

CONSTRUCTION OF A QUANTUM HEISENBERG GROUP

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ABSTRACT. In this paper, we give a construction of a (C^* -algebraic) quantum Heisenberg group. This is done by viewing it as the dual quantum group of the specific non-compact quantum group (A, Δ) constructed earlier by the author. Our definition of the quantum Heisenberg group is different from the one considered earlier by Van Daele. To establish our object of study as a locally compact quantum group, we also give a discussion on its Haar weight, which is no longer a trace. In the latter part of the paper, we give some additional discussion on the duality mentioned above.

INTRODUCTION. Among the simplest while useful of non-compact groups is the Heisenberg Lie group H , which is two-step nilpotent. Our goal in this paper is to construct a version of a quantum Heisenberg group (i. e. a “quantized $C_0(H)$ ”), which would be an example of a non-compact, C^* -algebraic quantum group.

Our example is certainly not genuinely new. Already in the early 90’s, Van Daele in [18] gave a construction of a quantum Heisenberg group, which was actually one of the first examples of a non-compact quantum group in the C^* -algebra setting. Similar example but with a different approach (via geometric quantization) was given by Szymczak and Zakrzewski [15]. Meanwhile, a dual counterpart to these examples was given by Rieffel [13]. Ours is different from these, but it is true that we were strongly motivated by these early examples.

In [5], we constructed a specific non-compact quantum group (A, Δ) , by deformation quantization of a certain non-linear Poisson structure. The construction was based on a generalization of Rieffel’s approach (as given in [12] and [13]). And we saw in our previous papers that (A, Δ) can be considered as a “quantum Heisenberg group algebra” (i. e. a “quantized $C^*(H)$ ”). Naturally, we are interested in its dual counterpart. The dual quantum group, to be denoted by $(\hat{A}, \hat{\Delta})$ in the below, will be our main object of study in this paper. It will be our candidate to be a quantum Heisenberg group.

By general theory (for instance, see [10]), the dual object of a locally compact quantum group is again a locally compact quantum group.

This means that the proof of $(\hat{A}, \hat{\Delta})$ being a locally compact quantum group is more or less automatic from the proof of (A, Δ) being one. For this reason, we did not find pressing needs for giving a separate presentation on $(\hat{A}, \hat{\Delta})$ until now, and we instead have been only giving indications of its existence on several occasions in our previous papers [5], [6], [8]. However, as we are trying to develop some applications of these quantum groups (some of the program were already carried out in [6] and [7]), and also when we try to construct the “quantum double” (work in preparation), it became necessary to clarify the notion of our quantum Heisenberg group.

It is true that ours is not one of the attention-grabbing examples. But it is modestly interesting on its own, just as an ordinary Heisenberg group is an interesting object of study in various branches of mathematics. So in this article, we plan to carry out a careful construction of the quantum Heisenberg group $(\hat{A}, \hat{\Delta})$, including its non-tracial Haar weight. We will try to make the discussion as detailed as possible, even if we may have to repeat some of our earlier results. On the other hand, note that even for the case of the (simpler) quantum Heisenberg group of Van Daele, so far no explicit discussion in the C^* -algebra setting on the Haar weight has been given.

Here is a quick summary of how this paper is organized. In section 1, we review our example (A, Δ) . Although the information at the Poisson–Lie group level played a significant role in its construction, that angle will be de-emphasized here for the purpose of brevity. Among the useful tools that appear is the multiplicative unitary operator U_A . The description of our quantum Heisenberg group $(\hat{A}, \hat{\Delta})$ is given in section 2. After giving a realization of the underlying C^* -algebra \hat{A} , we will construct its quantum group structures, including comultiplication, antipode, and Haar weight. We will make a point that $(\hat{A}, \hat{\Delta})$ is reasonable to be considered as a “quantized $C_0(H)$ ”.

In section 3, we give a light discussion on the duality between (A, Δ) and $(\hat{A}, \hat{\Delta})$. And towards the end, we mention some other related quantum Heisenberg group algebras and quantum Heisenberg groups, namely the “opposite” and “co-opposite” versions of (A, Δ) and $(\hat{A}, \hat{\Delta})$. Most of the results here are straight from the general theory, but several of these will be useful in our future applications, including the quantum double construction.

1. THE HILBERT SPACE \mathcal{H} . THE QUANTUM GROUP (A, Δ) .

Let H be the $(2n + 1)$ -dimensional Heisenberg Lie group. The underlying space for this Lie group is \mathbb{R}^{2n+1} , and the multiplication on it

is defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \beta(x, y')),$$

for $x, y, x', y' \in \mathbb{R}^n$ and $z, z' \in \mathbb{R}$. Here $\beta(,)$ is the usual inner product on \mathbb{R}^n , used here for a possible future generalization.

As indicated in the Introduction, we wish to obtain our *quantum Heisenberg group* (“quantized $C_0(H)$ ”), as the dual object to the non-compact quantum group (A, Δ) constructed earlier by the author. So let us give here a brief review of definitions involving (A, Δ) . See [5],[8], for more detailed discussion.

To begin with, we need to establish our underlying Hilbert space \mathcal{H} . To do this, we first consider the Heisenberg group H as a finite-dimensional vector space. As above, typical elements in H are written as (x, y, z) . Next, let H^* be the dual vector space of H , whose typical elements will be written as (p, q, r) . Note that in [5], we used the notation \mathfrak{g} for the space H^* and considered it as $\mathfrak{g} = \mathfrak{h}^*$, where \mathfrak{h} is the Lie algebra (so a vector space) corresponding to H . Since $H = \mathfrak{h}$ as vector spaces (by virtue of being nilpotent), this is equivalent.

We take the natural Lebesgue measure $dxdydz$ on H , which would be the Haar measure for the group H . Whereas on H^* , we consider the dual Plancherel Lebesgue measure $dpdqdr$, corresponding to the chosen Haar measure on H . Then we can define the Fourier transform from $L^2(H)$ to $L^2(H^*)$, as follows:

$$(\mathcal{F}\xi)(p, q, r) = \int_H \bar{e}(p \cdot x + q \cdot y + r \cdot z)\xi(x, y, z) dxdydz.$$

Here \cdot denotes the dual pairing, and $e()$ is the function defined by $e(t) = e^{2\pi it}$. So $\bar{e}(t) = e^{-2\pi it}$. By our choice of measures, the Fourier transform is a unitary operator whose inverse is the following:

$$(\mathcal{F}^{-1}\zeta)(x, y, z) = \int_{H^*} e(p \cdot x + q \cdot y + r \cdot z)\zeta(x, y, z) dpdqdr.$$

The Fourier inversion theorem (the unitarity of the Fourier transform) holds such that we have: $\mathcal{F}^{-1}(\mathcal{F}\xi) = \xi$ and $\mathcal{F}(\mathcal{F}^{-1}\zeta) = \zeta$, at the level of L^2 -functions as well as at the level of Schwartz functions.

By Fourier transform, we can regard $L^2(H)$ and $L^2(H^*)$ as more or less the same. Actually, it is more convenient to work with the L^2 -functions in the (x, y, r) variables, which we denote by \mathcal{H} . That is, $\mathcal{H} = L^2(H/Z \times H^*/Z^\perp)$, where $Z = \{(0, 0, z)\}$'s in H . By using the partial Fourier transform in the third variable (defined similarly as above), we can see that \mathcal{H} is isomorphic to $L^2(H)$ (as well as to $L^2(H^*)$). All our constructions will be carried out over the Hilbert space \mathcal{H} .

As a C^* -algebra, A is isomorphic to the twisted crossed product C^* -algebra $C^*(H/Z, C_0(H^*/Z^\perp), \sigma)$, with the twisting given by a certain cocycle term σ . To be more precise, consider \mathcal{A} , which is the space of Schwartz functions in the (x, y, r) variables having compact support in the r variable. Clearly, $\mathcal{A} \subseteq C_0(H/Z \times H^*/Z^\perp)$ as well as $\mathcal{A} \subseteq \mathcal{H}$. For $f, g \in \mathcal{A}$, define:

$$(L_f g)(x, y, r) := \int f(\tilde{x}, \tilde{y}, r) g(x - \tilde{x}, y - \tilde{y}, r) \bar{e}[\eta_\lambda(r) \beta(\tilde{x}, y - \tilde{y})] d\tilde{x} d\tilde{y}.$$

Remark. In the definition above, $\lambda \in \mathbb{R}$ is a fixed constant, which determines a certain non-linear Poisson structure when $\lambda \neq 0$. The expression $\eta_\lambda(r)$ is defined such that $\eta_\lambda(r) = \frac{e^{2\lambda r} - 1}{2\lambda}$, which reflects the non-linear flavor. When $\lambda = 0$, we take $\eta_{\lambda=0}(r) = r$. We are not planning to explicitly mention the Poisson structure here. But in section 1 of [5], we gave a discussion on how it is related with a so-called ‘‘classical r -matrix’’ element. Finally, the expression $\bar{e}[\eta_\lambda(r) \beta(\tilde{x}, y - \tilde{y})]$ is the cocycle term, indicated by σ above.

