delta})$, reviewing the results from our previous papers. Since the multiplicative unitary operators will play a central role in the later sections, we chose to give characterizations of (A, Δ) and $(\hat{A}, \hat{\Delta})$ as subalgebras in $\mathcal{B}(\mathcal{H})$, via a multiplicative unitary operator U_A.

Our main construction of the quantum double $D(A) = (A_D, \Delta_D)$ is carried out in Section 3. The definition is given in terms of multiplicative unitary operators, but we provide justification that it is compatible with the definition in the purely algebraic setting. Reflecting the fact that the quantum double construction is closed in the category of "Kac algebras", we note that the antipodal map S_D for our example satisfies ${S_D}^2 \cong \mathrm{Id}$. In Section 4, we look at the dual of the quantum double $(\widehat{A_D}, \widehat{\Delta_D})$. Here, $\widehat{A_D} \cong A \otimes \hat{A}^{\mathrm{op}}$ as a C^*-algebra, but its coalgebra structure is twisted.

In Section 5, discussion is given on Haar weights for both of the dual objects $(\widehat{A_D}, \widehat{\Delta_D})$ and (A_D, Δ_D). We see that $(\widehat{A_D}, \widehat{\Delta_D})$ is unimodular, while (A_D, Δ_D) is not. The existence of the legitimate Haar weights assures us that both are (C^*-algebraic) locally compact quantum groups.

In Section 6, we find an operator $\mathcal{R}$ in the multiplier algebra $M(A_D \otimes A_D)$, which can be considered as a "quantum universal R-matrix". We only give its construction here. Its possible applications to representation theory and its connection with the Poisson structure at the classical limit level will be postponed to a future occasion.

Terminology. Let $\mathcal{H}$ be a Hilbert space. A unitary operator $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is said to be *multiplicative*, if it satisfies the "pentagon equation":

$$V_{12} V_{13} V_{23} = V_{23} V_{12} \ \big(\in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})\big).$$

Here, the notation V_{13} indicates that the operator V acts only on the first and third copies of $\mathcal{H}$, while letting the second copy unchanged. Similar comments hold for the others. For a systematic discussion on multiplicative unitary operators, see the paper by Baaj and Skandalis [1].

1. QUANTUM DOUBLE IN THE PURELY ALGEBRAIC FRAMEWORK

In this section, we will work only with finite-dimensional Hopf algebras. The goal here is to collect some useful results from the purely algebraic setting, which will guide us in our main construction at the level of (C^*-algebraic) locally compact quantum groups. Most of the results below are standard ones. See, for instance, [7], [22], [4], [20], [15].

Given a (finite-dimensional) Hopf algebra B, the *quantum double* $D(B)$ is a certain "double crossed product" algebra, $D(B) = B^{*\mathrm{op}} \bowtie B$, where $B^{*\mathrm{op}}$ is same as the dual Hopf algebra B^* but equipped with the opposite multiplication. The Hopf algebras B and $B^{*\mathrm{op}}$ mutually act by (generalized) coadjoint actions. A more precise description is given below.

Definition 1. (1) The (left) coadjoint action of B on $B^{*\mathrm{op}}$ is defined by

$$f \triangleright \phi = \mathrm{Ad}^*_f(\phi) = \sum \phi_{(2)} \bigl\langle f, (S\phi_{(1)})\phi_{(3)} \bigr\rangle.$$

Similarly, we can define the coadjoint action $\triangleleft$ of B^* on B, which we may view as a right action of $B^{*\mathrm{op}}$. That is,

$$f \triangleleft \phi = \mathrm{Ad}^*_\phi(f) = \sum f_{(2)} \bigl\langle (Sf_{(1)})f_{(3)}, \phi \bigr\rangle.$$

(2) The "quantum double" $D(B) = B^{*\mathrm{op}} \bowtie B$ is such that as a space it is isomorphic to $B^* \otimes B$, and is equipped with the multiplication:

$$(\phi \otimes f) \times (\psi \otimes g) := \sum \phi \cdot_{\mathrm{op}} (f_{(1)} \triangleright \psi_{(1)}) \otimes (f_{(2)} \triangleleft \psi_{(2)})g,$$

and the tensor product comultiplication:

$$\Delta_D(\phi \otimes f) := \sum \phi_{(1)} \otimes f_{(1)} \otimes \phi_{(2)} \otimes f_{(2)}.$$

In the above, we are using the standard Sweedler notation (See [22].). That is, we write Δf as $\Delta f = \sum f_{(1)} \otimes f_{(2)}$, and by the coassociativity we have: $(\Delta \otimes \mathrm{id})(\Delta f) = (\mathrm{id} \otimes \Delta)(\Delta f) = \sum f_{(1)} \otimes f_{(2)} \otimes f_{(3)}$. Since we are considering finite-dimensional algebras, $\otimes$ denotes the algebraic tensor product. Meanwhile, S is the antipode (co-inverse) and $\langle\ ,\ \rangle$ is the dual pairing.

The verification of $\triangleright$ and $\triangleleft$ being actions are not difficult, using the coassociativity and the property of the antipode map. As the name suggests, they are generalizations of the coadjoint actions of groups (similar to taking conjugates). The actions make $B^{*\mathrm{op}}$ as a B-module algebra, and B as a $B^{*\mathrm{op}}$-module algebra.

Moreover, $(B^{*\mathrm{op}}, B)$ forms a "matched pair" of Hopf algebras (in the sense of Majid), from which the above definition of $D(B)$ arises. See [19], [20] for details.

Remark. Instead of giving the definition of the quantum double in this way, we could also formulate the definition in terms of a "skew-pairing" between the Hopf algebras B and $B^{*\mathrm{op}}$, which would be the more common approach. See [20], [15]. However, we chose to give our definition as in Definition 1 above, mainly to point out prominently the two actions $\triangleright$ and $\triangleleft$.

We note also that above definition of $D(B)$ is different from Drinfeld's original form [7], containing B and the "co-opposite dual" $B^{*\mathrm{cop}}$. Ours is actually the one proposed by Majid (Theorem 7.1.1 of [20]), which is easily shown to be equivalent by using the antipode of B^*. There are also several other (equally valid) versions. Throughout this paper, due to reasons related with possible future applications, our preferred version of the quantum double $D(B)$ will be as in Definition 1.

It is well known that the quantum double construction leads to a "quasitriangular" Hopf algebra. In case of $D(B)$ as defined here, its quasitriangular structure is given by $R = \sum_j \psi^j \otimes f_j$, where $\{f_j\}$ is a basis for B and $\{\psi^j\}$ its dual basis.

Before we wrap up, let us briefly mention a special case, which will be a motivating model. Consider an ordinary finite group G. Let $B = \mathbb{C}G$ be the group algebra of G, and $B^* = C(G)$ be the algebra of functions on G (note that $B^* = B^{*\mathrm{op}}$ for being commutative), with their natural Hopf algebra structures. Then $D(B)$ becomes the crossed product algebra $C(G) \rtimes G$, given by the conjugate action. In the case of a locally compact group (not necessarily finite), this example was studied by Koornwinder and Muller in [17], [16]. This is a rather simple situation, but it has an interesting interpretation as an algebra of quantum observables of a quantum system (in which a particle is constrained to move on conjugacy classes in G). See Example 6.1.8 of [20]. Meanwhile, some genuine physical applications can be found in [6], [3], where the quantum double is used as a generalized symmetry object.

2. THE QUANTUM HEISENBERG GROUP ALGEBRA (A, Δ) AND THE QUANTUM HEISENBERG GROUP $(\hat{A}, \hat{\Delta})$

We now turn our attention to the C^*-algebra setting. Specifically, let us consider the non-compact quantum groups (A, Δ) and $(\hat{A}, \hat{\Delta})$, which were constructed in [9], [12], [11]. They are mutually dual locally compact quantum groups (in the sense of [18] or [21]).

As we saw in our previous papers, (A,Δ) is regarded as a "quantum Heisenberg group algebra" (i.e. "quantized $C^*(H)$"), while $(\hat{A},\hat{\Delta})$ is viewed as a "quantum Heisenberg group" (i.e. "quantized $C_0(H)$"). Originally, they were obtained by deformation quantization of the (mutually dual) pair of Poisson–Lie groups (G,H), where H is the Heisenberg Lie group and G is its dual Poisson–Lie group carrying a certain non-linear Poisson structure (See [9], for the description of the non-linear Poisson bracket on G and the construction of (A,Δ) as its deformation quantization.).

However, for the purpose of this article, we will de-emphasize the deformation process or the role of Poisson geometry. Instead, our descriptions of (A,Δ) and $(\hat{A},\hat{\Delta})$ will be given in terms of a multiplicative unitary operator U_A. We will postpone to a separate occasion the discussion of the relationships between the Poisson-Lie groups G, H, $D=G\bowtie H$ (described in [10], [13]) and the quantum groups A, $\hat{A}$, $D(A)$ (to be constructed below).

Both C^*-algebras A and $\hat{A}$ are realized as operator algebras contained in $\mathcal{B}(\mathcal{H})$. Here the Hilbert space $\mathcal{H}$ is defined by $L^2(H/Z\times H^*/Z^{\perp})$, where H is the $(2n+1)$-dimensional Heisenberg Lie group (considered naturally as a vector space); Z is the center of H (which is a subspace of H); while H^* is the dual vector space of H; and $Z^{\perp}\subseteq H^*$ is the orthogonal complement of the subspace Z. This means that $\mathcal{H}$ is the space of L^2-functions in the (x,y,r) variables, where $(x,y)\in H/Z(\cong\mathbb{R}^{2n})$ and $r\in H^*/Z^{\perp}(\cong\mathbb{R})$. By partial Fourier transform in the r variable, we have: $\mathcal{H}\cong L^2(H)$ as a Hilbert space.

Consider now the unitary operator $U_A\in\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, as defined in Proposition 3.1 of [9] (See also Proposition 2.2 of [12]):

$$\begin{aligned} U_A\xi(x,y,r;x',y',r') = (e^{-\lambda r'})^n\bar{e}\big[\eta_\lambda(r')\beta(e^{-\lambda r'}x, y'-e^{-\lambda r'}y)\big] \\ \xi(e^{-\lambda r'}x, e^{-\lambda r'}y, r+r'; x'-e^{-\lambda r'}x, y'-e^{-\lambda r'}y, r'). \end{aligned}$$

Note here that we are using a fairly standard notation of $e(t)=e^{2\pi it}$, so $\bar{e}(t)=e^{-2\pi it}$. And $\beta(\ ,\ )$ is the usual inner product. On the other hand, we need some explanation about the (fixed) constant $\lambda\in\mathbb{R}$. It is the constant that determines the aforementioned non-linear Poisson structure when $\lambda\neq 0$ (See [9].). The expression $\eta_\lambda(r)$ is defined such that $\eta_\lambda(r)=\dfrac{e^{2\lambda r}-1}{2\lambda}$, which reflects the non-linear flavor (When $\lambda=0$, we take $\eta_{\lambda=0}(r)=r$ which is linear.).

The unitary operator U_A is multiplicative (satisfying the pentagon equation) and is regular. Therefore, by following Baaj and Skandalis [1], we can define a pair of C^*-bialgebras (A,Δ) and $(\hat{A},\hat{\Delta})$. First, we have:

$$A=\overline{\big\{(\omega\otimes\mathrm{id})(U_A):\omega\in\mathcal{B}(\mathcal{H})_*\big\}}^{\|\ \|},$$

where the $L(\omega)=(\omega\otimes\mathrm{id})(U_A)$ are the "left slices" of U_A by the linear forms $\omega\in\mathcal{B}(\mathcal{H})_*$.

For an alternative characterization of A, consider $\mathcal{A}$, which is the space of Schwartz functions in the (x,y,r) variables having compact support in r. There is the following "regular representation" L of $\mathcal{A}$, on $\mathcal{B}(\mathcal{H})$.

$$(L_f\xi)(x,y,r):=\int f(\tilde{x},\tilde{y},r)\xi(x-\tilde{x},y-\tilde{y},r)\bar{e}\big[\eta_\lambda(r)\beta(\tilde{x},y-\tilde{y})\big]\,d\tilde{x}d\tilde{y}.$$

We have shown in [12] that $A=\overline{L(\mathcal{A})}^{\|\ \|}$. This means that $\mathcal{A}$ is a (norm dense) *-subalgebra of A, and we can regard the functions $f\in\mathcal{A}$ same as the operators L_f. More specifically, the multiplication and the involution on $\mathcal{A}$ take the following form (given by $L_{f\times g}=L_fL_g$ and $L_{f^*}=(L_f)^*$).

$$(f\times_A g)(x,y,r)=\int f(\tilde{x},\tilde{y},r)g(x-\tilde{x},y-\tilde{y},r)\bar{e}\big[\eta_\lambda(r)\beta(\tilde{x},y-\tilde{y})\big]\,d\tilde{x}d\tilde{y}, \tag{2.1}$$
$$f^*(x,y,r)=\bar{e}\big[\eta_\lambda(r)\beta(x,y)\big]\overline{f(-x,-y,r)}.$$

Meanwhile, the multiplicative unitary operator also defines the *comultiplication* $\Delta:A\to M(A\otimes A)$. For $a\in A$, we define Δa by $\Delta a=U_A(a\otimes 1){U_A}^*$. At the level of functions in $\mathcal{A}$, the equation $\Delta(L_f)=(L\otimes L)_{(\Delta f)}$ gives us the following:

$$\begin{aligned}(\Delta f)(x,y,r;x',y',r')&\\ =\int f(x',y',r+r')\bar{e}\big[\tilde{p}\cdot(e^{\lambda r'}x'-x)+\tilde{q}\cdot(e^{\lambda r'}y'-y)\big]\,d\tilde{p}d\tilde{q},&\end{aligned} \tag{2.2}$$

which is a Schwartz function having compact support in r and r'.

There is also the antipodal map $S:A\to A$, defined by $S(a)=\hat{J}a^*\hat{J}$, where $\hat{J}$ is the following involutive operator on $\mathcal{H}$.

$$\hat{J}\xi(x,y,r)=(e^{\lambda r})^n\overline{\xi(e^{\lambda r}x,e^{\lambda r}y,-r)}.$$

See Proposition 2.4 of [12]. Then $S(L_f)=L_{S(f)}$ gives us the following:

$$\big(S(f)\big)(x,y,r)=(e^{2\lambda r})^n\bar{e}\big[\eta_\lambda(r)\beta(x,y)\big]f(-e^{\lambda r}x,-e^{\lambda r}y,-r), \tag{2.3}$$

at the level of functions in $\mathcal{A}$.

Turning our focus to the dual object of (A,Δ), we now consider the "right slices" of U_A. That is, let us now consider the C^*-algebra $\hat{A}$ generated by the operators $\rho(\omega) = (\mathrm{id} \otimes \omega)(U_A)$, for $\omega \in \mathcal{B}(\mathcal{H})_*$. We have:

$$\hat{A} = \overline{\{(\mathrm{id} \otimes \omega)(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\ \|}.$$

The comultiplication $\hat{\Delta} : \hat{A} \to M(\hat{A} \otimes \hat{A})$ is given by $\hat{\Delta} b = {U_A}^*(1 \otimes b)U_A$.

