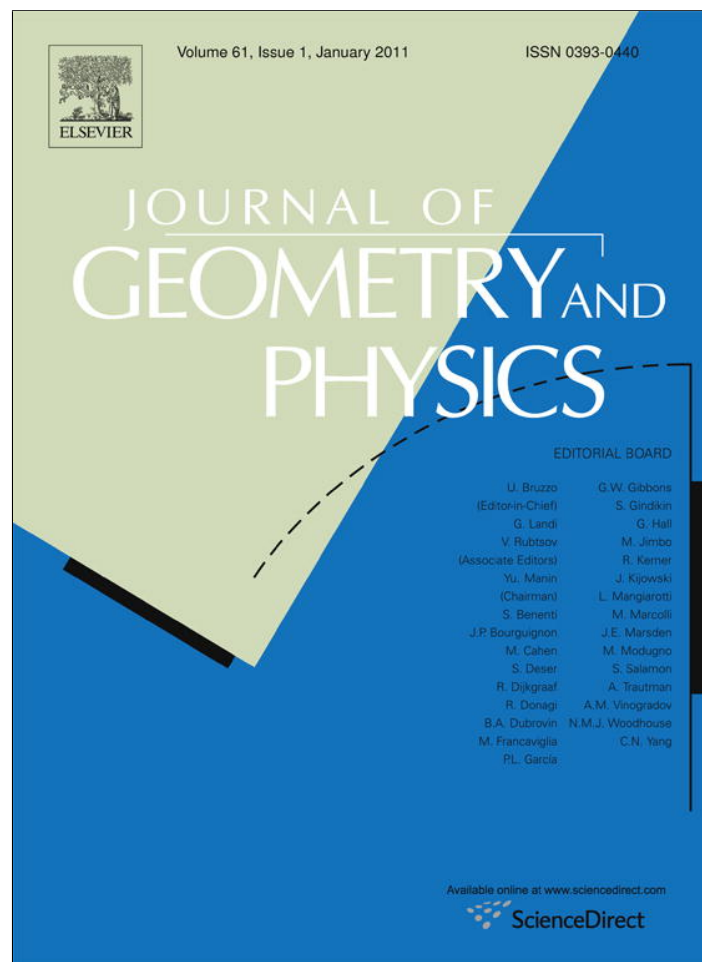


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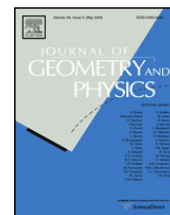
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# A $(2n + 1)$ -dimensional quantum group constructed from a skew-symmetric matrix

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## ABSTRACT

Beginning with a skew-symmetric matrix, we define a certain Poisson–Lie group. Its Poisson bracket can be viewed as a cocycle perturbation of the linear (or “Lie–Poisson”) Poisson bracket. By analyzing this Poisson structure, we gather enough information to construct a  $C^*$ -algebraic locally compact quantum group, via the “cocycle bicrossed product construction” method. The quantum group thus obtained is shown to be a deformation quantization of the Poisson–Lie group, in the sense of Rieffel.

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## 1. Introduction

It is generally understood that quantum groups are obtained by “quantizing” ordinary groups. On the other hand, it is not always clear what we mean by this statement. Typically, in the often used “generators and relations method” of constructing quantum groups, there exists a certain deformation parameter  $q$  such that, when  $q = 1$ , the quantum group degenerates to the universal enveloping algebra of an ordinary group or the function algebra of an ordinary group. See [1,2], and other examples. While this is nice, the method of generators and relations is at best an indirect method, meaning that the correspondence information about how the pointwise product on the function algebra is deformed to an operator product is usually not apparent.

There are also some technical issues when working with the  $q$ -relations among the generators. It is less of a problem in the case of a purely algebraic setting of quantized universal enveloping (QUE) algebras or that of compact quantum groups. However, when one wishes to construct a non-compact quantum group, the generators (coordinate functions of the group) tend to be unbounded, so things are more complicated. There are ways to handle the difficulties (see [3], where Woronowicz works with the notion of unbounded operators “affiliated” with  $C^*$ -algebras), but, in general, it is usually better to look for some other methods of construction.

One useful approach not relying on the generators is the method of deformation quantization. Here, the aim is to deform the (commutative) algebra of functions on a Poisson manifold, in the direction of the Poisson bracket. See [4,5]. In the  $C^*$ -algebra framework, the corresponding notion is the “strict deformation quantization” by Rieffel [6], or its more generalized versions developed later by other authors. To obtain a quantum group, one would begin with a suitable

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Poisson–Lie group  $G$  (a Lie group equipped with a compatible Poisson bracket) and perform the deformation quantization on the function space  $C_0(G)$ —for both its algebra and coalgebra structures.

Some of the non-compact quantum groups obtained by deformation quantization are [7–11]. In these examples, there exists a very close relationship between a quantum group and its Poisson–Lie group counterpart. Indeed, the information at the level of Poisson–Lie groups or Lie bialgebras plays a key role in the construction of the quantum groups.

The interplay between the Poisson data and the quantum group can go further. For instance, as for the example constructed by the author [11], the information at the classical (Poisson) level was useful not only in the construction of the quantum group but also in studying its representation theory, in relation to the dressing orbits. See [12].