In this way, we define the ‘‘regular representation’’ L , and obtain the C^* -algebra A as the norm closure in $\mathcal{B}(\mathcal{H})$ of $L(\mathcal{A})$. We will, in many occasions, regard $f \in \mathcal{A}$ and L_f as the same, and consider \mathcal{A} as a (dense) subalgebra of A . Actually, \mathcal{A} is a $*$ -subalgebra of A , whose multiplication is given by $L_{f \times g} = L_f L_g$. The involution $f \mapsto f^*$ can be described by $L_{f^*} = (L_f)^*$. We have the following:

$$(f \times_A g)(x, y, r) = \int f(\tilde{x}, \tilde{y}, r) g(x - \tilde{x}, y - \tilde{y}, r) \bar{e}[\eta_\lambda(r) \beta(\tilde{x}, y - \tilde{y})] d\tilde{x} d\tilde{y}.$$

$$f^*(x, y, r) = \bar{e}[\eta_\lambda(r) \beta(x, y)] \overline{f(-x, -y, r)}. \quad (1.1)$$

See Propositions 2.8 and 2.9 of [5].

In [8] (in Proposition 2.2), we gave another characterization of A , in terms of a certain unitary operator $U_A \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. Namely,

$$A = \overline{\{(\omega \otimes \text{id})(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}, \quad (1.2)$$

where the $L(\omega) = (\omega \otimes \text{id})(U_A) \in \mathcal{B}(\mathcal{H})$ are the ‘‘left slices’’ of U_A by the linear forms $\omega \in \mathcal{B}(\mathcal{H})_*$. The unitary operator U_A is ‘‘multiplicative’’ (in the sense of Baaj and Skandalis [1]), and is defined by

$$U_A \xi(x, y, r, x', y', r') = (e^{-\lambda r'})^n \bar{e}[\eta_\lambda(r') \beta(e^{-\lambda r'} x, y' - e^{-\lambda r'} y)]$$

$$\xi(e^{-\lambda r'} x, e^{-\lambda r'} y, r + r', x' - e^{-\lambda r'} x, y' - e^{-\lambda r'} y, r').$$

Using the multiplicative unitary operator U_A , we can also define the *comultiplication* $\Delta : A \rightarrow M(A \otimes A)$. For $a \in A$, we have:

$$\Delta a = U_A(a \otimes 1)U_A^*.$$

The comultiplication is a non-degenerate $*$ -homomorphism satisfying the coassociativity condition: $(\Delta \otimes \text{id})(\Delta a) = (\text{id} \otimes \Delta)(\Delta a)$. In case $f \in \mathcal{A}$, the equation $\Delta(L_f) = (L \otimes L)_{\Delta f}$ gives us the following:

$$\begin{aligned} & \Delta f(x, y, r; x', y', r') \\ &= \int f(x', y', r + r') \bar{e}[\tilde{p} \cdot (e^{\lambda r'} x' - x) + \tilde{q} \cdot (e^{\lambda r'} y' - y)] d\tilde{p}d\tilde{q}, \end{aligned} \quad (1.3)$$

which is a Schwartz function having compact support in r and r' .

We have further shown in [5] and [8] that the C^* -bialgebra (A, Δ) , together with the additional structures on it, namely counit ε , antipode S , and the Haar weight φ_A , becomes a locally compact (C^* -algebraic) quantum group, in the sense of Kustermans and Vaes [10]. In particular, a rigorous discussion on the definition of φ_A as a C^* -algebra weight and its left invariance property was given in section 3 of [8].

Remark. We have been arguing in our earlier papers that (A, Δ) is, in a sense, a “quantum Heisenberg group algebra” (For instance, see [6], where we studied its representation theory.). To make a brief case here, let $\lambda = 0$ (so $\eta_\lambda(r) = r$). Let us now re-write the expression for $L_f g$ at the level of (x, y, z) variables, by using the partial Fourier transform in the third variable and by using the Fourier inversion theorem. Then we have (for convenience, we are not distinguishing a function from its partial Fourier transform):

$$\begin{aligned} & (L_f g)(x, y, z) \\ &= \int f(\tilde{x}, \tilde{y}, \tilde{z}) \bar{e}(\tilde{z}r) g(x - \tilde{x}, y - \tilde{y}, \tilde{z}) \bar{e}(\tilde{z}r) \bar{e}[r\beta(\tilde{x}, y - \tilde{y})] e(rz) d\tilde{z}d\tilde{x}d\tilde{y}dr \\ &= \int f(\tilde{x}, \tilde{y}, \tilde{z}) g(x - \tilde{x}, y - \tilde{y}, z - \tilde{z} - \beta(\tilde{x}, y - \tilde{y})) d\tilde{x}d\tilde{y}d\tilde{z}. \end{aligned}$$

This is just the left regular representation of $C^*(H)$, given by the convolution product. The involution can be also realized as the natural one on the convolution algebra.

2. THE QUANTUM HEISENBERG GROUP $(\hat{A}, \hat{\Delta})$.

Since (A, Δ) can be viewed as a “quantized $C^*(H)$ ”, it is natural to consider its dual object as the candidate for the quantum Heisenberg group. Suggested by the general theory of locally compact quantum groups [10], and taking advantage of the theory of multiplicative unitary operators [1], we will define $(\hat{A}, \hat{\Delta})$ in terms of our fundamental multiplicative unitary operator U_A .

Definition 2.1. Consider the “right slices” of U_A , which are the operators $\rho(\omega) \in \mathcal{B}(\mathcal{H})$ defined by $\rho(\omega) = (\text{id} \otimes \omega)(U_A)$, for $\omega \in \mathcal{B}(\mathcal{H})_*$. We will define \hat{A} as the C^* -algebra generated by the $\rho(\omega)$:

$$\hat{A} = \overline{\{(\text{id} \otimes \omega)(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}.$$

For a typical element $b \in \hat{A}$, define $\hat{\Delta}b$ by $\hat{\Delta}b = U_A^*(1 \otimes b)U_A$. In this way, we obtain the *comultiplication* $\hat{\Delta} : \hat{A} \rightarrow M(\hat{A} \otimes \hat{A})$, which is a non-degenerate C^* -homomorphism satisfying the coassociativity.

The general theory of multiplicative unitary operators assures us that $(\hat{A}, \hat{\Delta})$ is a C^* -bialgebra (or a quantum semigroup) [1], [21], [10]. But to be more specific and to be more accessible in our future applications, let us give here an explicit realization of the C^* -algebra \hat{A} :

Proposition 2.2. *Let \hat{A} be the space of Schwartz functions in the (x, y, r) variables having compact support in the r variable. For $f \in \hat{A}$, define the operator $\rho_f \in \mathcal{B}(\mathcal{H})$ by*

$$(\rho_f \zeta)(x, y, r) = \int (e^{\lambda \tilde{r}})^n f(x, y, \tilde{r}) \zeta(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}) d\tilde{r}.$$

Then the C^ -algebra \hat{A} is generated by the operators ρ_f .*

Proof. Let us work with the standard notation $\omega_{\xi, \eta}$, where $\xi, \eta \in \mathcal{H}$. It is defined by $\omega_{\xi, \eta}(T) = \langle T\xi, \eta \rangle$, for $T \in \mathcal{B}(\mathcal{H})$. It is well known that linear combinations of the $\omega_{\xi, \eta}$ are (norm) dense in $\mathcal{B}(\mathcal{H})_*$. So consider $(\text{id} \otimes \omega_{\xi, \eta})(U_A) \in \mathcal{B}(\mathcal{H})$. Without loss of generality, we can assume that ξ and η are continuous functions having compact support. Then for $\zeta \in \mathcal{H}$, we have:

$$\begin{aligned} ((\text{id} \otimes \omega_{\xi, \eta})(U_A))\zeta(x, y, r) &= \int (U_A(\zeta \otimes \xi))(x, y, r; \tilde{x}, \tilde{y}, \tilde{r}) \overline{\eta(\tilde{x}, \tilde{y}, \tilde{r})} d\tilde{x}d\tilde{y}d\tilde{r} \\ &= \int (e^{\lambda \tilde{r}})^n f(x, y, \tilde{r}) \zeta(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}) d\tilde{r}, \end{aligned}$$

where

$$f(x, y, \tilde{r}) = \int \bar{e}[\eta_\lambda(\tilde{r})\beta(x, y - e^{-\lambda \tilde{r}} \tilde{y})] \xi(\tilde{x} - e^{\lambda \tilde{r}} x, \tilde{y} - e^{\lambda \tilde{r}} y, -\tilde{r}) \overline{\eta(\tilde{x}, \tilde{y}, -\tilde{r})} d\tilde{x}d\tilde{y}.$$