There is also an alternative characterization of $\hat{A}$. For this, consider $\hat{\mathcal{A}}$, which is again the space of Schwartz functions in the (x, y, r) variables having compact support in r. Define the "regular representation" ρ of $\hat{\mathcal{A}}$ on $\mathcal{B}(\mathcal{H})$, given by

$$(\rho_\phi \xi)(x, y, r) := \int (e^{\lambda \tilde{r}})^n \phi(x, y, \tilde{r}) \xi(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r})\, d\tilde{r}.$$

We saw in [11] that $\hat{A} = \overline{\rho(\hat{\mathcal{A}})}^{\|\ \|}$. As before, we can regard the functions $\phi \in \hat{\mathcal{A}}$ same as the operators ρ_ϕ, and $\hat{\mathcal{A}}$ is considered as a dense *-subalgebra of $\hat{A}$. On $\hat{\mathcal{A}}$, the multiplication and the involution take the following form (via $\rho_{\phi \times \psi} = \rho_\phi \rho_\psi$ and $\rho_{\phi^*} = (\rho_\phi)^*$).

$$(\phi \times_{\hat{A}} \psi)(x, y, r) = \int \phi(x, y, \tilde{r}) \psi(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r})\, d\tilde{r}, \tag{2.4}$$
$$\phi^*(x, y, r) = \overline{\phi(e^{\lambda r} x, e^{\lambda r} y, -r)}.$$

Meanwhile, we have the following description of the comultiplication, obtained by the equation $\hat{\Delta}(\rho_\phi) = (\rho \otimes \rho)_{(\hat{\Delta}\phi)}$.

$$\begin{aligned} &(\hat{\Delta}\phi)(x, y, r; x', y', r') \\ &\quad = \int \phi(x + x', y + y', \tilde{r}) e\big[\eta_\lambda(\tilde{r}) \beta(x, y')\big] e\big[\tilde{r}(z + z')\big] \bar{e}[zr + z'r']\, d\tilde{r} dz dz'. \end{aligned} \tag{2.5}$$

The antipode $\hat{S} : \hat{A} \to \hat{A}$ is given by $\hat{S} = Jb^*J$, where J is the operator on $\mathcal{H}$ defined by

$$J\xi(x, y, r) = \bar{e}\big[\eta_\lambda(r) \beta(x, y)\big] \overline{\xi(-x, -y, r)}.$$

Then at the level of functions in $\hat{\mathcal{A}}$, the expression $\hat{S}(\rho_\phi) = \rho_{\hat{S}(\phi)}$ gives us the following:

$$\big(\hat{S}(\phi)\big)(x, y, r) = \bar{e}\big[\eta_\lambda(r) \beta(x, y)\big] \phi(-e^{\lambda r} x, -e^{\lambda r} y, -r). \tag{2.6}$$

We have further shown in our previous papers that the two C^*-bialgebras (A, Δ) and $(\hat{A}, \hat{\Delta})$ are indeed examples of non-compact, C^*-algebraic quantum groups (together with the necessary ingredients like Haar weights). They are mutually dual objects in the framework of locally compact quantum groups. See [12] and [11]. Since the square of the antipode map is identity for both of them (which can be easily seen from the definitions of S and $\hat{S}$ given above), they are cases of *Kac C^*-algebras* (as in [25]).

Unlike in the purely algebraic or finite-dimensional setting, no proper dual pairing exists between A and $\hat{A}$. However, at least at the level of the dense subalgebras $\mathcal{A}$ and $\hat{\mathcal{A}}$, there does exist a suitable dual pairing, defined as follows:

$$\langle f, \phi \rangle = \int f(x,y,r)\phi(e^{\lambda r}x, e^{\lambda r}y, -r)\,dxdydr, \tag{2.7}$$

for $f(=L_f) \in \mathcal{A}$ and $\phi(=\rho_\phi) \in \hat{\mathcal{A}}$. See Proposition 3.1 of [11], which is just an immediate consequence of Definition 1.3 of [1]. As we have shown in Proposition 3.1 of [11], this dual pairing satisfies all the necessary properties for it to be considered as the correct analog of the pairing in the (purely algebraic) framework of Hopf algebras.

Since we are planning to construct the quantum double, we also need to clarify the "opposite" and "co-opposite" versions of (A, Δ) and $(\hat{A}, \hat{\Delta})$, again in the C^*-algebra framework. For this purpose, it is useful to know that we can form a *Kac system* (in the sense of Baaj and Skandalis) from our multiplicative unitary operator U_A. The following observation was made in Section 3 of [11].

Proposition 2.1. *Let $j \in \mathcal{B}(\mathcal{H})$ be defined by $j = \hat{J}J = J\hat{J}$, where J and $\hat{J}$ are the anti-unitary operators as appeared in the definitions of the antipode maps. Then j is an (involutive) unitary operator given by*

$$j\xi(x,y,r) = (e^{\lambda r})^n \bar{e}\big[\eta_\lambda(r)\beta(x,y)\big]\xi(-e^{\lambda r}x, -e^{\lambda r}y, -r).$$

Moreover, the triple $(\mathcal{H}, U_A, j)$ forms a "Kac system" (as in Section 6 of [1]*). In particular, the following unitary operators are all multiplicative:*

$$\begin{aligned}
&U_A \in M(\hat{A} \otimes A),\\
&\widehat{U_A} = \Sigma(j \otimes 1)U_A(j \otimes 1)\Sigma \in M(A \otimes \hat{A}^{\mathrm{op}}),\\
&\widetilde{U_A} = (j \otimes j)\widehat{U_A}(j \otimes j) = (j \otimes 1)(\Sigma U_A \Sigma)(j \otimes 1) \in M(A^{\mathrm{op}} \otimes \hat{A}),\\
&\widehat{\widehat{U_A}} = \widetilde{\widetilde{U_A}} = (j \otimes j)U_A(j \otimes j) \in M(\hat{A}^{\mathrm{op}} \otimes A^{\mathrm{op}}),
\end{aligned}$$

where Σ denotes the flip.

Remark. The results may be checked by a direct computation (See Proposition 3.2 of [11].). But this proposition is really due to the properties of the (anti-unitary) operators J and $\hat{J}$. See also 6.11 (d) of [1].

Using the multiplicative unitary operators U_A and its variations obtained in the above proposition, we can define several different versions of the quantum Heisenberg group algebra and the quantum Heisenberg group, in the form of $(A^{\mathrm{op}}, \Delta)$, $(A, \Delta^{\mathrm{cop}})$, $(A^{\mathrm{op}}, \Delta^{\mathrm{cop}})$, as well as $(\hat{A}^{\mathrm{op}}, \hat{\Delta})$, $(\hat{A}, \hat{\Delta}^{\mathrm{cop}})$, $(\hat{A}^{\mathrm{op}}, \hat{\Delta}^{\mathrm{cop}})$.

For instance, $(\hat{A}^{\mathrm{op}}, \hat{\Delta})$ is determined by the multiplicative unitary operator $X = \Sigma \widehat{U_A}^{\,*} \Sigma$. To be more precise, we have:

$$\hat{A}^{\mathrm{op}} = \overline{\bigl\{(\mathrm{id} \otimes \omega)(X) : \omega \in \mathcal{B}(\mathcal{H})_*\bigr\}}^{\|\ \|} = \overline{\lambda(\hat{\mathcal{A}})}^{\|\ \|} \ \bigl(\subseteq \mathcal{B}(\mathcal{H})\bigr),$$

where $\lambda : \hat{\mathcal{A}} \to \mathcal{B}(\mathcal{H})$ is defined by

$$(\lambda_\phi \xi)(x, y, r) := \int \phi(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}) \xi(x, y, \tilde{r})\, d\tilde{r}.$$

Notice that $\lambda_\phi \lambda_\psi = \lambda_{\psi \times \phi}$, implementing the opposite multiplication. The comultiplication, given by $\hat{A}^{\mathrm{op}} \ni b \mapsto X^*(1 \otimes b)X \in M(\hat{A}^{\mathrm{op}} \otimes \hat{A}^{\mathrm{op}})$, stays the same at the function level: That is, $X^*(1 \otimes \lambda_\phi)X = (\lambda \otimes \lambda)_{(\hat{\Delta}\phi)}$, where $\hat{\Delta}\phi$ is same as in equation (2.5). The antipode also stays the same: $\hat{S}(\lambda_\phi) = \lambda_{\hat{S}(\phi)}$, as in equation (2.6).

We only gave here one possible description of $(\hat{A}^{\mathrm{op}}, \hat{\Delta})$, since $\hat{A}^{\mathrm{op}}$ is the one we immediately need for the definition of our quantum double. But See Proposition 3.5 of [11] for the others.

3. The quantum double

Since we know the expressions for various operations on $\mathcal{A}$ and $\hat{\mathcal{A}}$ (the equations (2.1), (2.2), (2.3), and (2.4), (2.5), (2.6)), as well as the expression for the dual pairing between them given by equation (2.7), we can use Definition 1 to write down the product on the quantum double, at the level of functions in $\mathcal{A} \odot \hat{\mathcal{A}}$ [the algebraic tensor product, without any completion]. So we have:

(3.1)

$$\begin{aligned}
&\bigl((\phi \otimes f) \times (\psi \otimes g)\bigr)(x, y, r; x', y', r') \\
&= \int \phi(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}) \psi(x - e^{\lambda(r' - \tilde{r})} \tilde{x} + e^{-\lambda \tilde{r}} \tilde{x}, y - e^{\lambda(r' - \tilde{r})} \tilde{y} + e^{-\lambda \tilde{r}} \tilde{y}, \tilde{r}) \\
&\qquad \bar{e}\bigl[\eta_\lambda(r') \beta(e^{-\lambda \tilde{r}} \tilde{x}, y')\bigr]\, e\bigl[\eta_\lambda(\tilde{r}) \beta(x, e^{-\lambda \tilde{r}} \tilde{y})\bigr]\, \bar{e}\bigl[\eta_\lambda(\tilde{r}) \beta(e^{\lambda(r' - \tilde{r})} \tilde{x}, y)\bigr] \\
&\qquad e\bigl[\eta_\lambda(r') \beta(\tilde{x}, \tilde{y})\bigr]\, e\bigl[\eta_\lambda(\tilde{r}) \beta(e^{-\lambda \tilde{r}} \tilde{x}, e^{-\lambda \tilde{r}} \tilde{y})\bigr]\, \bar{e}\bigl[\eta_\lambda(\tilde{r}) \beta(e^{\lambda(r' - \tilde{r})} \tilde{x}, e^{-\lambda \tilde{r}} \tilde{y})\bigr] \\
&\qquad f(\tilde{x}, \tilde{y}, r') g(x' - e^{-\lambda \tilde{r}} \tilde{x}, y' - e^{-\lambda \tilde{r}} \tilde{y}, r')\, d\tilde{x} d\tilde{y} d\tilde{r}.
\end{aligned}$$

Here, $\phi, \psi \in \hat{\mathcal{A}}$ and $f, g \in \mathcal{A}$. Computation is rather long, but not really difficult.

However, for us to be able to define $D(A)$ properly at the C^*-algebra level, it is again best to work with the multiplicative unitary operators. Since we wish to construct an object that will be considered as containing (A, Δ) and its "opposite dual" $(\hat{A}^{\mathrm{op}}, \hat{\Delta})$, with some actions involved, let us define the following unitary operator:

$$V_D = Z_{12} Y_{24} Z^*_{12} X_{13} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}). \tag{3.2}$$

Here $X = \Sigma \widehat{U_A}^* \Sigma$ is as defined in the previous section, $Y = \Sigma X^* \Sigma = \widehat{U_A}$, while $Z = Y^* \widehat{\widehat{Y}} = \widehat{\widehat{Y}} Y^*$ (We can see easily from Proposition 2.1 that $Y \in M(A \otimes \hat{A}^{\mathrm{op}})$ and $\widehat{\widehat{Y}} \in M(A^{\mathrm{op}} \otimes \hat{A})$. See also Corollary of Proposition 3.5 of [11].). The leg numbering notation is as before.

Main motivation for our choice comes from Section 8 of [1], and the formulation is essentially equivalent to the ones given in [28], [2] (though slightly different). Roughly speaking, the operator X gives $(\hat{A}^{\mathrm{op}}, \hat{\Delta})$ (as we saw in Section 2), the operator Y gives (A, Δ) (as in Proposition 3.5 of [11]), and the operator Z enables us to encode the generalized coadjoint actions. See Proposition 3.2 below, which comes after the following short lemma:

Lemma 3.1. *For any $a \in A$ and $b \in \hat{A}^{\mathrm{op}}$, we have:*

$$Z(a \otimes b)Z^* = Y^*(a \otimes b)Y.$$

Proof. The proof easily follows from the fact that $Z = Y^* \widehat{\widehat{Y}} = \widehat{\widehat{Y}} Y^*$, while $Y \in M(A \otimes \hat{A}^{\mathrm{op}})$ and $\widehat{\widehat{Y}} \in M(A^{\mathrm{op}} \otimes \hat{A})$. Actually, the result will still hold if $a \in M(A)$ and $b \in M(\hat{A}^{\mathrm{op}})$. ☐

Proposition 3.2. *Let $Z = Y^* \widehat{\widehat{Y}} = \widehat{\widehat{Y}} Y^* \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ be the operator defined above. Explicitly,*

$$\begin{aligned} Z\xi(x, y, r; x', y', r') = \bar{e}\big[\eta_\lambda(r)\beta(e^{\lambda r'}x', y - e^{\lambda r'}y')\big]\, e\big[\eta_\lambda(r)\beta(x - e^{\lambda r'}x', y')\big] \\ (e^{\lambda r})^n\, \xi(x - e^{\lambda r'}x' + x', y - e^{\lambda r'}y' + y', r; e^{\lambda r}x', e^{\lambda r}y', r'). \end{aligned}$$

Let $\alpha : A \to M(\hat{A}^{\mathrm{op}} \otimes A)$ and $\alpha' : \hat{A}^{\mathrm{op}} \to M(\hat{A}^{\mathrm{op}} \otimes A)$ be defined by

$$\alpha(a) := \Sigma Z^*(a \otimes 1)Z\Sigma \qquad \textit{and} \qquad \alpha'(b) := \Sigma Z^*(1 \otimes b)Z\Sigma.$$

Then α is a left coaction of (A, Δ) on the C^*-algebra $\hat{A}^{\mathrm{op}}$, while α' is a right coaction of $(\hat{A}^{\mathrm{op}}, \hat{\Delta})$ on A. That is, the maps α and α' are non-degenerate *-homomorphisms such that:

$$(\hat{\Delta} \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\alpha \qquad \textit{and} \qquad (\mathrm{id} \otimes \Delta)\alpha' = (\alpha' \otimes \mathrm{id})\alpha'.$$

Proof. Let $a \in A$. Then we have:

$$\begin{aligned}(\mathrm{id}\otimes\alpha)\alpha(a) &= \Sigma_{23}Z_{23}^*\Sigma_{12}Z_{12}^*(a\otimes 1\otimes 1)Z_{12}\Sigma_{12}Z_{23}\Sigma_{23} \\ &= \Sigma_{23}Y_{23}\Sigma_{12}Y_{12}(a\otimes 1\otimes 1)Y_{12}^*\Sigma_{12}Y_{23}^*\Sigma_{23} \\ &= X_{23}^*\Sigma_{23}X_{12}^*\Sigma_{12}(a\otimes 1\otimes 1)\Sigma_{12}X_{12}\Sigma_{23}X_{23} \\ &= X_{23}^*X_{13}^*(1\otimes 1\otimes a)X_{13}X_{23}.\end{aligned}$$

In the second equality, we are using Lemma 3.1. And in the third equality, we use $Y = \Sigma X^*\Sigma$. On the other hand, remembering that $\hat{\Delta}(b) = X^*(1\otimes b)X$, for $b \in \hat{A}^{\mathrm{op}}$, we have:

$$\begin{aligned}(\hat{\Delta}\otimes\mathrm{id})\alpha(a) &= X_{12}^*\Sigma_{23}Z_{23}^*(1\otimes a\otimes 1)Z_{23}\Sigma_{23}X_{12} \\ &= X_{12}^*X_{23}^*\Sigma_{23}(1\otimes a\otimes 1)\Sigma_{23}X_{23}X_{12} \\ &= X_{12}^*X_{23}^*(1\otimes 1\otimes a)X_{23}X_{12} \\ &= X_{23}^*X_{13}^*X_{12}^*(1\otimes 1\otimes a)X_{12}X_{13}X_{23} \\ &= X_{23}^*X_{13}^*(1\otimes 1\otimes a)X_{13}X_{23}.\end{aligned}$$

We again used Lemma 3.1 and $Y = \Sigma X^*\Sigma$ in the second equality above. In the fourth equality, the multiplicativity of X (satisfying the pentagon equation: $X_{12}X_{13}X_{23} = X_{23}X_{12}$) was used. In this way, we show that: $(\hat{\Delta}\otimes\mathrm{id})\alpha = (\mathrm{id}\otimes\alpha)\alpha$.