Despite many advantages, however, jumping from a Poisson–Lie group to the  $C^*$ -algebraic quantum group level is not always easy. Deformation quantization only provides the “spatial” quantization. Even with the guides suggested by the Poisson data, the actual construction of the structure maps like comultiplication, antipode, or Haar weight requires various specialized techniques. Often, a method that works for some examples may not work for others.

Considering the drawbacks to the geometric approach above, we proposed in [13] to enhance the “geometric” (deformation quantization) approach by combining it with a more “algebraic” framework of cocycle bicrossed products [14,15].

The bicrossed product method is relatively simple, but sufficiently general to include many interesting examples. Historically, it goes back to the group extension problems in the Kac algebra setting. For a comprehensive treatment on constructing quantum groups using this framework, see [15]. However, as is the case for any general method, having the framework is not enough to construct actual and specific examples. For this method to work, one needs to have a specific “matched pair” of groups (or quantum groups), together with a compatible cocycle.

Our proposal, as given in [13], is to begin first with a Poisson–Lie group and analyze its Poisson structure. The Poisson data will help us obtain a suitable matched pair and a compatible cocycle. If, in particular, the Poisson bracket is of the “cocycle perturbation of the linear Poisson bracket” type, in the sense of [16], then the deformation process can be made more precise. Finally, using the matched pair and the cocycle data, we will perform the cocycle bicrossed product construction.

Quantum groups obtained in this way tend to have (twisted) crossed products as their underlying  $C^*$ -algebras. Therefore, this program is usually best for constructing solvable-type quantum groups. This is because crossed products often model quantized spaces. (For instance, the “Weyl algebra”,  $C_0(\mathbb{R}^n) \rtimes_{\tau} \mathbb{R}^n$  with  $\tau$  being the translation, is the quantized phase space [17].) But, with some adjustments, the method could be adopted to construct other types of quantum groups.

Previously, [13] gave examples of some Poisson–Lie groups and implicitly indicated how one may be able to carry out the program, but it did not contain any detailed construction. Case (1) in [13] is related with the examples from [7,8,18], while Case (2) was studied in [11]. However, these earlier papers did not exactly take the approach that we are proposing here.

The reason behind writing this paper is that, in addition to giving an example of a quantum group, we wanted to expand on our work in [13] by providing a careful description of our construction method, taking Case (3) in that paper as a model. The author was initially content with the brief description as given in that paper, but while visiting Leuven during November 2008, it was suggested by Professor Alfons Van Daele that it would be beneficial to give a fuller description of the example and the method. This is done here. We expect that our program can be used to construct other new examples in a similar way. Moreover, since we would have a close, built-in connection between the Poisson–Lie group and the quantum group, we will be able to take advantage of the geometric data in further studying the quantum group and applications (for instance, dressing orbits on Poisson–Lie groups are closely related with the quantum group representations).

The paper is organized as follows. In Section 2, using a given skew-symmetric matrix  $J$ , we define the Poisson–Lie group  $G$  that we wish to quantize. Its Poisson bracket is non-linear, but can be regarded as a “cocycle perturbation” of the linear (Lie–Poisson type) Poisson bracket. The deformation quantization of the Poisson–Lie group  $(G, \{ , \})$  is carried out in Section 3. The Poisson data helps us to define a certain multiplicative unitary operator (in the sense of Baaj and Skandalis [19]), and it enables us to define the  $C^*$ -bialgebra  $(S, \Delta)$ . It is shown here that  $(S, \Delta)$  is a strict deformation quantization (in the sense of Rieffel [6,20]) of the Poisson–Lie group  $G$ . In Section 4, we realize that the construction we carry out in Section 3 is in fact a case of the cocycle bicrossed product construction, in the sense of [15]. The result is that the  $C^*$ -bialgebra is indeed a locally compact quantum group. To tie the loose ends, brief descriptions are given on the antipode map and the Haar weight on our quantum group  $(S, \Delta)$ .

## 2. The Poisson–Lie group $G$

### 2.1. The group

Let  $n$  be an integer such that  $n \geq 2$ , and let  $J = (J_{ik})_{1 \leq i, k \leq n}$  be an  $n \times n$  skew-symmetric matrix. So  $J_{ki} = -J_{ik}$ , for  $1 \leq i, k \leq n$ . Then consider the  $(2n + 1)$ -dimensional Lie algebra  $\mathfrak{g}$ , spanned by the basis elements  $\mathbf{p}_i, \mathbf{q}_i$  ( $i = 1, \dots, n$ ),  $\mathbf{r}$ , satisfying the following relations:

$$[\mathbf{p}_i, \mathbf{p}_j] = 0, \quad [\mathbf{q}_i, \mathbf{q}_j] = 0, \quad [\mathbf{p}_i, \mathbf{q}_j] = 0, \quad [\mathbf{p}_i, \mathbf{r}] = \sum_{k=1}^n J_{ik} \mathbf{q}_k, \quad [\mathbf{q}_i, \mathbf{r}] = 0,$$

for  $i, j = 1, \dots, n$ . Since  $J = (J_{ik})$  is skew, it is clear that  $[ , ]$  is a valid Lie bracket. Observe also that the  $\mathbf{q}_j$  are central and that  $\mathfrak{g}$  is a two-step nilpotent Lie algebra.