Since ξ and η are L^2 -functions, f is a continuous function. It would also have compact support. Meanwhile, since the choice of ξ and η is arbitrary and since the $\omega_{\xi, \eta}$ are dense in $\mathcal{B}(\mathcal{H})_*$, we can see that the collection of the f will form a total set in the space of continuous

functions in the (x, y, r) variables having compact support. It follows that we have:

$$\overline{\rho(\hat{\mathcal{A}})}^{\|\cdot\|} = \overline{\{(\text{id} \otimes \omega)(U_A) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|} = \hat{A}.$$

□

As in the case of A , we will often regard the functions $f \in \hat{\mathcal{A}}$ as the same as the operators $\rho_f \in \hat{A}$. In this way, $\hat{\mathcal{A}}$ is considered as a *-subalgebra of \hat{A} . The multiplication on it is defined by $\rho_{f \times g} = \rho_f \rho_g$:

$$(f \times_{\hat{A}} g)(x, y, r) = \int f(x, y, \tilde{r}) g(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}) d\tilde{r}. \quad (2.1)$$

While, the involution on $\hat{\mathcal{A}}$ is given by $\rho_{f^*} = (\rho_f)^*$:

$$f^*(x, y, r) = \overline{f(e^{\lambda r} x, e^{\lambda r} y, -r)}. \quad (2.2)$$

Remark. We can see that as a C^* -algebra, $\hat{A} \cong C_0(\mathbb{R}^{2n}) \rtimes_{\alpha} \mathbb{R}$, which is a crossed product algebra (together with the action of \mathbb{R} onto \mathbb{R}^{2n} given by $\alpha(r) : (x, y) \mapsto (e^{\lambda r} x, e^{\lambda r} y)$). Although our construction of \hat{A} here is given indirectly by relying on the duality, certainly there is a direct way of constructing $(\hat{A}, \hat{\Delta})$, giving a Hopf C^* -algebra structure on a crossed product algebra (It is actually easier than the case of (A, Δ) in [5]. See also the more general approach described in [16]). Meanwhile, we also note that when the parameter $\lambda = 0$, we have: $\hat{A}_{\lambda=0} \cong C_0(\mathbb{R}^{2n+1}) = C_0(H)$, giving us a mild justification that \hat{A} is a good candidate to become a “quantized $C_0(H)$ ”.

Turning our attention to the coalgebra structure, let us give here the description of the comultiplication $\hat{\Delta}$, at the level of functions:

Proposition 2.3. *For $f \in \hat{\mathcal{A}}$, let $\hat{\Delta}f$ be the Schwartz function in the $(x, y, r; x', y', r')$ variables having compact support in r and r' , given by*

$$\begin{aligned} & \hat{\Delta}f(x, y, r; x', y', r') \\ &= \int f(x + x', y + y', \tilde{r}) e[\eta_{\lambda}(\tilde{r})\beta(x, y')] e[\tilde{r}(z + z')] \bar{e}[zr + z'r'] d\tilde{r} dz dz'. \end{aligned}$$

The map $f \mapsto \hat{\Delta}f$ extends to the map $\hat{\Delta} : \hat{A} \rightarrow M(\hat{A} \otimes \hat{A})$, which is the comultiplication given in Definition 2.1.

Proof. For $f \in \hat{\mathcal{A}}$ and for $\xi \in \mathcal{H} \otimes \mathcal{H}$, we have:

$$\begin{aligned} & U_A^*(1 \otimes \rho_f) U_A \xi(x, y, r; x', y', r') \\ &= \int (e^{2\lambda \tilde{r}})^n f(x + x', y + y', \tilde{r}) e[\eta_{\lambda}(\tilde{r})\beta(x, y')] \xi(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}; e^{\lambda \tilde{r}} x', e^{\lambda \tilde{r}} y', r' - \tilde{r}) d\tilde{r}. \end{aligned}$$

By straightforward calculation, we can check without difficulty that $(\rho \otimes \rho)_{\hat{\Delta}_f} \xi(x, y, r; x', y', r') = U_A^*(1 \otimes \rho_f) U_A \xi(x, y, r; x', y', r')$. This means that $f \mapsto \hat{\Delta}f$ coincides with the comultiplication $\rho_f \mapsto \hat{\Delta}(\rho_f)$, at the level of the dense subalgebra. It obviously extends to the comultiplication on \hat{A} . \square

We will skip the proofs of the various properties of $\hat{\Delta}$, referring instead to general theory. For instance, the coassociativity of $\hat{\Delta}$ follows from the unitary operator U_A being multiplicative. Meanwhile, we see again that when the parameter $\lambda = 0$, the expression at the level of functions in the $(x, y, z; x', y', z')$ variables for $\hat{\Delta}f$, obtained by using partial Fourier transform, is just:

$$\hat{\Delta}f(x, y, z; x', y', z') = f(x + x', y + y', z + z' + \beta(x, y')),$$

recovering the usual comultiplication on $C_0(H)$.

Next, let us consider the *antipode* \hat{S} . The main result is summarized below:

Proposition 2.4. *For $f \in \hat{\mathcal{A}}$, let $\hat{S}(f)$ be the function in $\hat{\mathcal{A}}$ defined by*

$$(\hat{S}(f))(x, y, r) = \bar{e}[\eta_\lambda(r)\beta(x, y)]f(-e^{\lambda r}x, -e^{\lambda r}y, -r).$$

Then \hat{S} can be extended to the anti-automorphism $\hat{S} : \hat{A} \rightarrow \hat{A}$. It is the antipodal map, satisfying: $\hat{S}(\hat{S}(b)^)^* = b$ and $(\hat{S} \otimes \hat{S})(\hat{\Delta}b) = \chi(\hat{\Delta}(\hat{S}(b)))$, where χ denotes the flip. We also have: $\hat{S}^2 \equiv \text{Id}$.*

Proof. The definition is suggested by [1] and [21]. It is equivalent to the map $\hat{S} : (\text{id} \otimes \omega)(U_A) \mapsto (\text{id} \otimes \omega)(U_A^*)$, for $\omega \in \mathcal{B}(\mathcal{H})_*$. Using same kind of the technique we used in the proof of Proposition 2.2, we could obtain the above expression for $\hat{S}(f)$. As before, this should be interpreted as $\hat{S}(\rho_f) = \rho_{\hat{S}(f)}$. Meanwhile, a straight calculation shows that \hat{S} can be equivalently written as $\hat{S}(b) = Jb^*J$, for $b \in \hat{A}$, where J is the anti-unitary operator defined by

$$J\xi(x, y, r) = \bar{e}[\eta_\lambda(r)\beta(x, y)]\overline{\xi(-x, -y, r)}.$$

Due to this characterization, the remaining properties are easy to verify. \square

Remark. The notation for the operator J introduced in the proof is motivated by the modular theory, and it is essentially the involution on \mathcal{A} (as defined in (1.1)). Indeed, remembering that the space \mathcal{A} is dense in \mathcal{H} with respect to the Hilbert space norm, we see that J is just the extension of the map $f \mapsto f^*$ in \mathcal{A} .

The correct formulation of \hat{S} being the legitimate antipode relies on the existence of an appropriate Haar weight (to be constructed shortly). However, we may still point out that if $\lambda = 0$, the antipode at the level of functions in the (x, y, z) variables is just:

$$(\hat{S}(f))(x, y, z) = f(-x, -y, -z + \beta(x, y)) = f((x, y, z)^{-1}).$$

Here again, we used the partial Fourier transform.

Since \hat{S} is already an anti-automorphism such that $\hat{S}^2 \equiv \text{Id}$, its “polar decomposition” (in the sense of [11] and [10]) is trivial: That is, $\hat{S} = \hat{R}$ (the “unitary antipode”), and $\hat{\tau} \equiv \text{Id}$ (the “scaling group”). Meanwhile, $\hat{S}^2 \equiv \text{Id}$ suggests that our example will be a kind of a “Kac C^* -algebra” [17], [4], which is expected since (A, Δ) was one such.