To prove the condition for α', we may use the fact that $\Delta a = Y^*(1\otimes a)Y$, for $a \in A$ (see Proposition 3.5 of [11]), and proceed similarly as above. $\square$

Remark. The coactions α and α' are essentially "coadjoint coactions", which (dually) correspond to the "coadjoint actions" given in Definition 1. Indeed, at least at the level of functions in $\mathcal{A}$ and $\hat{\mathcal{A}}$, it is possible to show that:

$$\big\langle \alpha(a), f\otimes\phi \big\rangle = \langle a, f \triangleright \phi\rangle \qquad \text{and} \qquad \big\langle \alpha'(b), f\otimes\phi \big\rangle = \langle b, f \triangleleft \phi\rangle,$$

where $a, f \in \mathcal{A}$ and $b, \phi \in \hat{\mathcal{A}}$, while $\langle\ ,\ \rangle$ is the dual pairing given in equation (2.7).

In the ensuing paragraphs, we will show that the operator V_D as defined in equation (3.2) is actually multiplicative, and make our case that the C^*-bialgebra it determines is exactly the quantum double $D(A)$ we are looking for. In particular, we will see that the C^*-algebra contains as a dense subalgebra $\hat{\mathcal{A}} \odot \mathcal{A}$, with its product defined in equation (3.1).

The multiplicativity of V_D (i. e. satisfying the "pentagon equation") could be shown directly, but the computation will be rather long and tedious due to the fact that we have to work with 18 variables. So we present here an alternative way, where the crucial point is that the operators X and Y give rise to a "matched pair" (See Definition 8.13 of [1]) of Kac systems.

Lemma 3.3. *Let the notation be as above.*

1. *The triples* $(\mathcal{H}, X, j)$ *and* $(\mathcal{H}, Y, j)$*, together with the operator* Z*, form a matched pair of Kac systems.*
2. *The operator* $V := Z^*_{12}X_{13}Z_{12}Y_{24}$ *is multiplicative.*
3. $Z_{34}V = Z_{34}Z^*_{12}X_{13}Z_{12}Y_{24} = Y_{24}Z^*_{12}X_{13}Z_{12}Z_{34}$.

Proof. (1) is the result of Theorem 8.17 of [1], from which the multiplicativity of V follows (By Definition 8.15 of [1], the operator V determines the "Z-tensor product" of the matched pair.). See also our Proposition 4.2 and Corollary 4.3 in Section 4 below. Meanwhile, by Proposition 8.10 of [1], condition (2) is equivalent to condition (3) (We can also check (3) directly from the definitions.). $\square$

Corollary 3.4. *The unitary operator* V_D *defined in* (3.2) *is multiplicative.*

Proof. By (3) of Lemma 3.3, we have: $Z_{12}Z_{34}VZ^*_{34}Z^*_{12} = Z_{12}Y_{24}Z^*_{12}X_{13} = V_D$. Re-writing this expression as $V_D = (Z \otimes Z)V(Z^* \otimes Z^*)$ and noting that Z is unitary, we see clearly that V_D is also a multiplicative unitary operator (by being unitarily equivalent to V). $\square$

By the general theory of multiplicative unitary operators [1], [27], the operator V_D will let us define a C^*-bialgebra, on which we build the necessary ingredients for it to become a locally compact quantum group. Specifically, let us consider the C^*-bialgebra (A_D, Δ_D), which is generated by the "right slices" of V_D, as follows:

Definition 2. Let A_D be the C^*-algebra contained in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, defined by

$$A_D = \overline{\left\{(\mathrm{id} \otimes \mathrm{id} \otimes \Omega)(V_D) : \Omega \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})_*\right\}}^{\|\ \|}.$$

Let us also define the *comultiplication* $\Delta_D : A_D \to M(A_D \otimes A_D)$, by $\Delta_D(x) := {V_D}^*(1 \otimes 1 \otimes x)V_D$, for $x \in A_D$. It is a non-degenerate C^*-homomorphism satisfying the coassociativity condition: $(\Delta_D \otimes \mathrm{id})\Delta_D = (\mathrm{id} \otimes \Delta_D)\Delta_D$. So A_D is a C^*-bialgebra. Moreover, $\Delta_D(A_D)(A_D \otimes 1)$ and $\Delta_D(A_D)(1 \otimes A_D)$ are dense subsets in $A_D \otimes A_D$.

For the last statement (the density conditions), see Theorem 1.5 and Section 5 of [27] (It is a non-trivial result.). Our goal now is to show that (A_D, Δ_D) is exactly the quantum double $D(A)$, analogous to Definition 1. Let us first give a more concrete C^*-algebraic realization of A_D.

Proposition 3.5. *Let* $\pi : A \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ *and* $\pi' : \hat{A}^{\mathrm{op}} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ *be defined by*

$$\pi(a) := Z(1 \otimes a)Z^* \qquad \text{and} \qquad \pi'(b) := b \otimes 1.$$

Then A_D *is the* C^**-algebra generated by the operators* $\pi(a)\pi'(b)$*, for* $a \in A$*,* $b \in \hat{A}^{\mathrm{op}}$*.*

Proof. For $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$, we have:

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \omega')(V_D) &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \omega')(Z_{12} Y_{24} Z_{12}^* X_{13}) \\ &= Z\big[1 \otimes (\mathrm{id} \otimes \omega')(Y)\big] Z^* \big[(\mathrm{id} \otimes \omega)(X) \otimes 1\big] = \pi(a)\pi'(b),\end{aligned}$$

where $a = (\mathrm{id} \otimes \omega')(Y)$ and $b = (\mathrm{id} \otimes \omega)(X)$. This is valid, because by Proposition 3.5 (2) of [11] and by Section 6 (Appendix) of [12], we have: $a = (\mathrm{id} \otimes \omega')(Y) = (\mathrm{id} \otimes \omega')(\widehat{U_A}) \in A$ and $b = (\mathrm{id} \otimes \omega)(X) = (\mathrm{id} \otimes \omega)(\Sigma \widehat{U_A}^* \Sigma) \in \hat{A}^{\mathrm{op}}$. In fact, the operators $(\mathrm{id} \otimes \omega')(Y)$, $\omega' \in \mathcal{B}(\mathcal{H})_*$, generate A; while the operators $(\mathrm{id} \otimes \omega)(X)$, $\omega \in \mathcal{B}(\mathcal{H})_*$, generate $\hat{A}^{\mathrm{op}}$.

Since the operators $(\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \omega')(V_D)$ generate A_D (Definition 2), the result of the proposition follows. ☐

Corollary 3.6. *The maps* π *and* π' *are* C^**-algebra homomorphisms. Namely,*

$$\pi : A \to M(A_D) \qquad \text{and} \qquad \pi' : \hat{A}^{\mathrm{op}} \to M(A_D).$$

Remark. The corollary is obvious from the definitions of π and π'. Here, $M(A_D)$ denotes the multiplier algebra of A_D. Later, when we clarify the co-algebra structure on A_D, they will actually become C^*-bialgebra homomorphisms.

Proposition 3.7. *Let* $\Pi : \hat{\mathcal{A}} \odot \mathcal{A} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ *be defined by*

$$\Pi(\phi \otimes f) := \pi'(\lambda_\phi)\pi(L_f), \qquad \text{for } \phi \in \hat{\mathcal{A}}, f \in \mathcal{A}.$$

Then $A_D = \overline{\Pi(\hat{\mathcal{A}} \odot \mathcal{A})}^{\|\ \|}$.

Proof. We know from Section 2 that $A = \overline{L(\mathcal{A})}^{\|\ \|}$ and $\hat{A}^{\mathrm{op}} = \overline{\lambda(\hat{\mathcal{A}})}^{\|\ \|}$. So this result is an immediate consequence of Proposition 3.5, with the aid of the fact that the C^*-algebras are closed under involution. ☐

We observe that Π above determines a multiplication on $\hat{\mathcal{A}} \odot \mathcal{A}$, given by $\Pi(\phi \otimes f)\Pi(\psi \otimes g) = \Pi\big((\phi \otimes f) \times (\psi \otimes g)\big)$, making $\hat{\mathcal{A}} \odot \mathcal{A}$ a (dense) subalgebra of A_D. It turns out that the product obtained in this way exactly coincides with the one given in (3.1). See below.

Proposition 3.8. *Let $\hat{\mathcal{A}} \odot \mathcal{A}$ be given the multiplication, as in* (3.1)*. Then we have:*

$$\Pi(\phi \otimes f)\Pi(\psi \otimes g) = \Pi\big((\phi \otimes f) \times (\psi \otimes g)\big),$$

for $\phi, \psi \in \hat{\mathcal{A}}$ and $f, g \in \mathcal{A}$.

Proof. For $\phi \in \hat{\mathcal{A}}$ and $\xi \in \mathcal{H} \otimes \mathcal{H}$,

$$\begin{aligned}\pi'(\lambda_\phi)\xi(x,y,r;x',y',r') &= (\lambda_\phi \otimes 1)\xi(x,y,r;x',y',r') \\ &= \int \phi(e^{\lambda\tilde{r}}x, e^{\lambda\tilde{r}}y, r-\tilde{r})\xi(x,y,\tilde{r};x',y',r')\,d\tilde{r}.\end{aligned}$$

Whereas for $f \in \mathcal{A}$ and $\xi \in \mathcal{H} \otimes \mathcal{H}$,

$$\begin{aligned}&\pi(L_f)\xi(x,y,r;x',y',r') = Z(1 \otimes L_f)Z^*\xi(x,y,r;x',y',r') \\ &= (e^{\lambda r})^n\,\bar{e}\big[\eta_\lambda(r)\beta(e^{\lambda r'}x', y - e^{\lambda r'}y')\big]\,e\big[\eta_\lambda(r)\beta(x - e^{\lambda r'}x', y')\big] \\ &\quad (1 \otimes L_f)Z^*\xi(x - e^{\lambda r'}x' + x', y - e^{\lambda r'}y' + y', r; e^{\lambda r}x', e^{\lambda r}y', r') \\ &= \int (e^{\lambda r})^n\,\bar{e}\big[\eta_\lambda(r)\beta(e^{\lambda r'}x', y - e^{\lambda r'}y')\big]\,e\big[\eta_\lambda(r)\beta(x - e^{\lambda r'}x', y')\big] \\ &\qquad f(\tilde{x},\tilde{y},r')\bar{e}\big[\eta_\lambda(r')\beta(\tilde{x}, e^{\lambda r}y' - \tilde{y})\big] \\ &\qquad Z^*\xi(x - e^{\lambda r'}x' + x', y - e^{\lambda r'}y' + y', r; e^{\lambda r}x' - \tilde{x}, e^{\lambda r}y' - \tilde{y}, r')\,d\tilde{x}d\tilde{y} \\ &= (\cdots) \\ &= \int f(\tilde{x},\tilde{y},r')\bar{e}\big[\eta_\lambda(r')\beta(e^{-\lambda r}\tilde{x}, y')\big]\,e\big[\eta_\lambda(r)\beta(x, e^{-\lambda r}\tilde{y})\big]\,\bar{e}\big[\eta_\lambda(r)\beta(e^{\lambda r' - \lambda r}\tilde{x}, y)\big] \\ &\qquad e\big[\eta_\lambda(r')\beta(\tilde{x},\tilde{y})\big]\,e\big[\eta_\lambda(r)\beta(e^{-\lambda r}\tilde{x}, e^{-\lambda r}\tilde{y})\big]\,\bar{e}\big[\eta_\lambda(r)\beta(e^{\lambda r' - \lambda r}\tilde{x}, e^{-\lambda r}\tilde{y})\big] \\ &\qquad \xi(x - e^{\lambda r' - \lambda r}\tilde{x} + e^{-\lambda r}\tilde{x}, y - e^{\lambda r' - \lambda r}\tilde{y} + e^{-\lambda r}\tilde{y}, r; x' - e^{-\lambda r}\tilde{x}, y' - e^{-\lambda r}\tilde{y}, r')\,d\tilde{x}d\tilde{y}.\end{aligned}$$

So we have:

$$\begin{aligned}&\Pi(\phi \otimes f)\xi(x,y,r;x',y',r') = \pi'(\lambda_\phi)\pi(L_f)\xi(x,y,r;x',y',r') \\ &= \int \phi(e^{\lambda\tilde{r}}x, e^{\lambda\tilde{r}}y, r - \tilde{r})f(\tilde{x},\tilde{y},r')\bar{e}\big[\eta_\lambda(r')\beta(e^{-\lambda\tilde{r}}\tilde{x}, y')\big] \\ &\qquad e\big[\eta_\lambda(\tilde{r})\beta(x, e^{-\lambda\tilde{r}}\tilde{y})\big]\,\bar{e}\big[\eta_\lambda(\tilde{r})\beta(e^{\lambda r' - \lambda\tilde{r}}\tilde{x}, y)\big] \\ &\qquad e\big[\eta_\lambda(r')\beta(\tilde{x},\tilde{y})\big]\,e\big[\eta_\lambda(\tilde{r})\beta(e^{-\lambda\tilde{r}}\tilde{x}, e^{-\lambda\tilde{r}}\tilde{y})\big]\,\bar{e}\big[\eta_\lambda(\tilde{r})\beta(e^{\lambda r' - \lambda\tilde{r}}\tilde{x}, e^{-\lambda\tilde{r}}\tilde{y})\big] \\ &\qquad \xi(x - e^{\lambda r' - \lambda\tilde{r}}\tilde{x} + e^{-\lambda\tilde{r}}\tilde{x}, y - e^{\lambda r' - \lambda\tilde{r}}\tilde{y} + e^{-\lambda\tilde{r}}\tilde{y}, \tilde{r}; x' - e^{-\lambda\tilde{r}}\tilde{x}, y' - e^{-\lambda\tilde{r}}\tilde{y}, r')\,d\tilde{x}d\tilde{y}d\tilde{r}.\end{aligned}$$

Using this and noting its resemblance to equation (3.1), we can see without trouble that: $\Pi(\phi \otimes f)\Pi(\psi \otimes g)\xi = \Pi\big((\phi \otimes f) \times (\psi \otimes g)\big)\xi$, for any $\xi \in \mathcal{H} \otimes \mathcal{H}$. ☐

The involution on A_D is inherited from that of $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$. At the level of the subalgebra $\hat{\mathcal{A}}\odot\mathcal{A}$, it takes the following form:

$$\begin{aligned}
&(\phi\otimes f)^*(x,y,r;x',y',r') \\
&= (e^{2\lambda r})^n\,\overline{f(-e^{\lambda r}x',-e^{\lambda r}y',r')}\,e\big[\eta_\lambda(r)\beta(x,y')\big]\,\bar{e}\big[\eta_\lambda(r)\beta(e^{\lambda r'}x',y)\big] \\
&\quad \bar{e}\big[\eta_\lambda(r'-r)\beta(e^{\lambda r}x',e^{\lambda r}y')\big]\,\bar{e}\big[\eta_\lambda(r)\beta(e^{\lambda r'}x',y')\big] \\
&\quad \overline{\phi(e^{\lambda r}x-e^{\lambda r'+\lambda r}x'+e^{\lambda r}x',e^{\lambda r}y-e^{\lambda r'+\lambda r}y'+e^{\lambda r}y',-r)}.
\end{aligned}\tag{3.3}$$

To be more precise, the definition of $(\phi\otimes f)^*\in\hat{\mathcal{A}}\odot\mathcal{A}$ above has been chosen so that we have: $\Pi\big((\phi\otimes f)^*\big)=\big[\Pi(\phi\otimes f)\big]^*=\pi(L_f)^*\pi'(\lambda_\phi)^*$.