It is not difficult to describe the corresponding Lie group. The group  $G$  has  $\mathbb{R}^{2n+1}$  as its underlying space, and the multiplication on it is defined by

$$(p, q, r)(p', q', r') = \left( p + p', q + q' + r' \sum_{i,k=1}^n J_{ik} p_i \mathbf{q}_k, r + r' \right). \tag{2.1}$$

Here,  $p, q, p', q' \in \mathbb{R}^n$  and  $r, r' \in \mathbb{R}$ . For convenience, we are regarding  $p \in \mathbb{R}^n$  as  $p = p_1 \mathbf{p}_1 + p_2 \mathbf{p}_2 + \dots + p_n \mathbf{p}_n$ , and similarly for the other variables. In other words, the multiplication law in (2.1) could be also written as

$$(p, q, r)(p', q', r') = \left( p_1 + p'_1, \dots, p_n + p'_n; q_1 + q'_1 + r' \sum_{i=1}^n J_{i1} p_i, \dots, q_n + q'_n + r' \sum_{i=1}^n J_{in} p_i; r + r' \right).$$

The identity element is  $e = (0, 0, 0)$ , while the inverse element for  $(p, q, r) \in G$  is

$$(p, q, r)^{-1} = \left( -p, -q + r \sum_{i,k=1}^n J_{ik} p_i \mathbf{q}_k, -r \right).$$

The group  $G$  is a (connected and simply connected) exponential solvable Lie group corresponding to  $\mathfrak{g}$ . We can identify  $G \cong \mathfrak{g}$  as vector spaces. Note that an ordinary Lebesgue measure becomes a Haar measure for  $G$ .

In the following section, we will further show that  $G$  is equipped with a compatible Poisson bracket, making it a Poisson–Lie group. Our aim in this paper is to construct a locally compact quantum group that can be considered as a “quantized  $C_0(G)$ ”.

### 2.2. Non-linear Poisson structure on $G$

By general theory on Poisson–Lie groups (see [21]), any compatible Poisson structure on  $G$  canonically determines a dual Poisson–Lie group, and vice versa. In fact, in our case, it is in some sense more convenient to consider first its dual counterpart  $H = G^*$ , which is shown to be a Poisson–Lie group. We can then regard  $G$  as the dual Poisson–Lie group of  $H$ . The following discussion was first reported in [13]; see Section 1, Case (3) of [13].

**Definition 2.1** (Heisenberg Lie Group). Let  $H$  be the  $(2n + 1)$ -dimensional Heisenberg Lie group. Its underlying space is  $\mathbb{R}^{2n+1}$  and the multiplication on it is given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \beta(x, y')),$$

for  $x, x', y, y' \in \mathbb{R}^n$  and  $z, z' \in \mathbb{R}$ . Here,  $\beta(\cdot, \cdot)$  denotes the ordinary inner product. So  $\beta(x, y) = x \cdot y$ , for  $x, y \in \mathbb{R}^n$ .

Its Lie algebra counterpart is the Heisenberg Lie algebra  $\mathfrak{h}$ . It is generated by the basis elements  $\mathbf{x}_i, \mathbf{y}_i$  ( $i = 1, \dots, n$ ),  $\mathbf{z}$ , with the following relations:

$$[\mathbf{x}_i, \mathbf{y}_j] = \delta_{ij} \mathbf{z}, \quad [\mathbf{x}_i, \mathbf{x}_j] = [\mathbf{y}_i, \mathbf{y}_j] = 0, \quad [\mathbf{z}, \mathbf{x}_i] = [\mathbf{z}, \mathbf{y}_i] = 0.$$

**Remark.** For convenience, we will identify  $H \cong \mathfrak{h}$  as vector spaces. This is possible since  $H$  is an exponential solvable Lie group (it is actually nilpotent). And we choose a Lebesgue measure on  $H \cong \mathfrak{h}$ , which is in fact a Haar measure for  $H$ . As in Section 2.1, we will understand that  $x = x_1 \mathbf{x}_1 + \dots + x_n \mathbf{x}_n$ , and similarly for the other variables.

To describe the Poisson structure on  $H$ , it is equivalent to specify a “Lie bialgebra” structure  $(\mathfrak{h}, \delta)$ . See [22,21] for the general theory on Poisson–Lie groups and Lie bialgebras. In our case, the cobracket  $\delta : \mathfrak{h} \rightarrow \mathfrak{h} \wedge \mathfrak{h}$  is obtained from a certain classical  $r$ -matrix. Details are given in the following proposition. See also Section 5 of [13].

**Proposition 2.2.** Let  $r \in \mathfrak{h} \otimes \mathfrak{h}$  be defined by  $r = \sum_{i,k=1}^n J_{ik} \mathbf{x}_k \otimes \mathbf{x}_i$ . It is a skew solution of the “classical Yang–Baxter equation” (CYBE):

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

Therefore, it determines a “triangular” Lie bialgebra structure,  $\delta : \mathfrak{h} \rightarrow \mathfrak{h} \wedge \mathfrak{h}$ , by  $\delta(X) = \text{ad}_X(r)$ ,  $X \in \mathfrak{h}$ . To be specific, we have

$$\delta(\mathbf{x}_k) = 0, \quad \delta(\mathbf{y}_k) = \sum_{i=1}^n J_{ik} (\mathbf{x}_i \otimes \mathbf{z} - \mathbf{z} \otimes \mathbf{x}_i) = \sum_{i=1}^n J_{ik} \mathbf{x}_i \wedge \mathbf{z}, \quad \delta(\mathbf{z}) = 0,$$

for  $k = 1, \dots, n$ .