One remaining important structure to be constructed is the Haar weight. We begin with the linear functional $\hat{\varphi}$ defined at the dense function algebra level (i. e. on $\hat{\mathcal{A}}$), motivated by the Lebesgue measure on H :

Proposition 2.5. *On $\hat{\mathcal{A}}$, define the linear functional $\hat{\varphi}$ by*

$$\hat{\varphi}(f) = \int f(x, y, 0) dx dy.$$

Then $\hat{\varphi}$ defined as above is a faithful, positive linear functional. It is also unimodular, in the sense that $\hat{\varphi} \circ \hat{S} = \hat{\varphi}$.

Proof. Suppose $F \in \hat{\mathcal{A}}$ is a typical positive element such that $\rho_F = (\rho_f)(\rho_f)^*$ for some $f \in \hat{\mathcal{A}}$. Then we have:

$$\begin{aligned} \hat{\varphi}(F) &= \hat{\varphi}(f \times_{\hat{\mathcal{A}}} f^*) = \int (f \times_{\hat{\mathcal{A}}} f^*)(x, y, 0) dx dy \\ &= \int f(x, y, \tilde{r}) f^*(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, 0 - \tilde{r}) d\tilde{r} dx dy \\ &= \int f(x, y, \tilde{r}) \overline{f(x, y, \tilde{r})} d\tilde{r} dx dy = \|f\|_2^2. \end{aligned}$$

From this, the first part of the proposition is immediate. Meanwhile, for an arbitrary element $f \in \hat{\mathcal{A}}$, we have:

$$\hat{\varphi}(\hat{S}(f)) = \int (\hat{S}(f))(x, y, 0) dx dy = \int f(-x, -y, 0) dx dy = \hat{\varphi}(f),$$

giving us the proof of the unimodularity. \square

We need to find a C^* -algebra weight extending this linear functional. The following steps are more or less the same ones we took in [8] (One difference is that $\hat{\varphi}$ is no longer a trace.). First, let us consider the

GNS construction associated with $\hat{\varphi}$. We see below that the “regular representation” ρ of $\hat{\mathcal{A}}$ we have been using is essentially the GNS representation:

Proposition 2.6. *Let $\Gamma : \hat{\mathcal{A}} \rightarrow \mathcal{H}$ be defined by $\Gamma(f)(x, y, r) := (e^{\lambda r})^n f(x, y, r)$. Then for $f, g \in \hat{\mathcal{A}}$, we have:*

$$\langle \Gamma(f), \Gamma(g) \rangle_{\mathcal{H}} = \hat{\varphi}(g^* \times_{\hat{\mathcal{A}}} f),$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the inner product on \mathcal{H} , conjugate in the second place. From this, we see that Γ gives the Hilbert space isomorphism between $\mathcal{H}_{\hat{\varphi}}$ and \mathcal{H} , where $\mathcal{H}_{\hat{\varphi}}$ is the GNS Hilbert space for $\hat{\varphi}$. Meanwhile, consider the non-degenerate $*$ -representation $\pi_{\hat{\varphi}} : \hat{\mathcal{A}} \rightarrow \mathcal{B}(\mathcal{H})$, given by $(\pi_{\hat{\varphi}}(f))(\Gamma(g)) := \Gamma(f \times_{\hat{\mathcal{A}}} g)$. It turns out that $\pi_{\hat{\varphi}}$ coincides with the representation ρ .

Proof. For $f, g \in \hat{\mathcal{A}}$,

$$\begin{aligned} \hat{\varphi}(g^* \times_{\hat{\mathcal{A}}} f) &= \int g^*(x, y, \tilde{r}) f(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, 0 - \tilde{r}) d\tilde{r} dx dy \\ &= \int \overline{g(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, -\tilde{r})} f(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, -\tilde{r}) d\tilde{r} dx dy \\ &= \int (e^{2\lambda \tilde{r}})^n \overline{g(x, y, \tilde{r})} f(x, y, \tilde{r}) d\tilde{r} dx dy = \langle \Gamma(f), \Gamma(g) \rangle_{\mathcal{H}}. \end{aligned}$$

Since the GNS Hilbert space $\mathcal{H}_{\hat{\varphi}}$ is obtained by completing $\hat{\mathcal{A}}$ with respect to the inner product $(f, g) \mapsto \hat{\varphi}(g^* \times_{\hat{\mathcal{A}}} f)$, we see easily that Γ (now extended to $\mathcal{H}_{\hat{\varphi}}$) provides the Hilbert space isomorphism $\Gamma : \mathcal{H}_{\hat{\varphi}} \cong \mathcal{H}$. The representation $\pi_{\hat{\varphi}}$ being non-degenerate is immediate, remembering that $\Gamma(\hat{\mathcal{A}})$ is dense in \mathcal{H} . Now to learn about the representation $\pi_{\hat{\varphi}}$, consider $f, g \in \hat{\mathcal{A}}$. Let us write $\zeta = \Gamma(g) \in \mathcal{H}$. Then:

$$\begin{aligned} (\pi_{\hat{\varphi}}(f))\zeta(x, y, r) &= (\Gamma(f \times_{\hat{\mathcal{A}}} g))(x, y, r) = (e^{\lambda r})^n (f \times_{\hat{\mathcal{A}}} g)(x, y, r) \\ &= \int (e^{\lambda r})^n f(x, y, \tilde{r}) g(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}) d\tilde{r} \\ &= \int (e^{\lambda \tilde{r}})^n f(x, y, \tilde{r}) \zeta(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}) d\tilde{r} \\ &= (\rho_f \zeta)(x, y, r), \end{aligned}$$

recovering the representation ρ . □

By (essential) uniqueness of GNS construction, we see from the above proposition that $(\mathcal{H}, \Gamma, \pi_{\hat{\varphi}} = \rho)$ is the GNS triple associated with $\hat{\varphi}$. The consequence is that the algebra $\hat{\mathcal{A}}$ (to be more precise, $\Gamma(\hat{\mathcal{A}}) \subseteq \mathcal{H}$) is a “left Hilbert algebra” (See literature on modular theory [3], [14]).

One detail to note is that the involution on $\hat{\mathcal{A}}$ is not isometric with respect to the inner product (This reflects the fact that the functional $\hat{\varphi}$ is not a trace.). But it is still closable.

We denote by \hat{T} the closure of the involution on $\hat{\mathcal{A}}$. Then it is a closed, anti-linear map on \mathcal{H} having $\Gamma(\hat{\mathcal{A}})$ as a core for \hat{T} , such that $\hat{T}(\Gamma(f)) = \Gamma(f^*)$. By a simple calculation, we have:

$$\hat{T}\zeta(x, y, r) = (e^{2\lambda r})^n \overline{\zeta(e^{\lambda r}x, e^{\lambda r}y, -r)}.$$

We have the polar decomposition: $\hat{T} = \hat{J}\hat{\nabla}^{\frac{1}{2}}$, where $\hat{\nabla} = \hat{T}^*\hat{T}$ is the “modular operator” and \hat{J} is an anti-unitary operator. They are given as follows:

$$\hat{\nabla}f(x, y, r) = (e^{-2\lambda r})^n f(x, y, r), \quad \hat{J}f(x, y, r) = (e^{\lambda r})^n \overline{f(e^{\lambda r}x, e^{\lambda r}y, -r)}.$$

Note here that \hat{J} is exactly the anti-unitary operator which we used in our definition of the antipode S for (A, Δ) , given in [8]. Compare this with the remark we made following Proposition 2.4, pointing out the relationship between the operator J and the antipode \hat{S} of $(\hat{A}, \hat{\Delta})$. This aspect is one of many useful relationships between the (mutually dual) algebras \mathcal{A} and $\hat{\mathcal{A}}$. See [11] and [10].

Since we have a left Hilbert algebra structure on $\hat{\mathcal{A}}$, we can follow the standard modular theory ([3], [14]) to obtain a C^* -algebra weight extending $\hat{\varphi}$. The modular operator $\hat{\nabla}$ plays an important role in the formulation of the KMS property.

Theorem 2.7. *There is a faithful, lower semi-continuous weight $\hat{\varphi}_{\hat{A}}$ on the C^* -algebra \hat{A} , extending the linear functional $\hat{\varphi}$. It is also a KMS weight: With respect to the (norm-continuous) one-parameter group of automorphisms $\hat{\sigma}$ given by $\hat{\sigma}_t(b) = \hat{\nabla}^{it}b\hat{\nabla}^{-it}$, we have:*

$$\begin{aligned} \hat{\varphi}_{\hat{A}} \circ \hat{\sigma}_t &= \hat{\varphi}_{\hat{A}}, \quad \text{for all } t \in \mathbb{R}, \\ \hat{\varphi}_{\hat{A}}(b^*b) &= \hat{\varphi}_{\hat{A}}(\hat{\sigma}_{i/2}(b)\hat{\sigma}_{i/2}(b)^*), \quad \text{for all } b \in D(\hat{\sigma}_{i/2}). \end{aligned}$$

Remark. The notion of “KMS weight” we are using above is due to Kustermans [9], which is actually equivalent to the original notion given by Combes [3]. Since the weight $\hat{\varphi}_{\hat{A}}$ extends the functional $\hat{\varphi}$ on $\hat{\mathcal{A}}$, it is densely defined, giving us a “proper” KMS weight. Refer the discussion in §1.1 of [8] or literature on weight theory [2], [3], [14].