Next, let us turn our attention to the co-algebra structure on A_D. As in the previous proposition, we will see that at the level of functions in $\hat{\mathcal{A}}\odot\mathcal{A}$, the comultiplication on A_D exactly coincides with the one on $D(A)$, as given in Definition 1.

Proposition 3.9. *For $\phi\in\hat{\mathcal{A}}$ and $f\in\mathcal{A}$, we have:*

$$\begin{aligned}
\Delta_D\big(\Pi(\phi\otimes f)\big)&=\Delta_D\big(\pi'(\lambda_\phi)\pi(L_f)\big)=\big[(\pi'\otimes\pi')(\hat{\Delta}\phi)\big]\big[(\pi\otimes\pi)(\Delta f)\big] \\
&=(\Pi\otimes\Pi)\left(\sum\phi_{(1)}\otimes f_{(1)}\otimes\phi_{(2)}\otimes f_{(2)}\right).
\end{aligned}$$

Proof. Note that by Definition 2, we have:

$$\begin{aligned}
\Delta_D\big(\pi'(\lambda_\phi)\pi(L_f)\big)&={V_D}^*\big(1\otimes1\otimes\pi'(\lambda_\phi)\pi(L_f)\big)V_D \\
&=\big[{V_D}^*\big(1\otimes1\otimes\pi'(\lambda_\phi)\big)V_D\big]\big[{V_D}^*\big(1\otimes1\otimes\pi(L_f)\big)V_D\big].
\end{aligned}$$

But by definition of V_D and by definition of π', we have:

$$\begin{aligned}
{V_D}^*\big(1\otimes1\otimes\pi'(\lambda_\phi)\big)V_D&=X_{13}^*Z_{12}Y_{24}^*Z_{12}^*(1\otimes1\otimes\lambda_\phi\otimes1)Z_{12}Y_{24}Z_{12}^*X_{13} \\
&=X_{13}^*(1\otimes1\otimes\lambda_\phi\otimes1)X_{13} \\
&=\big[(\lambda\otimes\lambda)(\hat{\Delta}\phi)\big]_{13}=(\pi'\otimes\pi')(\hat{\Delta}\phi).
\end{aligned}$$

Similarly,

$$\begin{aligned}
{V_D}^*\big(1\otimes1\otimes\pi(L_f)\big)V_D&=X_{13}^*Z_{12}Y_{24}^*Z_{12}^*\big[Z_{34}(1\otimes L_f)_{34}Z_{34}^*\big]Z_{12}Y_{24}Z_{12}^*X_{13} \\
&=Z_{12}Z_{34}Y_{24}^*Z_{12}^*X_{13}^*\big[(1\otimes L_f)\big]_{34}X_{13}Z_{12}Y_{24}Z_{34}^*Z_{12}^* \\
&=Z_{12}Z_{34}Y_{24}^*\big[(1\otimes1\otimes1\otimes L_f)\big]Y_{24}Z_{34}^*Z_{12}^* \\
&=Z_{12}Z_{34}\big[(L\otimes L)(\Delta f)\big]_{24}Z_{34}^*Z_{12}^*=(\pi\otimes\pi)(\Delta f).
\end{aligned}$$

In the second equality above, we used the result of Lemma 3.3 (3).

Combining these results, we prove the proposition. □

Corollary 3.10. *The maps* $\pi : A \to M(A_D)$ *and* $\pi' : \hat{A}^{\mathrm{op}} \to M(A_D)$*, as defined earlier, are* C^**-bialgebra homomorphisms.*

Proof. We already know from Corollary 3.6 earlier that π and π' are C^*-algebra homomorphisms. Meanwhile, from the proof of Proposition 3.9, we see that: $(\pi \otimes \pi) \circ \Delta = \Delta_D \circ \pi$, and $(\pi' \otimes \pi') \circ \hat{\Delta} = \Delta_D \circ \pi'$. ☐

Propositions 3.7, 3.8, 3.9 support our assertion that (A_D, Δ_D) is indeed the C^*-algebraic analog of the quantum double $D(A) = \hat{A}^{\mathrm{op}} \bowtie A$, as given in Definition 1. To continue with our construction, we next define the antipodal map S_D on A_D.

Lemma 3.11. *With the notation as in Section 2, we have:*

$$Z(J \otimes \hat{J}) = (J \otimes \hat{J})Z \qquad \text{and} \qquad Z(\hat{J} \otimes J) = (\hat{J} \otimes J)Z^*.$$

Remark. The results of the lemma can be shown by a straightforward calculation. Only the first result is immediately needed, but the second result will be useful in Section 4.

Proposition 3.12. *Let* $\widehat{J_D}$ *be the involutive operator in* $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ *defined by* $\widehat{J_D} := J \otimes \hat{J}$*. Then let* $S_D : A_D \to A_D$ *by*

$$S_D(x) := \widehat{J_D} x^* \widehat{J_D} = (J \otimes \hat{J}) x^* (J \otimes \hat{J}).$$

In particular, if $f \in \mathcal{A}$ *and* $\phi \in \hat{\mathcal{A}}$*, we have:*

$$S_D\big(\Pi(\phi \otimes f)\big) = S_D\big(\pi'(\lambda_\phi)\pi(L_f)\big) = \pi\big(S(L_f)\big)\pi'\big(\hat{S}(\lambda_\phi)\big).$$

This defines the "antipode" on A_D*. It is an anti-automorphism on* A_D*, satisfying:* $S_D\big(S_D(x)^*\big)^* = x$ *and* $(S_D \otimes S_D)\big(\Delta_D(x)\big) = {\Delta_D}^{\mathrm{cop}}(S_D(x))$*, for* $x \in A_D$*. Here* ${\Delta_D}^{\mathrm{cop}} = \chi^{2\leftrightarrow 4}_{1\leftrightarrow 3} \circ \Delta_D$*, where* χ *denotes the flip.*

Proof. For $f \in \mathcal{A}$ and $\phi \in \hat{\mathcal{A}}$,

$$\begin{aligned} S_D\big(\pi'(\lambda_\phi)\pi(L_f)\big) &= (J \otimes \hat{J})\big(\pi'(\lambda_\phi)\pi(L_f)\big)^*(J \otimes \hat{J}) \\ &= (J \otimes \hat{J})\pi({L_f}^*)\pi'({\lambda_\phi}^*)(J \otimes \hat{J}) \\ &= (J \otimes \hat{J})Z(1 \otimes {L_f}^*)Z^*({\lambda_\phi}^* \otimes 1)(J \otimes \hat{J}) \\ &= Z(1 \otimes \hat{J}{L_f}^*\hat{J})(J \otimes \hat{J})Z^*(J \otimes \hat{J})(J{\lambda_\phi}^*J \otimes 1) \\ &= Z(1 \otimes \hat{J}{L_f}^*\hat{J})Z^*(J{\lambda_\phi}^*J \otimes 1) = \pi\big(S(L_f)\big)\pi'\big(\hat{S}(\lambda_\phi)\big). \end{aligned}$$

In the fourth and fifth equalities, we used the result of Lemma 3.11. In the last equality, we used the definitions of S and $\hat{S}$, given in terms of $\hat{J}$ and J (See Section 2.).

By definition, it is easy to see that S_D is an anti-automorphism and also that $S_D\big(S_D(x)^*\big)^* = x$, for $x \in A_D$. The last statement is also easy to verify, remembering the corresponding properties of S and $\hat{S}$. For instance, for $f \in \mathcal{A}$ and $\phi \in \hat{\mathcal{A}}$, we have:

$$
\begin{aligned}
&(S_D \otimes S_D)\big(\Delta_D(\Pi(\phi \otimes f))\big) \\
&= (S_D \otimes S_D)\big([(\pi' \otimes \pi')(\hat{\Delta}\phi)][(\pi \otimes \pi)(\Delta f)]\big) \\
&= (\pi \otimes \pi)\big((S \otimes S)(\Delta f)\big)(\pi' \otimes \pi')\big((\hat{S} \otimes \hat{S})(\hat{\Delta}\phi)\big) \\
&= \big[(\pi \otimes \pi)\big(\Delta^{\mathrm{cop}}(S(f))\big)\big]\big[(\pi' \otimes \pi')\big(\hat{\Delta}^{\mathrm{cop}}(\hat{S}(\phi))\big)\big] \\
&= {\Delta_D}^{\mathrm{cop}}\big(\pi(S(f))\pi'(\hat{S}(\phi))\big) = {\Delta_D}^{\mathrm{cop}}\big(S_D(\Pi(\phi \otimes f))\big).
\end{aligned}
$$

The third equality follows from the properties of S (in Proposition 2.4 of [12]) and of $\hat{S}$ (in Proposition 2.4 of [11]). The fourth equality follows from Proposition 3.9. □

For S_D to be correctly considered an antipode of (A_D, Δ_D), we further need the notion of Haar weight clarified. This will be done later in this paper (in Section 5). But for our immediate purposes, result of Proposition 3.12 is sufficient. In fact, it is not difficult to show that the definition of S_D given above is equivalent to the map,

$$S_D : (\mathrm{id} \otimes \mathrm{id} \otimes \Omega)(V_D) \mapsto (\mathrm{id} \otimes \mathrm{id} \otimes \Omega)({V_D}^*),$$

for $\Omega \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})_*$. The general theory assures us that once we establish the existence of the Haar weight, this map actually characterizes the antipode (See [1], [27], [18].).

According to the general theory of locally compact quantum groups [18], [21], the antipode allows the "polar decomposition". In our case, with S_D being an anti-automorphism, its polar decomposition is trivial: That is, $S_D \equiv R_D$ (the "unitary antipode"), and $\tau_D \equiv \mathrm{Id}$ (the "scaling group"). These observations, in addition to the fact that ${S_D}^2 \equiv \mathrm{Id}$, manifests that A_D is essentially a Kac C^*-algebra (in the sense of [25]). This is to be expected, since (A, Δ) and $(\hat{A}, \hat{\Delta})$ are also as such.

4. The dual of the quantum double

In the previous section, we considered the *quantum double* (A_D, Δ_D), together with its "antipode" S_D, all within the C^*-algebra framework. The discussion on its Haar weight (thereby establishing it as a locally compact quantum group) will be postponed until Section 5. In the present section, we will consider the dual object of (A_D, Δ_D).

As we can expect from the way A_D was constructed in Definition 2 (via the multiplicative unitary operator V_D), the dual object will be obtained by considering the "left slices" of V_D, as follows (See again, [1], [27].):

Definition 3. Let $\widehat{A_D}$ be the C^*-algebra contained in $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, defined by

$$\widehat{A_D} = \overline{\big\{(\Omega\otimes\mathrm{id}\otimes\mathrm{id})(V_D) : \Omega\in\mathcal{B}(\mathcal{H}\otimes\mathcal{H})_*\big\}}^{\|\ \|}.$$

In addition, define the *comultiplication* $\widehat{\Delta_D} : \widehat{A_D}\to M(\widehat{A_D}\otimes\widehat{A_D})$ by

$$\widehat{\Delta_D}(y) := V_D(y\otimes 1\otimes 1){V_D}^*, \qquad \text{for } y\in\widehat{A_D}.$$

It is a non-degenerate C^*-homomorphism satisfying the coassociativity condition: $(\widehat{\Delta_D}\otimes\mathrm{id})\widehat{\Delta_D} = (\mathrm{id}\otimes\widehat{\Delta_D})\widehat{\Delta_D}$. As before, $\widehat{\Delta_D}(\widehat{A_D})(\widehat{A_D}\otimes 1)$ and $\widehat{\Delta_D}(\widehat{A_D})(1\otimes\widehat{A_D})$ are dense subsets in $\widehat{A_D}\otimes\widehat{A_D}$.

By the multiplicativity of V_D, we know that $(\widehat{A_D},\widehat{\Delta_D})$ is a C^*-bialgebra, dual to (A_D,Δ_D). Let us now find a more explicit realization of $(\widehat{A_D},\widehat{\Delta_D})$. The proof is adapted from Proposition 8.14 of [1].

Proposition 4.1. *As a C^*-algebra, we have:*

$$\begin{aligned}\widehat{A_D} &= \overline{\big\{(\Omega\otimes\mathrm{id}\otimes\mathrm{id})\big(Y_{24}(f\otimes\phi\otimes 1\otimes 1)X_{13}\big) : \Omega\in\mathcal{B}(\mathcal{H}\otimes\mathcal{H})_*, f\in A, \phi\in\hat{A}^{\mathrm{op}}\big\}}^{\|\ \|}\\ &= A\otimes\hat{A}^{\mathrm{op}}.\end{aligned}$$

Proof. Given $\Omega\in\mathcal{B}(\mathcal{H}\otimes\mathcal{H})_*$ and for arbitrary $a\in A$, $b\in\hat{A}^{\mathrm{op}}$, define $\tilde{\Omega}\in\mathcal{B}(\mathcal{H}\otimes\mathcal{H})_*$ by $\tilde{\Omega} := (1\otimes b)\Omega(a\otimes 1)Z^*$. In particular, if $\Omega = \Omega_{\xi,\eta}$ (This is a standard notation: $\Omega_{\xi,\eta}(T) = \langle T\xi,\eta\rangle$, for $T\in\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ and $\xi,\eta\in\mathcal{H}\otimes\mathcal{H}$.), then we will have: $\widetilde{\Omega_{\xi,\eta}}(T) = \big\langle (a\otimes 1)Z^*T(1\otimes b)\xi,\eta\big\rangle$. With the new notation, we have:

$$\begin{aligned}(\tilde{\Omega}\otimes\mathrm{id}\otimes\mathrm{id})(V_D) &= (\tilde{\Omega}\otimes\mathrm{id}\otimes\mathrm{id})(Z_{12}Y_{24}Z_{12}^*X_{13})\\ &= (\Omega\otimes\mathrm{id}\otimes\mathrm{id})\big((a\otimes 1\otimes 1\otimes 1)Y_{24}Z_{12}^*X_{13}(1\otimes b\otimes 1\otimes 1)\big)\\ &= (\Omega\otimes\mathrm{id}\otimes\mathrm{id})\big(Y_{24}(a\otimes 1\otimes 1\otimes 1)Z_{12}^*(1\otimes b\otimes 1\otimes 1)X_{13}\big)\\ &= (\Omega\otimes\mathrm{id}\otimes\mathrm{id})\big(Y_{24}z_{12}X_{13}\big),\end{aligned}$$

where $z = (a\otimes 1)Z^*(1\otimes b)$. Note that since $Z^* = Y\hat{\hat{Y}}^*$, with $Y\in M(A\otimes\hat{A}^{\mathrm{op}})$ and $\hat{\hat{Y}}\in M(A^{\mathrm{op}}\otimes\hat{A})$ being elements of (two-sided) multiplier algebras, we see easily that $z\in A\otimes\hat{A}^{\mathrm{op}}$. The first equality of the proposition follows.

For the second part, recall first that $Y = \widehat{U_A}$ and $X = \Sigma\widehat{U_A}^*\Sigma$. Then by Proposition 3.5 (2) of [11], we have: $\hat{A}^{\text{op}} = \overline{\bigl\{(\omega\otimes\text{id})(Y) : \omega\in\mathcal{B}(\mathcal{H})_*\bigr\}}^{\|\ \|}$ and $A = \overline{\bigl\{(\omega\otimes\text{id})(X) : \omega\in\mathcal{B}(\mathcal{H})_*\bigr\}}^{\|\ \|}$. Since we have $Y \in M(A\otimes\hat{A}^{\text{op}}) \subseteq M\bigl(\mathcal{K}(\mathcal{H})\otimes\hat{A}^{\text{op}}\bigr)$ and $X\in M(\hat{A}^{\text{op}}\otimes A)\subseteq M\bigl(\mathcal{K}(\mathcal{H})\otimes A\bigr)$, we can see without trouble that $\overline{\bigl\{(k\otimes 1)Y(\phi\otimes 1) : \phi\in\hat{A}^{\text{op}}, k\in\mathcal{K}(\mathcal{H})\bigr\}}^{\|\ \|} = \mathcal{K}(\mathcal{H})\otimes\hat{A}^{\text{op}}$, and similarly that $\overline{\bigl\{(f\otimes 1)X(k\otimes 1) : f\in A, k\in\mathcal{K}(\mathcal{H})\bigr\}}^{\|\ \|} = \mathcal{K}(\mathcal{H})\otimes A$. Noting that $Y_{24}(f\otimes\phi\otimes 1\otimes 1)X_{13} = \bigl[Y(\phi\otimes 1)\bigr]_{24}\bigl[(f\otimes 1)X\bigr]_{13}$, we can show easily the second equality and thus obtain: $\widehat{A_D} = A\otimes\hat{A}^{\text{op}}$. $\square$

The proposition shows that in the case of $\widehat{A_D}$, by being just the (ordinary) tensor product $A\otimes\hat{A}^{\text{op}}$, there is no "twisting" in the algebra structure (Recall that in the case of A_D, it is the coalgebra structure that does not involve twisting. See Definition 1 and Proposition 3.9.).