**Proof.** Since  $\text{span}(\mathbf{x}_i : i = 1, 2, \dots, n)$  is an abelian subalgebra of  $\mathfrak{h}$ , the element  $r$  trivially satisfies the CYBE. It is also skew (i.e.,  $r^{12} + r^{21} = 0$ ), because  $J$  is a skew-symmetric matrix. This means that  $r$  is a triangular classical  $r$ -matrix.

By general theory (see, for instance, [23,21]), we thus obtain a coboundary Lie bialgebra structure, given by  $\delta(X) = \text{ad}_X(r)$ ,  $X \in \mathfrak{h}$ , where  $\text{ad}_X(a \otimes b) = [X, a] \otimes b + a \otimes [X, b]$ . We can verify the results of the proposition by straightforward computation.  $\square$

Corresponding to the cobracket  $\delta : \mathfrak{h} \rightarrow \mathfrak{h} \wedge \mathfrak{h}$  given above, we can define a Lie bracket on the dual space  $\mathfrak{h}^*$  of  $\mathfrak{h}$  by  $[\cdot, \cdot] = \delta^* : \mathfrak{h}^* \wedge \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ . That is,  $[\mu, \nu]$  is defined by

$$[\mu, \nu], X = \langle \delta^*(\mu \otimes \nu), X \rangle = \langle \mu \otimes \nu, \delta(X) \rangle, \tag{2.2}$$

where  $X \in \mathfrak{h}$ ,  $\mu, \nu \in \mathfrak{h}^*$ , and  $\langle \cdot, \cdot \rangle$  is the dual pairing between  $\mathfrak{h}^*$  and  $\mathfrak{h}$ . It turns out that the “dual” Lie algebra structure on  $\mathfrak{h}^*$  coincides with the Lie algebra  $\mathfrak{g}$  described in the previous section. See the proposition below (the proof is straightforward).

**Proposition 2.3.** *Let  $\mathfrak{g} = \mathfrak{h}^*$  be spanned by  $\mathbf{p}_i, \mathbf{q}_i$  ( $i = 1, \dots, n$ ),  $\mathbf{r}$ , which form the dual basis of  $\mathbf{x}_i, \mathbf{y}_i$  ( $i = 1, \dots, n$ ),  $\mathbf{z}$ . On  $\mathfrak{g}$ , the Lie algebra relations can be defined by Eq. (2.2). Then we have*

$$[\mathbf{p}_i, \mathbf{p}_j] = 0, \quad [\mathbf{q}_i, \mathbf{q}_j] = 0, \quad [\mathbf{p}_i, \mathbf{q}_j] = 0, \quad [\mathbf{p}_i, \mathbf{r}] = \sum_{k=1}^n J_{ik} \mathbf{q}_k, \quad [\mathbf{q}_i, \mathbf{r}] = 0,$$

for  $i, j = 1, \dots, n$ . This is the Poisson dual of the Lie bialgebra  $(\mathfrak{h}, \delta)$ .

Comparing the result of Proposition 2.3 with the definition of the Lie algebra structure on  $\mathfrak{g}$  given in Section 2.1, we can see clearly that they are indeed the same. This reinterpretation of our Lie algebra  $\mathfrak{g}$  means that  $\mathfrak{g}$  is actually a Lie bialgebra, being a dual Lie bialgebra of  $(\mathfrak{h}, \delta)$ . The cobracket on  $\mathfrak{g}$  is the dual map of the Lie bracket on  $\mathfrak{h}$ . A short calculation shows that the cobracket  $\theta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$  takes its values on the basis vectors of  $\mathfrak{g}$  as follows:

$$\theta(\mathbf{p}_i) = 0, \quad \theta(\mathbf{q}_i) = 0, \quad \theta(\mathbf{r}) = \sum_{i=1}^n (\mathbf{p}_i \otimes \mathbf{q}_i - \mathbf{q}_i \otimes \mathbf{p}_i) = \sum_{i=1}^n (\mathbf{p}_i \wedge \mathbf{q}_i).$$

We thus have the (Poisson dual) Lie bialgebra  $(\mathfrak{g}, \theta)$ . Let us now consider the corresponding Poisson–Lie group  $G$  and its Poisson bracket. See Proposition 2.4 below. As before, we are regarding  $p = p_1 \mathbf{p}_1 + p_2 \mathbf{p}_2 + \dots + p_n \mathbf{p}_n$ , and similarly for the other variables.

**Proposition 2.4.** *Let  $G$  be the  $(2n + 1)$ -dimensional Lie group, together with the multiplication law*

$$(p, q, r)(p', q', r') = \left( p + p', q + q' + r' \sum_{i,k=1}^n J_{ik} p_i \mathbf{q}_k, r + r' \right).$$

This gives us the Lie group corresponding to  $\mathfrak{g}$  from Proposition 2.3. The Poisson bracket on  $G$  is given by

$$\{f, g\}(p, q, r) = r(\beta(x, y') - \beta(x', y)) + \frac{r^2}{2} \sum_{i,k=1}^n J_{ik} (y_k y'_i - y_i y'_k),$$

for  $f, g \in C^\infty(G)$ . Here,  $df(p, q, r) = (x, y, z)$  and  $dg(p, q, r) = (x', y', z')$ , which are naturally viewed as elements of  $\mathfrak{h}$ .