Proof. The (non-degenerate) representation $\pi_{\hat{\varphi}} (= \rho)$ generates the von Neumann algebra $M_{\hat{A}} = \rho(\hat{\mathcal{A}})''$ in $\mathcal{B}(\mathcal{H})$. Since $\hat{\mathcal{A}}$ is a left Hilbert algebra, there is a standard way of defining a faithful, semi-finite, normal weight on $M_{\hat{A}}$ (See Theorem 2.11 of [3]. See also the discussion we made

in Theorem 3.6 of [8].). We then obtain our weight $\hat{\varphi}_{\hat{A}}$, by restricting this normal weight to the C^* -algebra $\hat{A} = \overline{\rho(\hat{\mathcal{A}})}^{\|\cdot\|} (\subseteq \rho(\hat{\mathcal{A}})'' = M_{\hat{A}})$. Because of the way it is constructed, it is not difficult to see that $\hat{\varphi}_{\hat{A}}$ extends the functional $\hat{\varphi}$ and is faithful.

The lower semi-continuity and the KMS property of $\hat{\varphi}_{\hat{A}}$ is a consequence of the fact that it is obtained from a normal weight at the von Neumann algebra level. In our case, the modular automorphism group is such that $\hat{\mathcal{A}}$ forms a core for the $\hat{\sigma}_t$ and that for $f \in \mathcal{A}$, we have:

$$(\hat{\sigma}_t(f))(x, y, r) = (e^{-2\lambda rit})^n f(x, y, r).$$

As before, this is interpreted as $\rho_{\hat{\sigma}_t(f)} = \hat{\sigma}_t(\rho_f) = \hat{\nabla}^{it} \rho_f \hat{\nabla}^{-it}$. To verify the KMS property, we just choose $f \in \hat{\mathcal{A}}$ can calculate. Since $(\hat{\sigma}_{i/2}(f))(x, y, r) = (e^{\lambda r})^n f(x, y, r)$, We have:

$$\hat{\varphi}(\hat{\sigma}_{i/2}(f)\hat{\sigma}_{i/2}(f)^*) = \int (e^{2\lambda \tilde{r}})^n f(x, y, \tilde{r}) \overline{f(x, y, \tilde{r})} d\tilde{r} dx dy = \hat{\varphi}(f^* \times_{\hat{A}} f).$$

Verification of $\hat{\varphi}(\hat{\sigma}_t(f)) = \hat{\varphi}(f)$ is also straightforward. \square

For $\hat{\varphi}_{\hat{A}}$ to be considered as the legitimate Haar weight (as well as to complete the discussion that $(\hat{A}, \hat{\Delta})$ is a locally compact quantum group), we need to establish its (left) invariance property. This will be done following the idea suggested by Van Daele [19], [20] (See also our discussion in section 3 of [8].). We first begin with a lemma.

Lemma 2.8. *Let M_A be the enveloping von Neumann algebra of A (That is, $M_A = L(\mathcal{A})''$), while $M_{\hat{A}} (= \rho(\hat{\mathcal{A}})'')$ is the enveloping von Neumann algebra of \hat{A} as appeared in the proof of Theorem 2.7. Then we have:*

- (1) $U_A \in M_{\hat{A}} \otimes M_A \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.
- (2) $M_A \cap M_{\hat{A}} = \mathbb{C}1$.
- (3) *The linear space $M_A M_{\hat{A}}$ is σ -strongly dense in $\mathcal{B}(\mathcal{H})$.*

Proof. The first statement follows from general theory of multiplicative unitary operators. It is also true that we have: $U_A \in M(\hat{A} \otimes A)$. For the next two statements, we may follow Proposition 2.5 of [20]. \square

The main strategy suggested by Van Daele is that there exists a faithful, semi-finite, normal weight ν on $\mathcal{B}(\mathcal{H})$ such that at least formally, $\nu(ab) = \varphi(a)\hat{\varphi}(b)$, for $a \in M_A$, $b \in M_{\hat{A}}$. Note here that for convenience, we are using the notation φ and $\hat{\varphi}$ for the weights. As long as there is not going to be confusion between the linear functionals and the weights, we will often use the simpler notation.

Proposition 2.9. *On $\mathcal{B}(\mathcal{H})$, consider the linear functional $\nu := \text{Tr}$. Then ν is a faithful, semi-finite, normal weight on $\mathcal{B}(\mathcal{H})$ such that for $a \in \mathfrak{N}_\varphi$ and $b \in \mathfrak{N}_{\hat{\varphi}}$,*

$$\nu(a^*b^*ba) = \varphi(a^*a)\hat{\varphi}(b^*b).$$

Remark. The notations \mathfrak{N}_φ and $\mathfrak{N}_{\hat{\varphi}}$ are the standard ones used in weight theory, which just ensure that the expression in the right hand side does make sense (i. e. finite). This result is actually quite general in nature, although ν should be in general a certain “weighted trace” instead of being the regular trace (See Definition 2.6 of [20]). The reason why the regular trace works in our case has to do with the fact that φ is a tracial weight on M_A (as shown in section 3 of [8]).

Proof. Let us pick two elements at the dense function algebra level, namely $a = L_a \in \mathcal{A}(\subseteq M_A)$ and $b = \rho_b \in \hat{\mathcal{A}}(\subseteq M_{\hat{A}})$. Then by definition of L_a and ρ_b given in earlier sections, we have:

$$\begin{aligned} & (a^*b^*ba)\xi(x, y, r) \\ &= \int \overline{a(-\tilde{x}, -\tilde{y}, r)}\bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y)](e^{\lambda\tilde{r}})^n \overline{b(e^{\lambda\tilde{r}}(x - \tilde{x}), e^{\lambda\tilde{r}}(y - \tilde{y}), -\tilde{r})} \\ & \quad (e^{\lambda\hat{r}})^n b(e^{\lambda\hat{r}}(x - \tilde{x}), e^{\lambda\hat{r}}(y - \tilde{y}), \hat{r})a(\hat{x}, \hat{y}, r - \tilde{r} - \hat{r}) \\ & \quad \bar{e}[\eta_\lambda(r - \tilde{r} - \hat{r})\beta(\hat{x}, e^{\lambda(\tilde{r}+\hat{r})}(y - \tilde{y}) - \hat{y})] \\ & \quad \xi(e^{\lambda(\tilde{r}+\hat{r})}(x - \tilde{x}) - \hat{x}, e^{\lambda(\tilde{r}+\hat{r})}(y - \tilde{y}) - \hat{y}, r - \tilde{r} - \hat{r}) d\tilde{x}d\tilde{y}d\tilde{r}d\hat{x}d\hat{y}. \end{aligned}$$

So if we let (ξ_i) be an orthonormal basis in \mathcal{H} , we would have, by using change of variables:

$$\begin{aligned} \nu(a^*b^*ba) &= \text{Tr}(a^*b^*ba) = \sum_i \langle (a^*b^*ba)\xi_i, \xi_i \rangle \\ &= \int \overline{a(-\tilde{x}, -\tilde{y}, r)}\bar{e}[\eta_\lambda(r)\beta(\tilde{x}, y)] \overline{b(e^{\lambda\tilde{r}}(x - \tilde{x}), e^{\lambda\tilde{r}}(y - \tilde{y}), -\tilde{r})} \\ & \quad b(e^{\lambda\tilde{r}}(x - \tilde{x}), e^{\lambda\tilde{r}}(y - \tilde{y}), -\tilde{r})a(-\tilde{x}, -\tilde{y}, r)\bar{e}[\eta_\lambda(r)\beta(-\tilde{x}, y)] d\tilde{x}d\tilde{y}d\tilde{r}dx dy dr \\ &= \int \overline{a(\tilde{x}, \tilde{y}, r)}a(\tilde{x}, \tilde{y}, r)(e^{-2\lambda\tilde{r}})^n \overline{b(x, y, \tilde{r})}b(x, y, \tilde{r}) d\tilde{x}d\tilde{y}d\tilde{r}dx dy dr \\ &= \varphi(a^* \times_A a)\hat{\varphi}(b^* \times_{\hat{A}} b). \end{aligned}$$

Since \mathcal{A} and $\hat{\mathcal{A}}$ generate the von Neumann algebras M_A and $M_{\hat{A}}$, while $M_A M_{\hat{A}}$ is dense in $\mathcal{B}(\mathcal{H})$, this characterizes ν . \square

The implication of this proposition is that for some well-chosen element $a \in A$, the map $b \mapsto \nu(a^*ba)$ is a scalar multiple of the weight $\hat{\varphi}(b)$. So proving the left invariance of $\hat{\varphi}$ will be equivalent to showing the left invariance of $\nu(a^* \cdot a)$. This is done in Theorem 2.11 below,

with a short lemma preceding it. The steps are very similar to the proof of Theorem 3.9 of [8].