On the other hand, we see below that the comultiplication on $\widehat{A_D}$ is equivalent to a "τ-tensor product", where τ is an "inversion". Recall first the definition of an inversion: See Definitions 8.1, 8.2 and Proposition 8.3 of [1] (See also [19].).

Definition 4. (1) Let (A,δ_A) and (B,δ_B) be two C^*-bialgebras. An *inversion* on A and B is a *-isomorphism $\tau : A\otimes B\to B\otimes A$ such that:

$$(\tau\otimes\text{id}_A)(\text{id}_A\otimes\tau)(\delta_A\otimes\text{id}_B) = (\text{id}_B\otimes\delta_A)\tau$$

and

$$(\text{id}_B\otimes\tau)(\tau\otimes\text{id}_B)(\text{id}_A\otimes\delta_B) = (\delta_B\otimes\text{id}_A)\tau,$$

where we used the same notation τ for its extension to the multiplier algebra $M(A\otimes B)$.

(2) Given an inversion τ on (A,δ_A) and (B,δ_B), we can define the map $\delta_\tau : A\otimes B\to M(A\otimes B\otimes A\otimes B)$ by

$$\delta_\tau := (\text{id}_A\otimes\tau\otimes\text{id}_B)(\delta_A\otimes\delta_B).$$

Then δ_τ is coassociative. It is the comultiplication associated with τ.

In our case, we can show that the operator Z provides an inversion on (A,Δ^{cop}) and $(\hat{A}^{\text{op}},\hat{\Delta}^{\text{cop}})$, where Δ^{cop} and $\hat{\Delta}^{\text{cop}}$ are co-opposite comultiplications.

Proposition 4.2. *The map $\tau : p\mapsto\Sigma ZpZ^*\Sigma$, where Σ is the flip, is an "inversion" on (A,Δ^{cop}) and $(\hat{A}^{\text{op}},\hat{\Delta}^{\text{cop}})$.*

Proof. Let $p \in A \otimes \hat{A}^{\mathrm{op}}$. Since $Y = \widehat{U_A} \in M(A \otimes \hat{A}^{\mathrm{op}})$, it follows that $Y^*pY \in A \otimes \hat{A}^{\mathrm{op}}$. Note also that $p\widehat{\widehat{Y}} = \widehat{\widehat{Y}}p$, since $\widehat{\widehat{Y}} \in M(A^{\mathrm{op}} \otimes \hat{A})$. Therefore,

$$\tau(p) = \Sigma Z p Z^* \Sigma = \Sigma Y^* \widehat{\widehat{Y}} p \widehat{\widehat{Y}}^* Y \Sigma = \Sigma Y^* p Y \Sigma \in \hat{A}^{\mathrm{op}} \otimes A.$$

Since Z is a unitary operator, it is clear that $\tau : A \otimes \hat{A}^{\mathrm{op}} \to \hat{A}^{\mathrm{op}} \otimes A$ is a *-isomorphism.

To verify that τ is an inversion, we note that $\Delta^{\mathrm{cop}}(a) = X(a \otimes 1)X^*$, for $a \in A$, and that $\hat{\Delta}^{\mathrm{cop}}(b) = Y(b \otimes 1)Y^*$, for $b \in \hat{A}^{\mathrm{op}}$ (These are consequences of Proposition 3.5 (2) of [11].). Indeed, we have:

$$\begin{aligned}
(\mathrm{id}_B \otimes \tau)(\tau \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \delta_B)(p) &= \Sigma_{23} Y_{23}^* \Sigma_{12} Y_{12}^* Y_{23}(p_{12}) Y_{23}^* Y_{12} \Sigma_{12} Y_{23} \Sigma_{23} \\
&= \Sigma_{23} \Sigma_{12} Y_{13}^* Y_{12}^* Y_{23}(p_{12}) Y_{23}^* Y_{12} Y_{13} \Sigma_{12} \Sigma_{23} \\
&= \Sigma_{23} \Sigma_{12} Y_{23} Y_{12}^* (p_{12}) Y_{12} Y_{23}^* \Sigma_{12} \Sigma_{23} \\
&= \Sigma_{23} Y_{13} \Sigma_{12} Y_{12}^* (p_{12}) Y_{12} \Sigma_{12} Y_{13}^* \Sigma_{23} \\
&= \Sigma_{23} Y_{13} \big[\tau(p)\big]_{12} Y_{13}^* \Sigma_{23} = Y_{12} \Sigma_{23} \big[\tau(p)\big]_{12} \Sigma_{23} Y_{12}^* \\
&= Y_{12} \big[\tau(p)\big]_{13} Y_{12}^* = (\delta_B \otimes \mathrm{id}_A)\tau(p).
\end{aligned}$$

For convenience, we let $B = \hat{A}^{\mathrm{op}}$ and $\delta_B = \hat{\Delta}^{\mathrm{cop}}$. The first equality is by applying the definitions given above, and in the third equality, we used the fact that Y is multiplicative (i. e. $Y_{12} Y_{13} Y_{23} = Y_{23} Y_{12}$ is equivalent to $Y_{13}^* Y_{12}^* = Y_{23} Y_{12}^* Y_{23}^*$).

A similar computation will verify the other condition, thereby giving us the proof that τ is an inversion on $(A, \Delta^{\mathrm{cop}})$ and $(\hat{A}^{\mathrm{op}}, \hat{\Delta}^{\mathrm{cop}})$. ☐

Remark. This is actually re-writing the proof of Theorem 8.17 in [1]. We nevertheless chose to carry out the explicit computation here (instead of just referring), so that we can have a more tangible description of the twisted comultiplication below (following the corollary).

Corollary 4.3. *On $A \otimes \hat{A}^{\mathrm{op}}$, we have a coassociative comultiplication δ_τ, defined by*

$$\delta_\tau := (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_{\hat{A}^{\mathrm{op}}})(\Delta^{\mathrm{cop}} \otimes \hat{\Delta}^{\mathrm{cop}}).$$

Moreover, for $p \in A \otimes \hat{A}^{\mathrm{op}}$, we have: $\delta_\tau(p) = V(p \otimes 1 \otimes 1)V^*$, where $V = Z_{12}^* X_{13} Z_{12} Y_{24}$ is as defined in Lemma 3.3.

Proof. The first part is an immediate consequence of Proposition 4.2. Direct proof is possible for the second part (using similar method as in the above proof), but we will instead refer the reader to the proof of Lemma 8.9 in [1]. ☐

We are now ready to give a more specific description of the "twisted" comultiplication on $\widehat{A_D}$.

Proposition 4.4. *For $y \in \widehat{A_D}$, we have:*

$$\widehat{\Delta_D}(y) = (Z \otimes Z)\big[\delta_\tau(Z^* y Z)\big](Z^* \otimes Z^*).$$

Proof. Write $y = ZpZ^*$, for $p \in A \otimes \hat{A}^{\mathrm{op}}$. [Note that for any $y \in \widehat{A_D} = A \otimes \hat{A}^{\mathrm{op}}$, we have: $Z^* y Z \in A \otimes \hat{A}^{\mathrm{op}}$.] Then:

$$\begin{aligned}
\widehat{\Delta_D}(y) &= V_D(y \otimes 1 \otimes 1){V_D}^* = V_D(ZpZ^* \otimes 1 \otimes 1){V_D}^* \\
&= (Z \otimes Z)V(Z^* \otimes Z^*)[ZpZ^* \otimes 1 \otimes 1](Z \otimes Z)V^*(Z^* \otimes Z^*) \\
&= (Z \otimes Z)V(p \otimes 1 \otimes 1)V^*(Z^* \otimes Z^*) \\
&= (Z \otimes Z)\big[\delta_\tau(p)\big](Z^* \otimes Z^*) = (Z \otimes Z)\big[\delta_\tau(Z^* y Z)\big](Z^* \otimes Z^*).
\end{aligned}$$

The third equality uses Lemma 3.3 (See also the proof of the Corollary 3.4, where it is noted that $V_D = (Z \otimes Z)V(Z^* \otimes Z^*)$.). The next to last equality follows from the Corollary 4.3 above. □

Proposition 4.5. *We have the following C^*-bialgebra isomorphisms:*

$$\begin{aligned}
(\widehat{A_D}, \widehat{\Delta_D}) &\cong (A \otimes \hat{A}^{\mathrm{op}}, \delta_\tau), \qquad \textit{where } \delta_\tau = (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta^{\mathrm{cop}} \otimes \hat{\Delta}^{\mathrm{cop}}), \\
&\cong (A^{\mathrm{op}} \otimes \hat{A}, \delta_{\tau'}), \qquad \textit{where } \delta_{\tau'} = (\mathrm{id} \otimes \tau' \otimes \mathrm{id})(\Delta \otimes \hat{\Delta}).
\end{aligned}$$

Here $\tau : A \otimes \hat{A}^{\mathrm{op}} \to \hat{A}^{\mathrm{op}} \otimes A$ is as above, and $\tau' : A^{\mathrm{op}} \otimes \hat{A} \to \hat{A} \otimes A^{\mathrm{op}}$ is defined by $\tau'(q) = \Sigma Z^* q Z \Sigma$, which is also an inversion.

Proof. The first isomorphism follows from Proposition 4.4, given by the map $\widehat{A_D} \ni y \mapsto Z^* y Z \in A \otimes \hat{A}^{\mathrm{op}}$. The second isomorphism is given by the map $A \otimes \hat{A}^{\mathrm{op}} \ni p \mapsto (S \otimes \hat{S})(p) \in A^{\mathrm{op}} \otimes \hat{A}$ (Recall that in our case, the antipodes S and $\hat{S}$ are both anti-automorphisms.).

To verify the last description of $\delta_{\tau'}$, let $q \in A^{\mathrm{op}} \otimes \hat{A}$. Then:

$$\begin{aligned}
\delta_{\tau'}(q) &= (S \otimes \hat{S} \otimes S \otimes \hat{S})\big[\delta_\tau\big((S \otimes \hat{S})(q)\big)\big] \\
&= (S \otimes \hat{S} \otimes S \otimes \hat{S})\big[(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta^{\mathrm{cop}} \otimes \hat{\Delta}^{\mathrm{cop}})\big((S \otimes \hat{S})(q)\big)\big] \\
&= (S \otimes \hat{S} \otimes S \otimes \hat{S})\big[(\mathrm{id} \otimes \tau \otimes \mathrm{id})(S \otimes S \otimes \hat{S} \otimes \hat{S})\big((\Delta \otimes \hat{\Delta})(q)\big)\big] \\
&= \big(\mathrm{id} \otimes (\hat{S} \otimes S)\tau(S \otimes \hat{S}) \otimes \mathrm{id}\big)\big((\Delta \otimes \hat{\Delta})(q)\big).
\end{aligned}$$

But for any $q \in A^{\mathrm{op}} \otimes \hat{A}$, we have:

$$\begin{aligned}(\hat{S}\otimes S)\tau(S\otimes\hat{S})(q) &= (\hat{S}\otimes S)\tau\big((\hat{J}\otimes J)q^*(\hat{J}\otimes J)\big)\\ &= (\hat{S}\otimes S)\big[\Sigma Z\big((\hat{J}\otimes J)q^*(\hat{J}\otimes J)\big)Z^*\Sigma\big]\\ &= (J\otimes\hat{J})\big[\Sigma Z\big((\hat{J}\otimes J)q^*(\hat{J}\otimes J)\big)Z^*\Sigma\big]^*(J\otimes\hat{J})\\ &= \Sigma(\hat{J}\otimes J)Z(\hat{J}\otimes J)q(\hat{J}\otimes J)Z^*(\hat{J}\otimes J)\Sigma = \Sigma Z^* q Z\Sigma = \tau'(q).\end{aligned}$$

The next to last equality uses Lemma 3.11. By this, we have shown the second isomorphism. □

Remark. Note that we may also regard τ' as an inversion on (A, Δ) and $(\hat{A}^{\mathrm{op}}, \hat{\Delta})$ [The definition given in the above proposition is still valid.]. Then we will have: $\tau' = \chi\tau^{-1}\chi$, where χ is the flip. In this setting, the comultiplication $\delta_{\tau'}$ is equivalent to the co-opposite comultiplication $\delta_\tau^{\mathrm{cop}}$, in the sense that for any $p \in A \otimes \hat{A}^{\mathrm{op}}$, we have:

$$(Z^*\otimes Z^*)\big[\delta_{\tau'}(ZpZ^*)\big](Z\otimes Z) = \delta_\tau^{\mathrm{cop}}(p) = \chi_{1\leftrightarrow 3}^{2\leftrightarrow 4}\big(\delta_\tau(p)\big). \tag{4.1}$$

This may be shown by direct computation (which we do not carry out here), but it is really a consequence of our working with the Kac systems [1]. Then by definition of $\widehat{\Delta_D}$, it follows from equation (4.1) that:

$$\widehat{\Delta_D}^{\mathrm{cop}}(y) = (Z\otimes Z)\big[\delta_\tau^{\mathrm{cop}}(Z^* y Z)\big](Z^*\otimes Z^*) = \delta_{\tau'}(y),$$

giving us a very tidy description of $\widehat{\Delta_D}^{\mathrm{cop}} \equiv \delta_{\tau'}$.

Propositions 4.4 and 4.5 provide us a specific description of the comultiplication on $\widehat{A_D}$, and we see that $\widehat{\Delta_D}$ is equivalent to a certain "twisted" tensor product comultiplication. Now, let us look for the antipodal map on $(\widehat{A_D}, \widehat{\Delta_D})$. As the general theory suggests and similar to the cases of S, $\hat{S}$, and S_D (see [1], [27], [18], [21], as well as [12], [11], and the paragraph following Proposition 3.12 above), we wish to consider the following map:

$$\widehat{S_D} : (\Omega\otimes\mathrm{id}\otimes\mathrm{id})(V_D) \mapsto (\Omega\otimes\mathrm{id}\otimes\mathrm{id})({V_D}^*), \qquad \Omega\in\mathcal{B}(\mathcal{H}\otimes\mathcal{H})_*. \tag{4.2}$$

But note that by Lemma 3.3 (3),

$$\begin{aligned}{V_D}^* &= (Z_{12}Y_{24}Z_{12}^*X_{13})^* = (Z_{12}Z_{34}Z_{12}^*X_{13}Z_{12}Y_{24}Z_{34}^*Z_{12}^*)^*\\ &= Z_{12}Z_{34}Y_{24}^*Z_{12}^*X_{13}^*Z_{34}^* = Z_{34}(Z_{12}Y_{24}^*Z_{12}^*X_{13}^*)Z_{34}^*.\end{aligned}$$

Therefore, the equation (4.2) can be written as:

$$\widehat{S_D} : (\Omega\otimes\mathrm{id}\otimes\mathrm{id})(Z_{12}Y_{24}Z_{12}^*X_{13}) \mapsto Z\big[(\Omega\otimes\mathrm{id}\otimes\mathrm{id})(Z_{12}Y_{24}^*Z_{12}^*X_{13}^*)\big]Z^*.$$

Remembering the characterizations of $S : (\omega \otimes \mathrm{id})(X) \mapsto (\omega \otimes \mathrm{id})(X^*)$ and $\hat{S} : (\omega' \otimes \mathrm{id})(Y) \mapsto (\omega' \otimes \mathrm{id})(Y^*)$, the new expression suggests that

$$\widehat{S_D} : a \otimes b \mapsto Z\big(S(a) \otimes \hat{S}(b)\big)Z^*, \qquad a \in A, b \in \hat{A}^{\mathrm{op}}.$$

Having this result as a motivation, we treat this more precisely in the following proposition.