**Proof.** Construction of  $G$  from  $\mathfrak{g}$  was already described in Eq. (2.1). To find the expression for the Poisson bracket, we follow the standard procedure [21]. See also Proposition 2.3 of [13]. A similar computation (for a different Poisson structure) can be found in the proof of Theorem 2.2 of [11].

First, consider  $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ , the adjoint representation of  $G$  on  $\mathfrak{g}$ . Then we look for a map  $F : G \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ , that is, a group 1-cocycle on  $G$  for the  $\text{Ad}$ -representation, whose derivative at the identity element,  $df_e$ , coincides with  $\theta$  above. Or,  $dF_{(0,0,0)} = \theta$ . In general, integrating  $\theta$  to  $F$  is not always easy. However, in our case, it is not difficult to check that the following map  $F$  indeed satisfies the requirements above:

$$F(p, q, r) = r \sum_{i=1}^n (\mathbf{p}_i \wedge \mathbf{q}_i) - \frac{r^2}{2} \sum_{i,k=1}^n (J_{ik} \mathbf{q}_k \wedge \mathbf{q}_i).$$

Since we have the 1-cocycle  $F$ , the Poisson bivector field is then obtained by the right translation of  $F$ . To compute it, suppose that  $f, g \in C^\infty(G)$ . For  $(p, q, r) \in G$ , since  $df = df_e$  is the (linear) differential of the scalar-valued map  $f$  on  $G$ , and since  $\mathfrak{g}$  is the tangent space of  $G$  at its identity  $e = (0, 0, 0)$ , we can naturally identify  $df(p, q, r)$  as an element in  $\mathfrak{g}^* = \mathfrak{h}$ . Similarly for  $dg(p, q, r)$ . So, write  $df(p, q, r) = (x, y, z)$  and  $dg(p, q, r) = (x', y', z')$ . Noting that  $R_{(p,q,r)*}(\mathbf{p}_i) = \mathbf{p}_i + r \sum_{k=1}^n J_{ik} \mathbf{q}_k$  and  $R_{(p,q,r)*}(\mathbf{q}_i) = \mathbf{q}_i$  under the right translation, we have

$$\begin{aligned} \{f, g\}(p, q, r) &= \langle R_{(p,q,r)*} F(p, q, r), df(p, q, r) \wedge dg(p, q, r) \rangle \\ &= r(\beta(x, y') - \beta(x', y)) + \frac{r^2}{2} \sum_{i,k=1}^n J_{ik} (y_k y'_i - y_i y'_k). \quad \square \end{aligned}$$

We can see from Proposition 2.4 that we thus have a non-linear Poisson bracket on our group  $G$ . When  $J = O$  (zero matrix), it becomes linear, carrying only the part that comes from the Lie algebra structure on  $\mathfrak{h} = \mathfrak{g}^*$ . Ours is actually a “cocycle perturbation” of the linear Poisson bracket, as introduced in [16]. See Section 3.1 below for further discussion.

### 3. Deformation quantization of $G$

Now that we have described our Poisson–Lie group  $G$ , we wish to construct its quantum group counterpart. The Poisson data should guide our direction of quantization.

In Section 3.2 of [13], we obtained a  $C^*$ -bialgebra that can be reasonably considered as a quantum semigroup corresponding to  $G$ . The method was via a “cocycle bicrossed product” construction, as in [15] (see also Section 8 of [19]). However, the full construction of the quantum group was not carried out, for instance the existence proof of an appropriate Haar weight. In addition, it will be desirable to show a more comprehensive relationship between the Poisson–Lie group and the quantum group, including the deformation picture. We will fill in these gaps as we review and improve our quantum group construction.

#### 3.1. Poisson bracket of the cocycle perturbation type

The Poisson bracket on  $G$ , as obtained in Proposition 2.4 above, is of the “cocycle perturbation” type studied in Theorems 2.2 and 2.3 of [16]. Let us be more specific.

Since we are identifying  $G \cong \mathfrak{g}$ , our Poisson bracket on  $G$  may be also regarded as a Poisson bracket on  $\mathfrak{g} = \mathfrak{h}^*$ , where  $\mathfrak{h}$  is the Heisenberg Lie algebra noted earlier. Let  $\mathfrak{z}$  denote the center of  $\mathfrak{h}$ , spanned by the basis element  $\mathbf{z} \in \mathfrak{h}$ , and let us write  $\mathfrak{q} = \mathfrak{z}^\perp \subseteq \mathfrak{g}$ . Then we may regard the  $x, y, x', y' \in \mathbb{R}^n$  as elements of  $\mathfrak{h}/\mathfrak{z} = \text{span}(\mathbf{x}_i, \mathbf{y}_i : i = 1, \dots, n)$  and the  $r \in \mathbb{R}$  as elements of  $\mathfrak{g}/\mathfrak{q}$ .

Consider the vector space  $V = C^\infty(\mathfrak{g}/\mathfrak{q})$ , and give it the trivial  $U(\mathfrak{h}/\mathfrak{z})$ -module structure. Suggested by the Poisson bracket expression given in Proposition 2.4, let  $\omega : \mathfrak{h}/\mathfrak{z} \times \mathfrak{h}/\mathfrak{z} \rightarrow V$  be defined by

$$\omega((x, y), (x', y'); r) = r(\beta(x, y') - \beta(x', y)) + \frac{r^2}{2} \sum_{i,k=1}^n J_{ik}(y_k y'_i - y_i y'_k). \tag{3.1}$$

Then  $\omega$  is clearly a skew-symmetric bilinear map, and is a Lie algebra cocycle for  $\mathfrak{h}/\mathfrak{z}$ , trivially, since  $\mathfrak{h}/\mathfrak{z}$  is an abelian Lie algebra.