Lemma 2.10. *Let (ξ_l) be an orthonormal basis for \mathcal{H} . For $\zeta \in \mathcal{H}$, consider the element $w_k = (\omega_{\zeta, \xi_k} \otimes \text{id})(U_A) \in \mathcal{B}(\mathcal{H})$. Then we have:*

$$\sum_k \langle w_k \xi_l, w_k \xi_j \rangle = \langle \zeta, \zeta \rangle \langle \xi_l, \xi_j \rangle.$$

Remark. The well-known definition of the forms of the type $\omega_{\zeta, \xi} \in \mathcal{B}(\mathcal{H})_*$ was given in the proof of Proposition 2.2. Meanwhile, we know from (1.2) that $w_k \in A$ (See also [1] and Proposition 2.2 of [8]).

Proof. We take advantage of the fact that (ξ_k) is an orthonormal basis, and that U_A is a unitary operator. We have:

$$\sum_k \langle w_k \xi_l, w_k \xi_j \rangle = \langle U_A(\zeta \otimes \xi_l), U_A(\zeta \otimes \xi_j) \rangle = \langle \zeta \otimes \xi_l, \zeta \otimes \xi_j \rangle = \langle \zeta, \zeta \rangle \langle \xi_l, \xi_j \rangle.$$

□

Theorem 2.11. *For any positive element $b \in \hat{A}$ such that $\hat{\varphi}(b) < \infty$, and for positive $\omega \in \hat{A}^*$, we have:*

$$\hat{\varphi}((\omega \otimes \text{id})(\hat{\Delta}b)) = \omega(1)\hat{\varphi}(b).$$

Proof. As suggested above in our comments following Proposition 2.9, we may prove this for $\nu(a^* \cdot a)$, where $a \in \mathcal{A}(\subseteq \mathfrak{N}_\varphi)$ is a fixed element.

Let $b \in \mathfrak{M}_{\hat{\varphi}}^+$ and let $\omega \in A^*_+$. Without loss of generality, we may assume that ω is the vector state of the form $\omega = \omega_{\zeta, \zeta}$, for $\zeta \in \mathcal{H}$. We then have:

$$\begin{aligned} (\omega \otimes \text{id})(\hat{\Delta}b) &= (\omega_{\zeta, \zeta} \otimes \text{id})(U_A^*(1 \otimes b)U_A) \\ &= \sum_k [(\omega_{\xi_k, \zeta} \otimes \text{id})(U_A^*)] b [(\omega_{\zeta, \xi_k} \otimes \text{id})(U_A)] = \sum_k w_k^* b^{\frac{1}{2}} b^{\frac{1}{2}} w_k. \end{aligned}$$

Here (ξ_k) is an orthonormal basis for \mathcal{H} , and the sums above are convergent in the σ -weak topology on $M_{\hat{A}}$ (See Lemma 3.8 of [8]). Also for convenience, we wrote $w_k = (\omega_{\zeta, \xi_k} \otimes \text{id})(U_A)$.

Let us use the result of the previous lemma and calculate:

$$\begin{aligned} \nu(a^*(\omega_{\zeta, \zeta} \otimes \text{id})(\hat{\Delta}b)a) &= \sum_k \nu(a^* w_k^* b^{\frac{1}{2}} b^{\frac{1}{2}} w_k a) \\ &= \sum_{k,l} \langle w_k b^{\frac{1}{2}} a \xi_l, w_k b^{\frac{1}{2}} a \xi_l \rangle & \nu &:= \text{Tr} [\text{trace on } \mathcal{B}(\mathcal{H})] \\ &= \sum_l \langle \zeta, \zeta \rangle \langle b^{\frac{1}{2}} a \xi_l, b^{\frac{1}{2}} a \xi_l \rangle & &\text{by Lemma 2.10} \\ &= \langle \zeta, \zeta \rangle \text{Tr}(a^* b a) = \|\omega\| \nu(a^* b a) = \omega(1) \nu(a^* b a). \end{aligned}$$

Since $\nu(a^*ba) = \varphi(a^*a)\hat{\varphi}(b)$, and since $\varphi(a^*a)$ is a positive constant, this will give us the proof that $\hat{\varphi}$ is left invariant. \square

The left invariance we have just verified is a weak form, but by general theory [10], this is actually sufficient. This establishes the proof that $\hat{\varphi}$ is a legitimate Haar weight for $(\hat{A}, \hat{\Delta})$, in the sense that it is a proper, faithful, KMS weight which is left invariant. In our case, unlike the case of φ_A , the Haar weight $\hat{\varphi}_{\hat{A}}$ is actually unimodular (Recall Proposition 2.5). Since this is the case, no extra discussion is necessary for the “right Haar weight” or the “modular function”. Summarizing the results of this section, we have the following theorem:

Theorem 2.12. *The pair $(\hat{A}, \hat{\Delta})$, together with its additional structures including the antipode and the (unimodular) Haar weight, is a C^* -algebraic locally compact quantum group, in the sense of Kustermans and Vaes.*

As we have made our case throughout this section, we may now regard $(\hat{A}, \hat{\Delta})$ as the *quantum Heisenberg group* (i. e. “quantized $C_0(H)$ ”). On the other hand, we remark here that our example is different (and slightly more complicated) from the earlier example of a quantum Heisenberg group obtained by Van Daele [18].

For instance, the Haar weight in the earlier example (although it was not explicitly constructed in that paper) is a trace, while ours is non-tracial. The dual object of Van Daele’s example is the example by Rieffel [13], while in our case, (A, Δ) of [5], [8] plays that role. These differences can be understood more clearly if we consider the classical limits and compare the Poisson structures: Our examples (A, Δ) and $(\hat{A}, \hat{\Delta})$ were obtained by quantizing a certain non-linear Poisson structure, while the examples of Rieffel’s ([13]) and Van Daele’s ([18]) correspond to a linear Poisson structure.

3. DUALITY

The relationship between our two quantum groups (A, Δ) and $(\hat{A}, \hat{\Delta})$ is essentially the same as the relationship between $C^*(H)$ and $C_0(H)$. Actually, it is a general fact that given a locally compact quantum group (B, Δ) , one can construct the dual quantum group $(\hat{B}, \hat{\Delta})$ within the category of locally compact quantum groups, and that the generalized Pontryagin-type duality holds: That is, $(\hat{\hat{B}}, \hat{\hat{\Delta}}) \cong (B, \Delta)$. Refer [11], [10] for the general discussion on the duality of locally compact quantum groups.

Our goal in this section is to see how the general theory is reflected in the case of our specific examples. Most of the results below are more

or less obvious and are direct consequences of general theory. On the other hand, several of these will be useful in our future applications.

3.1. The dual pairing between \mathcal{A} and $\hat{\mathcal{A}}$. Our quantum groups (A, Δ) and $(\hat{A}, \hat{\Delta})$ are obtained as two Hopf C^* -algebras associated with the multiplicative unitary operator U_A (as in [1]). But unlike in the case of (finite-dimensional) Hopf algebras, we do not actually have a dual pairing at the level of C^* -algebras A and \hat{A} . What we do have is the dual pairing at the dense function algebra level of \mathcal{A} and $\hat{\mathcal{A}}$. This is described in the following proposition.

Proposition 3.1. (1) *The dual pairing exists between \mathcal{A} and $\hat{\mathcal{A}}$ such that for $f(=L_f) \in \mathcal{A}$ and $g(=\rho_g) \in \hat{\mathcal{A}}$, we have:*

$$\langle f, g \rangle = \int f(x, y, r) g(e^{\lambda r} x, e^{\lambda r} y, -r) dx dy dr.$$

This is equivalent to the following pairing suggested by the multiplicative unitary operator:

$$\langle L(\omega), \rho(\omega') \rangle = (\omega \otimes \omega')(U_A) = \omega(\rho(\omega')) = \omega'(L(\omega)),$$

where $L(\omega) = (\omega \otimes \text{id})(U_A) \in A$ and $\rho(\omega') = (\text{id} \otimes \omega')(U_A) \in \hat{A}$ are as defined earlier with $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$.