Proposition 4.6. *Let J_D be defined by $J_D := Z(\hat{J} \otimes J) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. By Lemma 3.11, we have: $J_D = Z(\hat{J} \otimes J) = (\hat{J} \otimes J)Z^*$, and it follows that J_D is involutive. We then define $\widehat{S_D} : \widehat{A_D} \to \widehat{A_D}$ by*

$$\widehat{S_D}(y) := J_D y^* J_D = Z(\hat{J} \otimes J)y^*(\hat{J} \otimes J)Z^* = (\hat{J} \otimes J)Z^* y^* Z(\hat{J} \otimes J).$$

Then $\widehat{S_D}$ defines the "antipode" on $\widehat{A_D}$.

Proof. Note first that the definition given in the proposition is same as the one suggested in the previous paragraph:

$$\widehat{S_D}(y) = J_D y^* J_D = Z(\hat{J} \otimes J)y^*(\hat{J} \otimes J)Z^* = Z\big((S \otimes \hat{S})(y)\big)Z^*.$$

In the last equality, we used the definitions of the maps S and $\hat{S}$ as given in Section 2.

From its definition, it is immediate that $\widehat{S_D}$ is an anti-automorphism on $\widehat{A_D}$, satisfying: $\widehat{S_D}\big(\widehat{S_D}(y)^*\big)^* = y$, for $y \in \widehat{A_D}$. Next, let us also prove:

$$(\widehat{S_D} \otimes \widehat{S_D})\big(\widehat{\Delta_D}(y)\big) = \widehat{\Delta_D}^{\mathrm{cop}}\big(\widehat{S_D}(y)\big) = \chi^{2\leftrightarrow 4}_{1\leftrightarrow 3}\big[\widehat{\Delta_D}(\widehat{S_D}(y))\big]. \tag{4.3}$$

For this, let us consider without loss of generality $y = ZpZ^* = Z(a \otimes b)Z^*$, where, $a \in A$ and $b \in \hat{A}^{\mathrm{op}}$. Then by the remark following Proposition 4.5, the right hand side of equation (4.3) can be realized as being equal to $\delta_{\tau'}\big(\widehat{S_D}(y)\big) = \delta_{\tau'}\big((S \otimes \hat{S})(p)\big)$. Meanwhile, the left hand side can be written as follows:

$$\begin{aligned}
&(\widehat{S_D} \otimes \widehat{S_D})\big(\widehat{\Delta_D}(y)\big)\\
&= (\hat{J} \otimes J \otimes \hat{J} \otimes J)(Z^* \otimes Z^*)\big[\widehat{\Delta_D}(y)^*\big](Z \otimes Z)(\hat{J} \otimes J \otimes \hat{J} \otimes J)\\
&= (\hat{J} \otimes J \otimes \hat{J} \otimes J)\big[\delta_\tau(p)^*\big](\hat{J} \otimes J \otimes \hat{J} \otimes J) = (S \otimes \hat{S} \otimes S \otimes \hat{S})\big[\delta_\tau(a \otimes b)\big]\\
&= (S \otimes \hat{S} \otimes S \otimes \hat{S})\big[(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta^{\mathrm{cop}} a \otimes \hat{\Delta}^{\mathrm{cop}} b)\big]\\
&= (\mathrm{id} \otimes \tau' \otimes \mathrm{id})\big[(S \otimes S \otimes \hat{S} \otimes \hat{S})\big(\Delta^{\mathrm{cop}} a \otimes \hat{\Delta}^{\mathrm{cop}} b\big)\big]\\
&= (\mathrm{id} \otimes \tau' \otimes \mathrm{id})\big[\Delta\big(S(a)\big) \otimes \hat{\Delta}\big(\hat{S}(b)\big)\big] = \delta_{\tau'}\big(S(a) \otimes \hat{S}(b)\big) = \delta_{\tau'}\big((S \otimes \hat{S})(p)\big).
\end{aligned}$$

In this way, we verify the equation (4.3). Note that in the second equality above, we used the characterization of $\widehat{\Delta_D}$ given in Proposition 4.4. In the fifth equality, we used the result: $(\hat{S} \otimes S)\big[\tau(p)\big] = \tau'\big[(S \otimes \hat{S})(p)\big]$, which earlier appeared in the proof of Proposition 4.5. The sixth equality is using the properties of the antipode maps S and $\hat{S}$. $\square$

As in the case of (A_D, Δ_D) and its antipode S_D, the properties of the antipodal map $\widehat{S_D}$ noted above (including $\widehat{S_D}^2 \equiv \mathrm{Id}$) manifests that $(\widehat{A_D}, \widehat{\Delta_D})$ is again a Kac C^*-algebra (with the existence of its Haar weight, to be given below).

5. Haar weight

To show that the quantum double (A_D, Δ_D) and its dual $(\widehat{A_D}, \widehat{\Delta_D})$ constructed above are indeed locally compact quantum groups, we need a discussion on their Haar weights. These Haar weights will be described in terms of the Haar weights on (A, Δ) and $(\hat{A}, \hat{\Delta})$, which we recall below:

Lemma 5.1. *On $\mathcal{A}$, define a linear functional φ by*

$$\varphi(a) = \int a(0,0,r)\,dr.$$

It can be extended to a faithful, lower semi-continuous, tracial weight (still denoted by φ) on the C^*-algebra A. The weight φ satisfies the "left invariance property": For any $a \in A_+$ such that $\varphi(a) < \infty$, and for $\omega \in A_+^*$, we have:

$$\varphi\big((\omega \otimes \mathrm{id})(\Delta a)\big) = \omega(1)\varphi(a).$$

Lemma 5.2. *On $\hat{\mathcal{A}}$, define a linear functional $\hat{\varphi}$ by*

$$\hat{\varphi}(b) = \int b(x,y,0)\,dxdy.$$

It can be extended to a faithful, lower semi-continuous, KMS weight (still denoted by $\hat{\varphi}$) on the C^*-algebra $\hat{A}$. The weight $\hat{\varphi}$ also satisfies the "left invariance property": For any $b \in \hat{A}_+$ such that $\hat{\varphi}(b) < \infty$, and for $\omega \in \hat{A}_+^*$, we have:

$$\hat{\varphi}\big((\omega \otimes \mathrm{id})(\hat{\Delta} b)\big) = \omega(1)\hat{\varphi}(b).$$

Moreover, we have the following (unimodular) property: $\hat{\varphi} \circ \hat{S} = \hat{\varphi}$.

Remark. In both cases above, $\omega(1) = \|\omega\|$. These results are described in [12] (Section 3) and in [11] (Section 2), respectively. In particular, the left invariance properties are proved in Theorem 3.9 of [12] and in Theorem 2.11 of [11]. The invariance properties written above are in a weak form, but the general theory assures us that they are actually sufficient (See [18], and see also Section 1 of [12].). Note finally that $\hat{\varphi}$ is invariant under $\hat{S}$ (so $(\hat{A}, \hat{\Delta})$ is unimodular), while it is not the case for φ and S.

The invariance properties stay the same when we consider instead (A^{op}, Δ) and $(\hat{A}^{\text{op}}, \hat{\Delta})$. On the other hand, see the following corollary for the cases having co-opposite comultiplications.

Corollary 5.3. *Since the antipode $S : A \to A$ satisfies $(S \otimes S)\Delta = \Delta^{\text{cop}} \circ S$, it follows from Lemma 5.1 that $\varphi \circ S$ is left invariant for Δ^{cop}. That is,*

$$(\varphi \circ S)\big((\omega \otimes \mathrm{id})(\Delta^{\text{cop}} a)\big) = \omega(1)\varphi\big(S(a)\big).$$

Similarly, from Lemma 5.2 and by using the property of the antipode $\hat{S}$, we see that $\hat{\varphi}$ is left invariant for $\hat{\Delta}^{\text{cop}}$. That is,

$$\hat{\varphi}\big((\omega \otimes \mathrm{id})(\hat{\Delta}^{\text{cop}} b)\big) = (\hat{\varphi} \circ \hat{S})\big((\omega \otimes \mathrm{id})(\hat{\Delta}^{\text{cop}} b)\big) = \omega(1)\hat{\varphi}\big(S(b)\big) = \omega(1)\hat{\varphi}(b).$$

Proof. The verification is straightforward. In the second case, we are using the fact that $\hat{\varphi}$ is invariant under $\hat{S}$. □

Let us begin our discussion by considering the Haar weight on $(\widehat{A_D}, \widehat{\Delta_D})$, whose definition is given below. [There is actually a simpler characterization for $\widehat{\varphi_D}$, as can be found in Corollary 5.6 below. But our choice of the definition has been made so that the proofs of the later propositions are a little simpler.]

Definition 5. Let $\widehat{\varphi_D}$ be the faithful, lower semi-continuous weight on $\widehat{A_D}$, defined by

$$\widehat{\varphi_D}(y) := \big((\varphi \circ S) \otimes \hat{\varphi}\big)(Z^* y Z).$$

Since $\varphi \circ S$ and $\hat{\varphi}$ are densely defined weights on the C^*-algebras A and $\hat{A}^{\text{op}}$, and since they are both faithful and lower semi-continuous (i. e. they are "proper" weights), we can define their tensor product $(\varphi \circ S) \otimes \hat{\varphi}$. See Definition 1.27 of [18]. Therefore, $\widehat{\varphi_D}$ is a proper weight on the C^*-algebra $\widehat{A_D}$. See the following lemma.

Lemma 5.4. *For arbitrary proper weights φ and ψ on C^*-algebras A and B, consider the tensor product weight $\varphi \otimes \psi$ on $A \otimes B$. Then the set $\mathfrak{N}_\varphi \odot \mathfrak{N}_\psi (\subseteq \mathfrak{N}_{\varphi\otimes\psi})$ forms a core for the GNS map $\Lambda_{\varphi\otimes\psi}$.*

Remark. For a more systematic discussion on tensor product weights, see Section 1.6 and Appendix of [18]. The result of this lemma actually goes back to Haagerup's density theorem [8]: Adapted to our case, if $X \in \mathfrak{N}_{\varphi\otimes\psi}$, there exists a sequence $\{X_n\}$ in $\mathfrak{N}_\varphi \odot \mathfrak{N}_\psi$ such that: $\lim_{n\to\infty} \Lambda_{\varphi\otimes\psi}(X_n) = \Lambda_{\varphi\otimes\psi}(X)$. Because of this density result, pretty much all the properties of a tensor product weight $\varphi \otimes \psi$ can be obtained by just working with the elements from the algebraic tensor product $\mathfrak{N}_\varphi \odot \mathfrak{N}_\psi$.

The weight $\widehat{\varphi_D}$ is shown to be left invariant for $(\widehat{A_D}, \widehat{\Delta_D})$, and is also $\widehat{S_D}$-invariant (i. e. unimodular). See Proposition 5.5 and Theorem 5.7 below.

Proposition 5.5. *The weight* $\widehat{\varphi_D}$ *is* $\widehat{S_D}$*-invariant:* $\widehat{\varphi_D} = \widehat{\varphi_D} \circ \widehat{S_D}$.

Proof. Let $y \in \widehat{A_D}$ be such that $y = Z(L_f \otimes \lambda_\phi)Z^*$, where $f \in \mathcal{A}$ and $\phi \in \hat{\mathcal{A}}$. Since such elements are dense in $\widehat{A_D}$ (and form a core for $\widehat{\varphi_D}$), our proof will be achieved if we just verify that $\widehat{\varphi_D}(y) = (\widehat{\varphi_D} \circ \widehat{S_D})(y)$.

By definition, we can see easily that:

$$\begin{aligned}\widehat{\varphi_D}(y) &= \big((\varphi \circ S) \otimes \hat{\varphi}\big)(Z^* y Z) = (\varphi \circ S)(L_f)\hat{\varphi}(\lambda_\phi) \\ &= \left(\int (e^{2\lambda r})^n f(0,0,-r)\,dr\right)\left(\int \phi(x,y,0)\,dxdy\right) \\ &= \int (e^{-2\lambda r})^n f(0,0,r)\phi(x,y,0)\,dxdydr.\end{aligned}$$

Meanwhile, by Proposition 4.6, we know that $\widehat{S_D}(y) = S(L_f)\hat{S}(\lambda_\phi)$. Therefore,

$$\widehat{\varphi_D}\big(\widehat{S_D}(y)\big) = \big((\varphi \circ S) \otimes \hat{\varphi}\big)\big(Z^* S(L_f)\hat{S}(\lambda_\phi)Z\big).$$

To compute this (so that we can compare the result with $\widehat{\varphi_D}(y)$ obtained above), let us begin by finding a suitable realization of $Z^* S(L_f)\hat{S}(\lambda_\phi)Z$. Let $\xi \in \mathcal{H} \otimes \mathcal{H}$ and compute:

$$\begin{aligned}&Z^* S(L_f)\hat{S}(\lambda_\phi)Z\xi(x,y,r;x',y',r') \\ &= \int (e^{-\lambda r})^n \bar{e}\big[\eta_\lambda(r)\beta(x - e^{-\lambda r}x', e^{-\lambda r}y')\big] e\big[\eta_\lambda(r)\beta(e^{\lambda r' - \lambda r}x', y - e^{-\lambda r}y')\big] \\ &\qquad S(f)(\tilde{x},\tilde{y},r)\hat{S}(\phi)(e^{\lambda\tilde{r} - \lambda r}x', e^{\lambda\tilde{r} - \lambda r}y', r' - \tilde{r}) \\ &\qquad \bar{e}\big[\eta_\lambda(r)\beta(\tilde{x}, y + e^{\lambda r' - \lambda r}y' - e^{-\lambda r}y' - \tilde{y})\big] \\ &\qquad \bar{e}\big[\eta_\lambda(r)\beta(e^{\lambda\tilde{r} - \lambda r}x', y + e^{\lambda r' - \lambda r}y' - e^{-\lambda r}y' - \tilde{y} - e^{\lambda\tilde{r} - \lambda r}y')\big] \\ &\qquad (e^{\lambda r})^n e\big[\eta_\lambda(r)\beta(x + e^{\lambda r' - \lambda r}x' - e^{-\lambda r}x' - \tilde{x} - e^{\lambda\tilde{r} - \lambda r}x', e^{-\lambda r}y')\big] \\ &\qquad \xi(x + e^{\lambda r' - \lambda r}x' - \tilde{x} - e^{\lambda\tilde{r} - \lambda r}x', y + e^{\lambda r' - \lambda r}y' - \tilde{y} - e^{\lambda\tilde{r} - \lambda r}y', r; x', y', \tilde{r})\,d\tilde{x}d\tilde{y}d\tilde{r} \\ &= \int F(\tilde{x},\tilde{y},r; e^{\lambda\tilde{r}}x', e^{\lambda\tilde{r}}y', \tilde{r} - r')\bar{e}\big[\eta_\lambda(r)\beta(\tilde{x}, y - \tilde{y})\big]\xi(x - \tilde{x}, y - \tilde{y}, r; x', y', \tilde{r})\,d\tilde{x}d\tilde{y}d\tilde{r} \\ &= (L \otimes \lambda)_F \xi(x,y,r;x',y',r'),\end{aligned}$$

where F is defined by

$$\begin{aligned}F(\tilde{x},\tilde{y},r;x',y',\tilde{r}) = (e^{2\lambda r})^n f(-e^{\lambda r}\tilde{x} - e^{-\lambda\tilde{r}}x' + x', -e^{\lambda r}\tilde{y} - e^{-\lambda\tilde{r}}y' + y', -r) \\ \bar{e}\bigl[\eta_\lambda(r)\beta(\tilde{x} + e^{-\lambda\tilde{r}-\lambda r}x' - e^{-\lambda r}x', e^{-\lambda\tilde{r}-\lambda r}y')\bigr] \\ \bar{e}\bigl[\eta_\lambda(-\tilde{r})\beta(e^{-\lambda r}x', e^{-\lambda r}y')\bigr] e\bigl[\eta_\lambda(r)\beta(e^{-\lambda r}x', \tilde{y})\bigr] \\ \bar{e}\bigl[\eta_\lambda(r)\beta(\tilde{x},\tilde{y})\bigr]\phi(-e^{-\lambda\tilde{r}-\lambda r}x', -e^{-\lambda\tilde{r}-\lambda r}y', \tilde{r}).\end{aligned}$$