Meanwhile, with  $\mathfrak{h}/\mathfrak{z}$  being abelian, the linear (or “Lie–Poisson”) Poisson bracket on  $(\mathfrak{h}/\mathfrak{z})^*$  is the trivial one. Therefore, our Poisson bracket on  $\mathfrak{h}^*$  is essentially the sum of the (trivial) linear Poisson bracket on  $(\mathfrak{h}/\mathfrak{z})^*$  and the cocycle  $\omega$ . We thus have the following conclusion.

**Proposition 3.1.** *Consider the Poisson bracket on  $G$ , obtained in Proposition 2.4, which is also regarded as defined on  $\mathfrak{g} = \mathfrak{h}^*$ . It is a “cocycle perturbation” of the linear Poisson bracket on  $\mathfrak{h}^*$ , in the sense of [16].*

**Proof.** The functions in  $V = C^\infty(\mathfrak{g}/\mathfrak{q})$  can be canonically realized as functions in  $C^\infty(\mathfrak{g})$ , by the “pull-back” using the natural projection of  $\mathfrak{g}$  onto  $\mathfrak{g}/\mathfrak{q}$ . In addition, the elements in  $\mathfrak{h}$  are linear functions on  $\mathfrak{g}$ . We thus have  $\mathfrak{h} + V \subseteq C^\infty(\mathfrak{g})$ , whereas  $\mathfrak{h} \cap V = \mathfrak{z}$ .

Meanwhile, the cocycle  $\omega$  on  $\mathfrak{h}/\mathfrak{z}$  (which takes values in  $V$ ) naturally determines a Lie bracket on  $\mathfrak{h}/\mathfrak{z} \oplus V$ , by central extension. Since  $\mathfrak{h} \cap V = \mathfrak{z}$ , we see that  $\mathfrak{h}/\mathfrak{z} \oplus V \cong \mathfrak{h} + V$ , as vector spaces. Under this spatial isomorphism, we can thus transfer the Lie bracket on  $\mathfrak{h}/\mathfrak{z} \oplus V$  to a Lie bracket on  $\mathfrak{h} + V$ , denoted by  $[\cdot, \cdot]_{\mathfrak{h}+V}$ . This Lie bracket is essentially a “perturbed Lie bracket” of the Lie bracket on  $\mathfrak{h}$ .

With  $\mathfrak{h} + V \subseteq C^\infty(\mathfrak{g})$ , we can give an alternative interpretation of our Poisson bracket in Proposition 2.4, as follows:

$$\{f, g\}(\mu) = [df(\mu), dg(\mu)]_{\mathfrak{h}+V}(\mu),$$

where  $\mu \in \mathfrak{g}$ . Here,  $df(\mu), dg(\mu) \in \mathfrak{h}(\subseteq \mathfrak{h} + V)$ , as shown in the proof of Proposition 2.4; the bracket operation in  $\mathfrak{h} + V$  is as described in the previous paragraph; and we are regarding an element in  $\mathfrak{h} + V$  as a function contained in  $C^\infty(\mathfrak{g})$ . Having come from the “perturbed Lie bracket” of the Lie bracket on  $\mathfrak{h}$ , our (non-linear) Poisson bracket is a “cocycle perturbation” of the linear Poisson bracket.

For a more detailed discussion, including some technicalities involving the cocycles, refer to Theorems 2.2 and 2.3 of [16], and the paragraphs about the theorems.  $\square$

**Remark.** When  $J = O$  (zero matrix), the cocycle  $\omega$  given in Eq. (3.1) becomes

$$\omega_{J=O}((x, y), (x', y'); r) = r(\beta(x, y') - \beta(x', y)).$$

It is a linear function on  $\mathfrak{g}/\mathfrak{q}$ , so we may write it as

$$\omega_{J=O}((x, y), (x', y')) = (\beta(x, y') - \beta(x', y))\mathbf{z},$$

where  $\mathbf{z}$  is the basis vector spanning  $\mathfrak{z}$ . In other words,  $\omega_{J=O}$  is a cocycle for  $\mathfrak{h}/\mathfrak{z}$  having values in  $\mathfrak{z}$ . It determines the Lie bracket on  $\mathfrak{h}$ , and, therefore, it corresponds to the linear (Lie–Poisson) Poisson bracket on  $\mathfrak{h}^*$ . What all this means is that the “perturbation” in our case is encoded by the matrix  $J$  and the associated cocycle  $\omega$ .



### 3.2. The bicrossed product construction

Since we realized our Poisson bracket as a cocycle perturbation of the linear Poisson bracket, we may follow the steps given in Section 3 of [16] to construct a deformation quantization of  $(C_0(G), \{, \})$ . The method would use the framework of twisted crossed product  $C^*$ -algebras, in the sense of Packer and Raeburn [24].