(2) *The dual pairing given above is compatible with the Hopf algebra structures on \mathcal{A} and $\hat{\mathcal{A}}$. Indeed, for $f, f_1, f_2 \in \mathcal{A}$ and $g, g_1, g_2 \in \hat{\mathcal{A}}$ (so $f = L_f, g = \rho_g, \dots$), we have:*

$$\begin{aligned} \langle f_1 \otimes f_2, \hat{\Delta}(g) \rangle &= \langle f_1 \times_{\mathcal{A}} f_2, g \rangle, \\ \langle f, g_1 \times_{\hat{\mathcal{A}}} g_2 \rangle &= \langle \Delta(f), g_1 \otimes g_2 \rangle, \\ \langle S(f), g \rangle &= \langle f, \hat{S}(g) \rangle, \quad \langle f, g^* \rangle = \overline{\langle S(f)^*, g \rangle}. \end{aligned}$$

Proof. As described in the first part of the proposition, our definition of the dual pairing was suggested by the theory of multiplicative unitary operators. For this, we use the same technique as in the proofs of Propositions 2.2 and 2.4. That is, consider $\omega_{\xi, \eta}$, with ξ, η being continuous functions with compact support (contained in \mathcal{H}), so that we can realize the expressions like $(\text{id} \otimes \omega_{\xi, \eta})(U_A)$ as continuous functions having compact support.

Once we take the above definition as our dual pairing, the verification of the statements in the second part is very much straightforward. All we need to do is to remember the expressions of various operations (for instance, equations (1.1), (1.3), (2.1), (2.2) and Propositions 2.3 and

2.4) and just carry out the calculations. For the first relation:

$$\begin{aligned}
 & \langle f_1 \otimes f_2, \hat{\Delta}(g) \rangle \\
 &= \int f_1(x, y, \tilde{r}) f_2(x', y', \tilde{r}) g(e^{\lambda \tilde{r}} x + e^{\lambda \tilde{r}} x', e^{\lambda \tilde{r}} y + e^{\lambda \tilde{r}} y', -\tilde{r}) \\
 &\quad e[\eta_\lambda(-\tilde{r})\beta(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y')] dx dy dx' dy' d\tilde{r} \\
 &= \int f_1(x, y, \tilde{r}) f_2(x' - x, y' - y, \tilde{r}) \bar{e}[\eta_\lambda(\tilde{r})\beta(x, y' - y)] \\
 &\quad g(e^{\lambda \tilde{r}} x', e^{\lambda \tilde{r}} y', -\tilde{r}) dx dy dx' dy' d\tilde{r} \\
 &= \int (f_1 \times_A f_2)(x', y', \tilde{r}) g(e^{\lambda \tilde{r}} x', e^{\lambda \tilde{r}} y', -\tilde{r}) dx' dy' d\tilde{r} = \langle f_1 \times_A f_2, g \rangle.
 \end{aligned}$$

The other relations can be verified similarly. Note that except the one involving the $*$ operation, the relations are exactly the ones we see from ordinary Hopf algebra theory. \square

3.2. Duality at the Poisson–Lie group level. We have not been much emphasizing the role of the Poisson geometry in this paper, but a brief discussion of the classical limit counterparts would be useful here. We have been arguing that (A, Δ) is a “quantized $C^*(H)$ ” (See remark at the end of section 1, as well as our previous papers [6], [7].). And we saw throughout section 2 that it is reasonable to consider $(\hat{A}, \hat{\Delta})$ as a “quantized $C_0(H)$ ”.

Meanwhile, in [5], we have made our case that (A, Δ) is also a “quantized $C_0(G)$ ”, where G is the dual Poisson–Lie group of H . For the case of $(\hat{A}, \hat{\Delta})$, we can actually regard it as a “quantized $C^*(G)$ ”. To illustrate just one aspect of this, recall the formula for the product on \hat{A} as given in (2.1). If we express this at the level of functions in the (p, q, r) variables (again by using the partial Fourier transform), it becomes:

$$(f \times_{\hat{A}} g)(p, q, r) = \int (e^{-2\lambda \tilde{r}})^n f(\tilde{p}, \tilde{q}, \tilde{r}) g(e^{-\lambda \tilde{r}} p - e^{-\lambda \tilde{r}} \tilde{p}, e^{-\lambda \tilde{r}} q - e^{-\lambda \tilde{r}} \tilde{q}, r - \tilde{r}) d\tilde{p} d\tilde{q} d\tilde{r}.$$

But if we assume that H^* has the group structure given by the multiplication law:

$$(p, q, r)(p', q', r') = (e^{\lambda r'} p + p', e^{\lambda r'} q + q', r + r'),$$

which is exactly the multiplication law for the dual Poisson–Lie group G of H as defined in [5], then the above expression for the product on \hat{A} can be written as:

$$(f \times_{\hat{A}} g)(p, q, r) = \int (e^{-2\lambda \tilde{r}})^n f(\tilde{p}, \tilde{q}, \tilde{r}) g((p, q, r)(\tilde{p}, \tilde{q}, \tilde{r})^{-1}) d\tilde{p} d\tilde{q} d\tilde{r}.$$

Since $(e^{-2\lambda\tilde{r}})^n d\tilde{p}d\tilde{q}d\tilde{r}$ is the right Haar measure for the group $G(=H^*)$, this means that it is really the convolution product. In other words, we notice that $\hat{A} \cong C^*(G)$ as a C^* -algebra, where $C^*(G)$ is realized as an operator algebra via the right regular representation of G .

These observations illustrate that the duality between (A, Δ) and $(\hat{A}, \hat{\Delta})$ is the quantum counterpart to the Poisson–Lie group duality between G and H . This point of view is certainly very useful in any applications involving our quantum groups. The duality picture will be enhanced when we consider the “quantum double” of our examples (Just as the “double Poisson–Lie group” $H \bowtie G$ and the “dressing orbits” play a useful role [6], [7].). In a future paper, we will give a discussion on the quantum double construction, again within the framework of C^* -algebraic, locally compact quantum groups.

3.3. The “opposite” and “co-opposite” Hopf C^* -algebras. By slightly modifying our fundamental multiplicative unitary operator U_A , we are able to construct a few different forms of the quantum Heisenberg group and the quantum Heisenberg group algebra. Borrowing terminologies from Hopf algebra theory, they will more or less correspond to “opposite” or “co-opposite” algebras, and “opposite dual” or “co-opposite dual” algebras.

Let $j \in \mathcal{B}(\mathcal{H})$ be defined by

$$j\xi(x, y, r) = (e^{\lambda r})^n \bar{e}[\eta_\lambda(r)\beta(x, y)]\xi(-e^{\lambda r}x, -e^{\lambda r}y, -r).$$

Then j is a unitary operator such that $j^2 = 1$. Note that the operator j can be written as $j = \hat{J}J = J\hat{J}$, where J and \hat{J} are the operators we saw earlier in our discussions on the antipode and the $*$ -operation. Incorporating the operator j to our fundamental multiplicative unitary operator U_A , we obtain the following:

Proposition 3.2. *The following operators are all regular multiplicative unitary operators (in the sense of Baaĵ and Skandalis) in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. Here Σ denotes the flip.*

$$\begin{aligned} U_A & & \widehat{U}_A &= \Sigma(j \otimes 1)U_A(j \otimes 1)\Sigma \\ \widetilde{U}_A &= (j \otimes 1)(\Sigma U_A \Sigma)(j \otimes 1) & \widehat{\widetilde{U}}_A &= \widetilde{\widetilde{U}}_A = (j \otimes j)U_A(j \otimes j) \end{aligned}$$

The same is true of the operators of the form $\Sigma X^ \Sigma$ for any of the multiplicative unitary operators X above.*

Remark. The verification is a straightforward calculation. What is really going on is that the triple (\mathcal{H}, U_A, j) forms a *Kac system*, in the terminology of Baaĵ and Skandalis (See section 6 of [1].).

We will obtain several C^* -bialgebras from these operators. Before we give descriptions of them, let us give the following definitions on “opposite” and “co-opposite” algebra/coalgebra structures.

Definition 3.3. (1) On \mathcal{A} , define instead the “opposite multiplication” by $(f, g) \mapsto g \times_{\mathcal{A}} f$. We keep the same involution. We will denote this opposite algebra by \mathcal{A}^{op} . Similarly, we can define $\hat{\mathcal{A}}^{\text{op}}$, whose multiplication is given by $(f, g) \mapsto g \times_{\hat{\mathcal{A}}} f$.