The expression for F is obtained by remembering the definitions of $S(f)$ and $\hat{S}(\phi)$ given in Section 2, and by using the change of variables. In this way, we obtained the following realization:

$$Z^*S(L_f)\hat{S}(\lambda_\phi)Z = (L\otimes\lambda)_F.$$

This means that:

$$\begin{aligned}\widehat{\varphi_D}\bigl(\widehat{S_D}(y)\bigr) &= \bigl((\varphi\circ S)\otimes\hat{\varphi}\bigr)(F) = \int (e^{-2\lambda r})^n F(0,0,r;x',y',0)\,dx'dy'dr \\ &= \int f(0,0,-r)\phi(-e^{-\lambda r}x', -e^{-\lambda r}y', 0)\,dx'dy'dr \\ &= \int f(0,0,r)\phi(x',y',0)(e^{-2\lambda r})^n\,dx'dy'dr.\end{aligned}$$

Combining the results, we see that $\widehat{\varphi_D}\bigl(\widehat{S_D}(y)\bigr) = \widehat{\varphi_D}(y)$, proving our assertion. □

Corollary 5.6. *For any $a\in A$ and $b\in\hat{A}^{\mathrm{op}}$, we have:*

$$\bigl((\varphi\circ S)\otimes\hat{\varphi}\bigr)\bigl(Z^*(a\otimes b)Z\bigr) = \varphi(a)\hat{\varphi}\bigl(\hat{S}(b)\bigr) = \varphi(a)\hat{\varphi}(b).$$

In particular, for any $p\in\widehat{A_D}$, we have: $\widehat{\varphi_D}(p) = (\varphi\otimes\hat{\varphi})(p)$. [This is the simpler characterization of $\widehat{\varphi_D}$ mentioned earlier.]

Proof. Consider $y = Z\bigl(S(a)\otimes\hat{S}(b)\bigr)Z^*$. Then by Proposition 4.6, we know that $\widehat{S_D}(y) = S\bigl(S(a)\bigr)\otimes\hat{S}\bigl(\hat{S}(b)\bigr) = a\otimes b$. It follows that:

$$\widehat{\varphi_D}\bigl(\widehat{S_D}(y)\bigr) = \bigl((\varphi\circ S)\otimes\hat{\varphi}\bigr)\bigl(Z^*(a\otimes b)Z\bigr).$$

On the other hand, by definition of $\widehat{\varphi_D}$ and by using the unimodularity of $\hat{\varphi}$, we have:

$$\widehat{\varphi_D}(y) = \bigl((\varphi\circ S)\otimes\hat{\varphi}\bigr)\bigl(S(a)\otimes\hat{S}(b)\bigr) = \varphi(a)\hat{\varphi}\bigl(\hat{S}(b)\bigr) = \varphi(a)\hat{\varphi}(b).$$

Since we should have $\widehat{\varphi_D}\bigl(\widehat{S_D}(y)\bigr) = \widehat{\varphi_D}(y)$ by Proposition 5.5, the first statement follows. The second statement is an immediate consequence. □

Theorem 5.7. *For any positive element $y \in \widehat{A_D}$ such that $\widehat{\varphi_D}(y) < \infty$, and for $\Omega \in \widehat{A_D}_+^*$, we have:*

$$\widehat{\varphi_D}\big((\Omega \otimes \mathrm{id} \otimes \mathrm{id})(\widehat{\Delta_D}(y))\big) = \Omega(1)\widehat{\varphi_D}(y).$$

Proof. For convenience, let us write $B = \hat{A}^{\mathrm{op}}$. Consider $y = Z(a \otimes b)Z^*$, where $a \in A_+$, $(\varphi \circ S)(a) < \infty$ and $b \in B_+$, $\hat{\varphi}(b) < \infty$. Assume also that Ω has the form $\Omega = \omega_1 \otimes \omega_2$, for some $\omega_1 \in A_+^*$ and $\omega_2 \in B_+^*$. Then:

$$\begin{aligned}
&\big((\Omega \otimes \mathrm{id} \otimes \mathrm{id})(\widehat{\Delta_D}(y))\big)\\
&= (\omega_1 \otimes \omega_2 \otimes \mathrm{id} \otimes \mathrm{id})\big((Z_{12}Z_{34})\big[\Sigma_{23}Z_{23}(\Delta^{\mathrm{cop}}a \otimes \hat{\Delta}^{\mathrm{cop}}b)Z_{23}^*\Sigma_{23}\big](Z_{12}Z_{34})^*\big)\\
&= (\tilde{\omega_1} \otimes \tilde{\omega_2} \otimes \mathrm{id} \otimes \mathrm{id})\big(Z_{34}\Sigma_{23}Z_{23}(\Delta^{\mathrm{cop}}a \otimes \hat{\Delta}^{\mathrm{cop}}b)Z_{23}^*\Sigma_{23}Z_{34}^*\big),
\end{aligned}$$

where $\tilde{\omega_1}$, $\tilde{\omega_2}$ are defined such that $(\tilde{\omega_1} \otimes \tilde{\omega_2})(\cdot) := (\omega_1 \otimes \omega_2)(Z \cdot Z^*)$.

But by Lemma 3.1, and by using $\hat{\Delta}^{\mathrm{cop}}b = Y(b \otimes 1)Y^*$, we have:

$$\begin{aligned}
&Z_{34}\Sigma_{23}Z_{23}(\Delta^{\mathrm{cop}}a \otimes \hat{\Delta}^{\mathrm{cop}}b)Z_{23}^*\Sigma_{23}Z_{34}^*\\
&= Y_{34}^*\Sigma_{23}Y_{23}^*\big[(\mathrm{id} \otimes \mathrm{id} \otimes \hat{\Delta}^{\mathrm{cop}})(\Delta^{\mathrm{cop}}a \otimes b)\big]Y_{23}\Sigma_{23}Y_{34}\\
&= \Sigma_{23}Y_{24}^*Y_{23}^*Y_{34}(\Delta^{\mathrm{cop}}a \otimes b \otimes 1)Y_{34}^*Y_{23}Y_{24}\Sigma_{23}\\
&= \Sigma_{23}Y_{34}Y_{23}^*(\Delta^{\mathrm{cop}}a \otimes b \otimes 1)Y_{23}Y_{34}^*\Sigma_{23}\\
&= \Sigma_{23}\big[(\mathrm{id} \otimes \mathrm{id} \otimes \hat{\Delta}^{\mathrm{cop}})(Y_{23}^*(\Delta^{\mathrm{cop}}a \otimes b)Y_{23})\big]\Sigma_{23}.
\end{aligned}$$

We are using the multiplicativity of Y (i. e. $Y_{23}Y_{24}Y_{34} = Y_{34}Y_{23}$) in the third equality. Therefore, by using the definition of $\widehat{\varphi_D}$ given in Definition 5 and in Corollary 5.6, we have:

$$\begin{aligned}
&\widehat{\varphi_D}\big((\Omega \otimes \mathrm{id} \otimes \mathrm{id})(\widehat{\Delta_D}(y))\big) = (\varphi \otimes \hat{\varphi})\big((\Omega \otimes \mathrm{id} \otimes \mathrm{id})(\widehat{\Delta_D}(y))\big)\\
&= (\varphi \otimes \hat{\varphi})\big((\tilde{\omega_1} \otimes \mathrm{id} \otimes \tilde{\omega_2} \otimes \mathrm{id})\big[(\mathrm{id} \otimes \mathrm{id} \otimes \hat{\Delta}^{\mathrm{cop}})(Y_{23}^*(\Delta^{\mathrm{cop}}a \otimes b)Y_{23})\big]\big).
\end{aligned}$$

Further computation shows the following:

$$\begin{aligned}
\widehat{\varphi_D}\big((\Omega \otimes \mathrm{id} \otimes \mathrm{id})(\widehat{\Delta_D}(y))\big) &= \tilde{\omega_2}(1)(\varphi \otimes \hat{\varphi})\big((\tilde{\omega_1} \otimes \mathrm{id})\big[Y_{23}^*(\Delta^{\mathrm{cop}}a \otimes b)Y_{23}\big]\big)\\
&= \tilde{\omega_2}(1)\big((\varphi \circ S) \otimes \hat{\varphi}\big)\big((\tilde{\omega_1} \otimes \mathrm{id})[\Delta^{\mathrm{cop}}a \otimes b]\big)\\
&= \tilde{\omega_2}(1)\tilde{\omega_1}(1)(\varphi \circ S)(a)\hat{\varphi}(b)\\
&= \omega_2(1)\omega_1(1)\widehat{\varphi_D}(y) = \Omega(1)\widehat{\varphi_D}(y).
\end{aligned}$$

Notice that the first and third equalities above use the left invariance properties of $\hat{\varphi}$ and $(\varphi \circ S)$ (See Corollary 5.3.). However, some care has to be given (using Lemma 5.4), if we want them to be perfectly valid. We will not go into the details here (to avoid our discussion from becoming too technical and lengthy), but for instance, we may follow the discussion similar to the proof of Lemma 3.5 and Corollary 3.6 of [28]. Meanwhile, the second equality is due to Lemma 3.1 and Corollary 5.6. Fourth and fifth equalities follow from the observation that

$$\begin{aligned}\tilde{\omega_1}(1)\tilde{\omega_2}(1) &= (\tilde{\omega_1}\otimes\tilde{\omega_2})(1\otimes 1) = (\omega_1\otimes\omega_2)\big(Z(1\otimes 1)Z^*\big)\\ &= (\omega_1\otimes\omega_2)(1\otimes 1) = \Omega(1).\end{aligned}$$

So far we proved the case when $y = Z(a\otimes b)Z^*$ and $\Omega = \omega_1\otimes\omega_2$. Extending the proof for general $y \in \widehat{A_D}_+$ and $\Omega \in \widehat{A_D}_+^*$ is not necessarily trivial. Nevertheless, we will again invoke Lemma 5.4 here and refer the reader instead to the papers mentioned above (See [28], [18].). □

Theorem 5.7 establishes the proof that $\widehat{\varphi_D}$ is a legitimate (invariant) Haar weight. By general theory [18], it is therefore the unique (up to multiplication by a scalar) Haar weight for $(\widehat{A_D}, \widehat{\Delta_D})$. In our case, we note that even if (A, Δ) was non-unimodular, $\widehat{\varphi_D}$ is actually unimodular for $(\widehat{A_D}, \widehat{\Delta_D})$ (Proposition 5.5). Since this is the case, we do not need any further discussion on the "modular function". Summarizing the results so far, we now state the following theorem:

Theorem 5.8. *The C^*-bialgebra $(\widehat{A_D}, \widehat{\Delta_D})$, together with its additional structure maps including the antipode $\widehat{S_D}$ and the (unimodular) Haar weight $\widehat{\varphi_D}$, is a C^*-algebraic locally compact quantum group, in the sense of Kustermans and Vaes (or of Masuda, Nakagami, and Woronowicz).*

As we have made our case throughout Section 4 and Section 5, we regard $(\widehat{A_D}, \widehat{\Delta_D})$ as the dual of the quantum double. Namely, $\widehat{D(A)}$.

Let us now look at the case of $D(A) = (A_D, \Delta_D)$. Since it is the dual object of $(\widehat{A_D}, \widehat{\Delta_D})$ associated with the multiplicative unitary operator V_D, and since $(\widehat{A_D}, \widehat{\Delta_D})$ is a legitimate locally compact quantum group (Theorem 5.7), we conclude immediately from general theory [18], [27], [21] that it is also a C^*-algebraic locally compact quantum group. This achieves our stated goal.

For the remainder of this section, we will just give an explicit description of the Haar weight φ_D of (A_D, Δ_D), whose existence (and uniqueness up to multiplication by a scalar) is assured from the above observation. The subalgebra $\hat{\mathcal{A}} \odot \mathcal{A} \subseteq A_D$ forms a core for the Haar weight.

Proposition 5.9. *For $\Pi(b\otimes a) = \pi'(\lambda_b)\pi(L_a) \in A_D$, where $b \in \hat{\mathcal{A}}$ and $a \in \mathcal{A}$, define:*

$$\varphi_D\big(\Pi(b\otimes a)\big) := \hat{\varphi}(\lambda_b)\varphi(L_a) = \int (b\otimes a)(x,y,0;0,0,r')\,dxdydr'.$$

This defines a linear functional on $\hat{\mathcal{A}} \odot \mathcal{A}$*. Then we have:*

$$\varphi_D\big(\Pi(\phi \otimes f)^*\Pi(b \otimes a)\big) = \hat{\varphi}({\lambda_\phi}^*\lambda_b)\varphi({L_f}^*L_a),$$

for $b, \phi \in \hat{\mathcal{A}}$ *and for* $a, f \in \mathcal{A}$*. Furthermore, we have:*

$$(\Omega \otimes \varphi_D)\big(\Delta_D(\Pi(b \otimes a))\big) = \Omega(1)\varphi_D\big(\Pi(b \otimes a)\big), \qquad \Omega \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})_*.$$

This will characterize the Haar weight on (A_D, Δ_D)*. In other words, the functional* φ_D *extends to a (unique)* C^**-algebra weight on* A_D*, which is left invariant.*

Proof. Note that $\Pi(\phi\otimes f)^*\Pi(b\otimes a) = \Pi\big((\phi\otimes f)^* \times (b\otimes a)\big)$, where the involution and multiplication on $\hat{\mathcal{A}} \odot \mathcal{A}$ are as given in equations (3.3) and (3.1). By a long but straightforward computation using the equations (3.3) and (3.1), we have:

$$\begin{aligned}
&\int \big((\phi \otimes f)^* \times (b \otimes a)\big)(x, y, 0; 0, 0, r')\, dxdydr' \\
&= \int \overline{\phi(x, y, \tilde{r})} b(x, y, \tilde{r}) \overline{f(\tilde{x}, \tilde{y}, r')} a(\tilde{x}, \tilde{y}, r')\, d\tilde{x}d\tilde{y}d\tilde{r}dxdydr' \\
&= \hat{\varphi}(b \times_{\hat{A}} \phi^*)\varphi(f^* \times_A a) = \hat{\varphi}({\lambda_\phi}^*\lambda_b)\varphi({L_f}^*L_a).
\end{aligned}$$

It thus follows that: $\varphi_D\big(\Pi(\phi \otimes f)^*\Pi(b \otimes a)\big) = \hat{\varphi}({\lambda_\phi}^*\lambda_b)\varphi({L_f}^*L_a)$.

Using this, we can give $\hat{\mathcal{A}} \odot \mathcal{A}$ a left Hilbert algebra structure. Moreover, we can show without difficulty that the GNS Hilbert space for φ_D is $\mathcal{H} \otimes \mathcal{H}$, while the GNS representation is Π. Following a standard procedure (see [5], and also [12], [11]), we can define a C^*-algebra weight on A_D extending the functional φ_D (The extended weight will be still denoted by φ_D.).