However, this method, while valid, gives only the deformation at the  $C^*$ -algebra level. Since we are interested in the construction of a quantum group, let us employ a different approach, following instead the one given in [13]. This approach is based on the “bicrossed product construction” of Vaes and Vainerman [15], as well as the earlier work by Baaj and Skandalis (Section 8 of [19]). Clarification of the deformation picture will be postponed to Section 3.3 below.

First, from our Lie algebra cocycle  $\omega$  given in Eq. (3.1), we obtain a continuous family of  $\mathbb{T}$ -valued group cocycles for the Lie group  $H/Z$  of  $\mathfrak{h}/\mathfrak{z}$ .

**Proposition 3.2.** Fix an element  $r \in \mathfrak{g}/\mathfrak{q}$ , and define the map  $\sigma^r : H/Z \times H/Z \rightarrow \mathbb{T}$  by

$$\sigma^r((x, y), (x', y')) = \bar{e}[r\beta(x, y)]\bar{e} \left[ \frac{r^2}{2} \sum_{i,k=1}^n J_{ik} y_k y'_i \right],$$

where  $e[t] = e^{2\pi it}$ , and so  $\bar{e}[t] = e^{-2\pi it}$ . Then each  $\sigma^r$  is a  $\mathbb{T}$ -valued, normalized group cocycle for  $H/Z$ . In addition,  $r \mapsto \sigma^r$  forms a continuous field of cocycles.

**Proof.** Let  $h = (x, y), h' = (x', y'), h'' = (x'', y'')$  be elements of  $H/Z$ , which is just an abelian group under addition. We can easily verify the cocycle identity, as follows:

$$\begin{aligned} \sigma^r(hh', h'')\sigma^r(h, h') &= \bar{e}[r(\beta(x, y'') + \beta(x', y'') + \beta(x, y'))]\bar{e} \left[ \frac{r^2}{2} \sum_{i,k=1}^n J_{ik}(y_k y''_i + y'_k y''_i + y_k y'_i) \right] \\ &= \sigma^r(h, h'h'')\sigma^r(h', h''). \end{aligned}$$

We also have  $\sigma^r(h, 0) = 1 = \sigma^r(0, h)$ , where  $0 = (0, 0)$  is the identity element of  $H/Z$ . From the definition, the continuity is quite clear.  $\square$

**Remark.** In general, constructing the group cocycle by “integrating” the Lie algebra cocycle is not necessarily easy. For a little more discussion on this matter, see Section 3 of [16].

Our Poisson bracket from Proposition 2.4 and the group cocycle arising from it, as obtained in Proposition 3.2 above, strongly suggest that it will be most convenient for us to work with the  $(x, y; r)$  variables, where  $(x, y) \in H/Z$  and  $r \in \mathfrak{g}/\mathfrak{q} = \mathfrak{h}^*/\mathfrak{z}^\perp$ . The dual space to  $H/Z$  is  $(\mathfrak{h}/\mathfrak{z})^* = \mathfrak{z}^\perp$ , whose elements are the  $(p, q)$ . Following this observation, we will break our group  $G$  into two parts, obtaining the following matched pair of groups.

**Definition 3.3.** Let  $G_1$  and  $G_2$  be subgroups of  $G$ , defined by

$$G_1 = \{(0, 0, r) : r \in \mathbb{R}\}, \quad G_2 = \{(p, q, 0) : p, q \in \mathbb{R}^n\}.$$

Clearly, as a space  $G \cong G_2 \times G_1$ , while  $G_1$  and  $G_2$  are closed subgroups of  $G$ , such that  $G_1 \cap G_2 = \{(0, 0, 0)\}$ . Moreover, any element  $(p, q, r) \in G$  can be (uniquely) expressed as a product,  $(p, q, r) = (0, 0, r)(p, q, 0)$ , with  $(0, 0, r) \in G_1$  and  $(p, q, 0) \in G_2$ . In other words, the groups  $G_1$  and  $G_2$  form a matched pair (or, couple assorti as in Section 8 of [19]).

From the matched pair  $(G_1, G_2)$ , we naturally obtain the group actions  $\alpha : G_1 \times G_2 \rightarrow G_2$  and  $\gamma : G_2 \times G_1 \rightarrow G_1$ , defined by

$$\alpha_r(p, q) := \left( p, q - r \sum_{i,k=1}^n J_{ik} p_i q_k \right), \quad \gamma_{(p,q)}(r) := r.$$

Here we are using the obvious identification of  $(p, q)$  with  $(p, q, 0)$ , and similarly for  $r$  and  $(0, 0, r)$ . Note that these actions are defined so that we have  $(\alpha_r(p, q))(\gamma_{(p,q)}(r)) = (p, q - r \sum_{i,k=1}^n J_{ik} p_i q_k, 0)(0, 0, r) = (p, q, r)$ .

Let us now convert the information we obtained so far into the language of Hilbert space operators and operator algebras. Recall that we chose a Lebesgue measure on  $H(= \mathfrak{h})$ , which is the Haar measure for  $H$ . On  $G(= \mathfrak{g} = \mathfrak{h}^*)$ , which is considered as the dual vector space of  $H$ , we give the dual Lebesgue measure. This will also be the Haar measure for  $G$ . These measures are chosen so that the Fourier transform becomes the unitary operator (from  $L^2(H)$  to  $L^2(G)$ ), and the Fourier inversion theorem holds. Similarly, a “partial” Fourier transform can be considered, for instance, between functions in the  $(p, q; r)$  variables and those in the  $(x, y; r)$  variables. See Remark 1.7 of [11].