(2) Define $R : \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}(\mathcal{H})$ by

$$(R_f \xi)(x, y, r) := \int f(\tilde{x}, \tilde{y}, r) \xi(x - \tilde{x}, y - \tilde{y}, r) \bar{e}[\eta_\lambda(r) \beta(x - \tilde{x}, \tilde{y})] d\tilde{x} d\tilde{y}.$$

It is a $*$ -representation of \mathcal{A}^{op} . Actually, \mathcal{A}^{op} is a pre- C^* -algebra, together with the C^* -norm $\|f\| := \|R_f\|$. We will denote by A^{op} the C^* -algebra completion in $\mathcal{B}(\mathcal{H})$ of \mathcal{A}^{op} . That is, $A^{\text{op}} = \overline{R(\mathcal{A}^{\text{op}})}^{\|\cdot\|}$.

(3) Define $\lambda : \hat{\mathcal{A}}^{\text{op}} \rightarrow \mathcal{B}(\mathcal{H})$ by

$$(\lambda_f \zeta)(x, y, r) := \int f(e^{\lambda \tilde{r}} x, e^{\lambda \tilde{r}} y, r - \tilde{r}) \zeta(x, y, \tilde{r}) d\tilde{r}.$$

It is a $*$ -representation of $\hat{\mathcal{A}}^{\text{op}}$. As above, we can define the C^* -algebra \hat{A}^{op} as $\hat{A}^{\text{op}} = \overline{\lambda(\hat{\mathcal{A}}^{\text{op}})}^{\|\cdot\|} (\subseteq \mathcal{B}(\mathcal{H}))$.

Remark. The above definitions resemble the characterizations of the C^* -algebras A and \hat{A} in $\mathcal{B}(\mathcal{H})$. And the roles played by $R(\mathcal{A}^{\text{op}})$ and $\lambda(\hat{\mathcal{A}}^{\text{op}})$ are exactly the same ones played by $L(\mathcal{A})$ and $\rho(\hat{\mathcal{A}})$. Meanwhile, on a related note concerning the enveloping von Neumann algebras, we have: $M_{\mathcal{A}^{\text{op}}} = M_{A'}$, and $M_{\hat{\mathcal{A}}^{\text{op}}} = M_{\hat{A}'}$.

Definition 3.4. (1) For the function f contained in \mathcal{A} (or in \mathcal{A}^{op}), define $\Delta^{\text{cop}} f$ by

$$\begin{aligned} & \Delta^{\text{cop}} f(x, y, r; x', y', r') \\ &= \int f(x, y, r + r') \bar{e}[\tilde{p} \cdot (e^{\lambda r} x - x') + \tilde{q} \cdot (e^{\lambda r} y - y')] d\tilde{p} d\tilde{q}, \end{aligned}$$

which is a Schwartz function having compact support in the r and the r' variables.

(2) For f contained in $\hat{\mathcal{A}}$ (or in $\hat{\mathcal{A}}^{\text{op}}$), let $\hat{\Delta}^{\text{cop}} f$ be the Schwartz function having compact support in r and r' , defined by

$$\begin{aligned} & \hat{\Delta}^{\text{cop}} f(x, y, r; x', y', r') \\ &= \int f(x + x', y + y', \tilde{r}) e[\eta_\lambda(\tilde{r}) \beta(x', y)] e[\tilde{r}(z + z')] \bar{e}[zr + z'r'] d\tilde{r} dz dz'. \end{aligned}$$

Remark. As the names suggest, these are the “co-opposite comultiplications” (Compare the above definitions with our earlier definitions of Δf and $\hat{\Delta} f$ given in (1.3) and Proposition 2.3.). Indeed, for $f \in \mathcal{A}$, we would have: $(L \otimes L)_{\Delta^{\text{cop}} f} = (\chi \circ \Delta)(L_f)$, where χ is the flip. Similar comment holds for $\hat{\Delta}^{\text{cop}}$. Meanwhile, just as were the cases of Δ and $\hat{\Delta}$, the above maps Δ^{cop} and $\hat{\Delta}^{\text{cop}}$ can be also extended to the C^* -algebra level (See Proposition 3.5 below.).

Let us turn our attention back to the multiplicative unitary operators in Proposition 3.2. For each of the multiplicative unitary operators V , we can consider $\overline{\{(\omega \otimes \text{id})(V) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}$ (the “left slices”) and $\overline{\{(\text{id} \otimes \omega)(V) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}$ (the “right slices”) contained in $\mathcal{B}(\mathcal{H})$. They are described below:

Proposition 3.5. *For $f \in \mathcal{A}$ and $g \in \hat{\mathcal{A}}$, we have:*

$$\begin{aligned} U_A(L_f \otimes 1)U_A^* &= \widehat{U}_A^*(1 \otimes L_f)\widehat{U}_A = (L \otimes L)_{\Delta f}, \\ \widehat{\widehat{U}}_A(R_f \otimes 1)\widehat{\widehat{U}}_A^* &= \widetilde{U}_A^*(1 \otimes R_f)\widetilde{U}_A = (R \otimes R)_{\Delta^{\text{cop}} f}, \\ U_A^*(1 \otimes \rho_g)U_A &= \widetilde{U}_A(\rho_g \otimes 1)\widetilde{U}_A^* = (\rho \otimes \rho)_{\hat{\Delta} g}, \\ \widehat{\widehat{U}}_A^*(1 \otimes \lambda_g)\widehat{\widehat{U}}_A &= \widehat{U}_A(\lambda_g \otimes 1)\widehat{U}_A^* = (\lambda \otimes \lambda)_{\hat{\Delta}^{\text{cop}} g}. \end{aligned}$$

From this, we obtain the following results (Here, the comultiplications are understood as defined at (extended to) the C^ -algebra level.):*

- (1) U_A determines two Hopf C^* -algebras (A, Δ) and $(\hat{A}, \hat{\Delta})$. And $\Sigma U_A^* \Sigma$ determines $(\hat{A}, \hat{\Delta}^{\text{cop}})$ and (A, Δ^{cop}) .
- (2) $\widehat{\widehat{U}}_A$ determines $(\hat{A}^{\text{op}}, \hat{\Delta}^{\text{cop}})$ and (A, Δ) , while $\Sigma \widehat{\widehat{U}}_A^* \Sigma$ determines (A, Δ^{cop}) and $(\hat{A}^{\text{op}}, \hat{\Delta})$.
- (3) \widetilde{U}_A determines $(\hat{A}, \hat{\Delta})$ and $(A^{\text{op}}, \Delta^{\text{cop}})$, while $\Sigma \widetilde{U}_A^* \Sigma$ determines (A^{op}, Δ) and $(\hat{A}, \hat{\Delta}^{\text{cop}})$.
- (4) $\widehat{\widehat{U}}_A$ determines $(A^{\text{op}}, \Delta^{\text{cop}})$ and $(\hat{A}^{\text{op}}, \hat{\Delta}^{\text{cop}})$, while $\Sigma \widehat{\widehat{U}}_A^* \Sigma$ determines $(\hat{A}^{\text{op}}, \hat{\Delta})$ and (A^{op}, Δ) .

Proof. Case (1) repeats the results of the sections 1 and 2. And case (2) was considered in Appendix (section 6) of [8].

To obtain the C^* -algebras corresponding to the multiplicative unitary operators, we adopt the method we used in Proposition 2.2. Their comultiplications can be read from one of the equations in the first part, which can be proved by a straightforward calculation (See also Proposition 6.8 of [1].). In addition, these equations justify our viewing the comultiplications as defined at the C^* -algebra level. \square

Corollary. *Let the notation be as in Proposition 3.5. We have:*

- (1) $U_A \in M(\hat{A} \otimes A)$.
- (2) $\widehat{U}_A \in M(A \otimes \hat{A}^{\text{op}})$.
- (3) $\widetilde{U}_A \in M(A^{\text{op}} \otimes \hat{A})$.
- (4) $\widehat{\widetilde{U}}_A \in M(\hat{A}^{\text{op}} \otimes A^{\text{op}})$.

Proof. The results are immediate consequences of Proposition 3.5. Refer to Proposition 3.6 of [1]. \square

At this moment, what we have in Proposition 3.5 are just C^* -bialgebras. But together with the appropriate Haar weights, they become locally compact quantum groups (We may work with the same Haar weights that we have used for (A, Δ) and $(\hat{A}, \hat{\Delta})$. In some cases, the roles of left Haar weight and the right Haar weight have to be reversed.). In this way, we obtain various versions of the quantum Heisenberg group and its dual (On the other hand, note that we have $(A, \Delta) \cong (A^{\text{op}}, \Delta^{\text{cop}})$, via the antipode.).

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