Meanwhile, at least at the level of the (dense) subalgebra $\hat{\mathcal{A}} \odot \mathcal{A}$, the verification of the left invariance of φ_D is not very difficult. Note that by Proposition 3.9, we can write:

$$\begin{aligned}
(\Omega \otimes \mathrm{id} \otimes \mathrm{id})\big(\Delta_D(\Pi(b \otimes a))\big) &= \sum (\Omega \otimes \mathrm{id} \otimes \mathrm{id})\big((\Pi \otimes \Pi)(b_{(1)} \otimes a_{(1)} \otimes b_{(2)} \otimes a_{(2)})\big) \\
&= \sum \big[\Omega\big(\pi'(b_{(1)})\pi(a_{(1)})\big)\big(\pi'(b_{(2)})\pi(a_{(2)})\big)\big],
\end{aligned}$$

where we are using Sweedler's notation for $\hat{\Delta}b$ and Δa. And, for convenience, we regard $b = \lambda_b$ and $a = L_a$. Then:

$$\begin{aligned}
(\Omega \otimes \varphi_D)\big(\Delta_D(\Pi(b \otimes a))\big) &= \sum \big[\Omega\big(\pi'(b_{(1)})\pi(a_{(1)})\big)\varphi_D\big(\pi'(b_{(2)})\pi(a_{(2)})\big)\big] \\
&= \sum \big[\Omega\big((b_{(1)} \otimes 1)Z(1 \otimes a_{(1)})Z^*\big)\hat{\varphi}(b_{(2)})\varphi(a_{(2)})\big].
\end{aligned}$$

Without loss of generality, assume that $\Omega = \Omega_{\xi,\eta}$, for $\xi, \eta \in \mathcal{H} \otimes \mathcal{H}$ (following the standard notation, as appeared in the proof of Proposition 4.1). Then the expression becomes:

$$
\begin{aligned}
&(\Omega\otimes\varphi_D)\bigl(\Delta_D(\Pi(b\otimes a))\bigr)=\sum\bigl[\bigl\langle(b_{(1)}\otimes 1)Z(1\otimes a_{(1)})Z^*\xi,\eta\bigr\rangle\hat{\varphi}(b_{(2)})\varphi(a_{(2)})\bigr]\\
&=\int\sum\bigl[(b_{(1)}\otimes 1)Z(1\otimes a_{(1)})Z^*\xi(x,y,r;x',y',r')\overline{\eta(x,y,r;x',y',r')}\\
&\qquad b_{(2)}(\tilde{x},\tilde{y},0)a_{(2)}(0,0,\tilde{r})\bigr]\,dxdydrdx'dy'dr'd\tilde{x}d\tilde{y}d\tilde{r}.
\end{aligned}
$$

We can compute this using the formulas we obtained in Section 2 for $b_{(1)}=\lambda(b_{(1)})$ and $a_{(1)}=L(a_{(1)})$, as well as the operator Z (obtained in Section 3). Next, note that $\hat{\Delta}b=\sum[b_{(1)}\otimes b_{(2)}]$ and that $\Delta a=\sum[a_{(1)}\otimes a_{(2)}]$, where we can use the equation (2.5) for $\hat{\Delta}b$ and the equation (2.2) for Δa. Then, by using change of variables, the expression becomes:

$$
\begin{aligned}
&(\Omega\otimes\varphi_D)\bigl(\Delta_D(\Pi(b\otimes a))\bigr)\\
&=\int b(e^{\lambda r}x+\tilde{x},e^{\lambda r}y+\tilde{y},0)a(0,0,r'+\tilde{r})\\
&\qquad\xi(x,y,r;x',y',r')\overline{\eta(x,y,r;x',y',r')}\,dxdydrdx'dy'dr'd\tilde{x}d\tilde{y}d\tilde{r}\\
&=\int b(\tilde{x},\tilde{y},0)a(0,0,\tilde{r})\xi(x,y,r;x',y',r')\overline{\eta(x,y,r;x',y',r')}\,dxdydrdx'dy'dr'd\tilde{x}d\tilde{y}d\tilde{r}\\
&=\hat{\varphi}(b)\varphi(a)\Omega_{\xi,\eta}(1)=\Omega(1)\varphi_D\bigl(\Pi(b\otimes a)\bigr).
\end{aligned}
$$

Since we already know the existence of the unique Haar weight from the discussion preceding the proposition, this invariance property at the dense subalgebra level is enough to assure us that φ_D is indeed the legitimate Haar weight for (A_D,Δ_D). ☐

Summarizing the results from Section 3 and the discussion on the Haar weight given here, we conclude the following:

Theorem 5.10. *The C^*-bialgebra (A_D,Δ_D), together with the Haar weight φ_D, is a C^*-algebraic locally compact quantum group. It is the (C^*-algebraic) "quantum double": $D(A)=\hat{A}^{\mathrm{op}}\bowtie A$.*

Remark. Unlike in the case of $(\widehat{A_D},\widehat{\Delta_D})$ and its Haar weight $\widehat{\varphi_D}$, we can show easily that φ_D is non-unimodular: That is, $\varphi_D\circ S_D\neq\varphi_D$. The same modular function operator for (A,Δ) (see Section 5 of [12]) will work as the modular function for (A_D,Δ_D).

6. Quantum Universal R-matrix

Just as in the case of the purely algebraic framework, our quantum double (A_D, Δ_D) is also equipped with a (quasi-triangular) "quantum universal R-matrix" type operator. The definition of a quantum R-matrix in the C^*-algebra framework is essentially same as in the more usual, Hopf algebra or QUE algebra setting (See [7], [4], [20], for the usual definition; And see Section 6 of [9], for the definition in the C^*-algebra setting.).

In this section, we will give a brief construction of a certain unitary operator $\mathcal{R} \in M(A_D \otimes A_D)$, which will be considered as the "quantum universal R-matrix" for the quantum group (A_D, Δ_D). Let us begin with a lemma, which actually follows from Lemma 3.3. The proof is adapted from Section 8 of [1].

Lemma 6.1. *Let the notations be as before, and let X, Y, and Z be the operators defined earlier. Then we have:*

1. $Z^*_{12}X_{14}Z_{12}Y_{25}Y_{45} = Y_{45}Y_{25}Z^*_{12}X_{14}Z_{12}$
2. $Z_{34}X_{14}Z^*_{34}X_{15}X_{35} = X_{35}X_{15}Z_{34}X_{14}Z^*_{34}$

Proof. Recall from Lemma 3.3 (3) that: $Z_{34}Z^*_{12}X_{13}Z_{12}Y_{24} = Y_{24}Z^*_{12}X_{13}Z_{12}Z_{34}$. It follows that: $Z^*_{12}X_{13}Z_{12}Y_{24}Z^*_{34} = Z^*_{34}Y_{24}Z^*_{12}X_{13}Z_{12}$. Remembering that $X \in M(\hat{A}^{\mathrm{op}} \otimes A)$, $Y \in M(A \otimes \hat{A}^{\mathrm{op}})$, and that $Z = \hat{\hat{Y}}Y^*$, where $\hat{\hat{Y}} \in M(A^{\mathrm{op}} \otimes \hat{A})$, this becomes: $Z^*_{12}X_{13}Z_{12}Y_{24}Y_{34} = Y_{34}Y_{24}Z^*_{12}X_{13}Z_{12}$. (The point is that $\hat{\hat{Y}}^*_{34}$ commutes with all the operators in the equation.) Certainly, this is equivalent to: $Z^*_{12}X_{14}Z_{12}Y_{25}Y_{45} = Y_{45}Y_{25}Z^*_{12}X_{14}Z_{12}$, obtaining (1).

For (2), recall first that $X = \Sigma Y^*\Sigma$. Then (1) above can be re-written as: $Z^*_{12}X_{14}Z_{12}X^*_{52}X^*_{54} = X^*_{54}X^*_{52}Z^*_{12}X_{14}Z_{12}$. It follows that: $X_{52}X_{54}Z^*_{12}X_{14}Z_{12} = Z^*_{12}X_{14}Z_{12}X_{54}X_{52}$. So we have: $Z_{12}X_{52}X_{54}Z^*_{12}X_{14} = X_{14}Z_{12}X_{54}X_{52}Z^*_{12}$, which is same as: $Z_{12}X_{52}Z^*_{12}X_{54}X_{14} = X_{14}X_{54}Z_{12}X_{52}Z^*_{12}$. But this is actually equivalent to (2) [Legs 1,2,4,5 are now considered as legs 3,4,5,1.]: $Z_{34}X_{14}Z^*_{34}X_{15}X_{35} = X_{35}X_{15}Z_{34}X_{14}Z^*_{34}$. ☐

We are now ready to give the description of our "quantum R-matrix" operator $\mathcal{R}$. Again, the definition is a slight modification of the one considered in Section 8 of [1].

Proposition 6.2. *Let $\mathcal{R} = Z_{34}X_{14}Z^*_{34}$. The following properties hold.*

1. $\mathcal{R} \in M(A_D \otimes A_D)$.
2. *We have:* $(\Delta_D \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$ *and* $(\mathrm{id} \otimes \Delta_D)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$.
3. *For any $x \in A_D$, we have:* $\mathcal{R}(\Delta_D(x))\mathcal{R}^* = \Delta^{\mathrm{cop}}_D(x)$.

(4) *The operator* $\mathcal{R}$ *satisfies the "quantum Yang–Baxter equation":* $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.

In (2) and (4) above, we are viewing $\mathcal{R}$ *as an operator in* $\mathcal{B}\big((\mathcal{H}\otimes\mathcal{H})\otimes(\mathcal{H}\otimes\mathcal{H})\big)$.

PROOF. (1) Recall that $X \in M(\hat{A}^{\mathrm{op}} \otimes A)$. So by naturally extending the C^*-algebra homomorphisms π' and π defined in Proposition 3.5 and Corollary 3.6, we can see that: $\mathcal{R} = Z_{34}X_{14}Z_{34}^* = (\pi' \otimes \pi)(X) \in M(A_D \otimes A_D)$.

(2) By using the characterization of $\mathcal{R}$ given above, we have:

$$\begin{aligned}(\Delta_D \otimes \mathrm{id})(\mathcal{R}) &= (\Delta_D \otimes \mathrm{id})\big((\pi' \otimes \pi)(X)\big) = (\pi' \otimes \pi' \otimes \pi)\big(\hat{\Delta} \otimes \mathrm{id})(X)\big) \\ &= (\pi' \otimes \pi' \otimes \pi)(X_{12}^* X_{23} X_{12}) = (\pi' \otimes \pi' \otimes \pi)(X_{13}X_{23}) \\ &= \big[(\pi' \otimes \pi' \otimes \pi)(X_{13})\big]\big[(\pi' \otimes \pi' \otimes \pi)(X_{23})\big] = \mathcal{R}_{13}\mathcal{R}_{23}.\end{aligned}$$

The second equality is due to $\Delta_D \circ \pi' = (\pi' \otimes \pi') \circ \hat{\Delta}$, which was observed in Corollary 3.10. Third equality is using the fact that $\hat{\Delta}b = X^*(1\otimes b)X$, for $b \in \hat{A}^{\mathrm{op}}$, while the next equality is the multiplicativity of X. The next to the last equality is using that π' and π are *-homomorphisms. Meanwhile, by remembering that $\Delta a = Y^*(1 \otimes a)Y$, for $a \in A$, and that $Y = \Sigma X^* \Sigma$, a similar computation will give us the other equation: $(\mathrm{id} \otimes \Delta_D)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$.

(3) Recall that $b = (\mathrm{id} \otimes \omega)(X) \in \hat{A}^{\mathrm{op}}$, and $a = (\mathrm{id} \otimes \omega')(Y) \in A$, for $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$, and that these operators generate $\hat{A}^{\mathrm{op}}$ and A, respectively. [This result follows from Proposition 3.5 (2) of [11] and Section 6 of [12]. It was also noted in the proof of Proposition 3.5 in the previous section.]

So consider $b = (\mathrm{id} \otimes \omega)(X)$ and compute. Then:

$$\begin{aligned}\mathcal{R}\big[\Delta_D\big(\pi'(b)\big)\big] &= \mathcal{R}\big[(\pi' \otimes \pi')(\hat{\Delta}b)\big] = (Z_{34}X_{14}Z_{34}^*)\big[X_{13}^*(1 \otimes 1 \otimes b \otimes 1)X_{13}\big] \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega)(Z_{34}X_{14}Z_{34}^*X_{13}^*X_{35}X_{13}) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega)(Z_{34}X_{14}Z_{34}^*X_{15}X_{35}) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega)(X_{35}X_{15}Z_{34}X_{14}Z_{34}^*) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega)(X_{31}^*X_{15}X_{31}Z_{34}X_{14}Z_{34}^*) \\ &= Y_{13}(b \otimes 1 \otimes 1 \otimes 1)Y_{13}^*Z_{34}X_{14}Z_{34}^* \\ &= \big[(\pi' \otimes \pi')(\hat{\Delta}^{\mathrm{cop}}b)\big]\mathcal{R} = \big[{\Delta_D}^{\mathrm{cop}}\big(\pi'(b)\big)\big]\mathcal{R}.\end{aligned}$$

The first equality is again using $\Delta_D \circ \pi' = (\pi' \otimes \pi') \circ \hat{\Delta}$. The fourth and sixth equalities follow from the multiplicativity of X, while the fifth equality is by Lemma 6.1 (2). The seventh equality is just using $Y = \Sigma X^* \Sigma$.

Next, consider $a = (\mathrm{id} \otimes \omega')(Y)$ and compute. Then:

$$\begin{aligned}
\mathcal{R}\big[\Delta_D\big(\pi(a)\big)\big] &= \mathcal{R}\big[(\pi \otimes \pi)(\Delta a)\big] \\
&= (Z_{34}X_{14}Z_{34}^*)\big[Z_{34}Z_{12}Y_{24}^*(1 \otimes 1 \otimes 1 \otimes a)Y_{24}Z_{12}^*Z_{34}^*\big] \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega)(Z_{34}X_{14}Z_{12}Y_{24}^*Y_{45}Y_{24}Z_{12}^*Z_{34}^*) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega)(Z_{34}X_{14}Z_{12}Y_{25}Y_{45}Z_{12}^*Z_{34}^*) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega)(Z_{34}Z_{12}Y_{45}Y_{25}Z_{12}^*X_{14}Z_{12}Z_{12}^*Z_{34}^*) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega)(Z_{34}Z_{12}Y_{42}^*Y_{25}Y_{42}Z_{12}^*X_{14}Z_{34}^*) \\
&= Z_{34}Z_{12}X_{24}(1 \otimes a \otimes 1 \otimes 1)X_{24}^*Z_{12}^*Z_{34}^*Z_{34}X_{14}Z_{34}^* \\
&= \big[(\pi \otimes \pi)(\Delta^{\mathrm{cop}}a)\big]\mathcal{R} = \big[{\Delta_D}^{\mathrm{cop}}\big(\pi(a)\big)\big]\mathcal{R}.
\end{aligned}$$

This is done in exactly same way as in the previous case. In particular, the first equality is using $\Delta_D \circ \pi = (\pi \otimes \pi) \circ \Delta$ (See Corollary of Proposition 3.9), while the fifth equality uses Lemma 6.1 (1).

Since A_D is known to be generated by the operators $\pi'(b)\pi(a)$, we conclude from these two results that we have: $\mathcal{R}\big[\Delta_D(x)\big]\mathcal{R}^* = {\Delta_D}^{\mathrm{cop}}(x)$, for any $x \in A_D$. [Note that by definition, $\mathcal{R}$ is unitary.]

(4) The last statement is an immediate consequence of results (2) and (3):

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{12}\big[(\Delta_D \otimes \mathrm{id})(\mathcal{R})\big] = \big[({\Delta_D}^{\mathrm{cop}} \otimes \mathrm{id})(\mathcal{R})\big]\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

First equality follows from (2); the second equality is from (3); and the third equality is from (2) with the legs 1 and 2 interchanged. ☐

Existence of a quantum R-matrix for a Hopf algebra (or a quantum group) is quite useful in the development of the representation theory (See, for instance, [10].). However, we will postpone to a future occasion any further discussion about the operator $\mathcal{R}$ and its applications. Some of these future discussions will be about the relationship between (A_D, Δ_D) and its "classical limit", the double Poisson–Lie group considered in [10], [13].

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