First, we define the multiplicative unitary operators  $X \in \mathcal{B}(L^2(G_1 \times G_1))$  and  $Y \in \mathcal{B}(L^2(G_2 \times G_2))$ , associated with the groups  $G_1$  and  $G_2$ . See [19]. Namely, define

$$X\xi(r; r') = \xi(r + r'; r'), \quad Y\zeta(p, q; p', q') = \zeta(p + p', q + q'; p', q'),$$

for  $\xi \in L^2(G_1 \times G_1)$  and  $\zeta \in L^2(G_2 \times G_2)$ . By Fourier transform,  $\mathcal{F} : L^2(G_2) \cong L^2(H/Z)$ , the operator  $Y$  can be also expressed as an operator in  $\mathcal{B}(L^2(H/Z \times H/Z))$ , in the  $(x, y)$  variables. In other words, for convenience, we will regard  $\mathcal{F}^{-1}Y\mathcal{F}$  to be the same as  $Y$ . We then have

$$Y\zeta(x, y; x', y') = \zeta(x, y; x' - x, y' - y), \quad \zeta \in L^2(H/Z).$$

**Remark.** By the theory of multiplicative unitary operators (see [19]), the operator  $X$  determines the (mutually dual)  $C^*$ -bialgebras  $C_0(G_1)$  and  $C^*(G_1)$ , and, similarly, the operator  $Y$  determines the  $C^*$ -bialgebras  $C_0(G_2)$  and  $C^*(G_2)$ . Working with the  $(x, y)$  variables, by the Fourier transform, we have  $C_0(G_2) \cong C^*(H/Z)$  and  $C^*(G_2) \cong C_0(H/Z)$ . Since the groups are abelian, all the computations are quite simple.

For convenience, a function  $f \in C_0(G_1)$  will be considered to be the same as the multiplication operator  $L_f \in \mathcal{B}(L^2(G_1))$ , defined by  $L_f\xi(r) = f(r)\xi(r)$ . Similarly for  $g \in C_0(G_2)$ , which will be also considered as the multiplication operator  $\lambda_g \in \mathcal{B}(L^2(G_2))$ . In the  $(x, y)$  variables, this is equivalent to saying that, for  $g \in C_c(H/Z) \subseteq C^*(H/Z)$ , the operator  $L_g \in \mathcal{B}(L^2(H/Z))$  is such that, for  $\zeta \in L^2(H/Z)$ , we have  $L_g\zeta(x, y) = \int g(\tilde{x}, \tilde{y})\zeta(x - \tilde{x}, y - \tilde{y})d\tilde{x}d\tilde{y}$ .

Next, we try to encode the actions  $\alpha$  and  $\gamma$  into an operator. Note that, at the level of the  $C^*$ -algebras  $C_0(G_1)$  and  $C_0(G_2)$ , the group actions  $\alpha$  and  $\gamma$  we defined above (though  $\gamma$  is trivial) are expressed as coactions  $\alpha : C_0(G_2) \rightarrow M(C_0(G_2) \otimes C_0(G_1))$  and  $\gamma : C_0(G_1) \rightarrow M(C_0(G_2) \otimes C_0(G_1))$ , given by

$$\alpha(g)(p, q; r) = g\left(p, q - r \sum_{i,k=1}^n J_{ik}p_i\mathbf{q}_k\right) = g(\alpha_r(p, q)), \quad \gamma(f)(p, q; r) = f(r) = f(\gamma_{(p,q)}(r)).$$

The coactions  $\alpha$  and  $\gamma$  can be realized using a certain unitary operator  $Z$ , as follows.

**Proposition 3.4.** Let  $Z \in \mathcal{B}(L^2(G)) = \mathcal{B}(L^2(G_2 \times G_1))$  be defined by

$$Z\xi(p, q; r) = \xi\left(p, q - r \sum_{i,k=1}^n J_{ik}p_i\mathbf{q}_k; r\right).$$

Then we have, for  $g \in C_0(G_2)$  and  $f \in C_0(G_1)$ ,

$$Z(\lambda_g \otimes 1)Z^* = (\lambda \otimes L)(\alpha(g)), \quad Z(1 \otimes L_f)Z^* = (\lambda \otimes L)(\gamma(f)).$$

**Proof.** The computations are straightforward.  $\square$

**Remark.** By using the operator realizations  $g = \lambda_g$  and  $f = L_f$ , as well as  $\alpha(g) = (\lambda \otimes L)(\alpha(g))$  and  $\gamma(f) = (\lambda \otimes L)(\gamma(f))$ , we may simply write the above result as  $\alpha(g) = Z(g \otimes 1)Z^*$  and  $\gamma(f) = Z(1 \otimes f)Z^*$ .

As indicated above, it is more convenient to work with the  $(x, y; r)$  variables. So from now on, consider the Hilbert space  $\mathcal{H} := L^2(H/Z \times G_1)$ , consisting of the  $L^2$ -functions in the  $(x, y; r)$  variables. Since we know, by the Fourier transform  $\mathcal{F} : L^2(G_2) \cong L^2(H/Z)$ , that  $C_0(G_2) \cong C^*(H/Z)$ , we may as well regard the coactions  $\alpha$  and  $\gamma$  to be on  $C^*(H/Z)$  and  $C_0(G_1)$ . (In that case, the definitions of  $\alpha$  and  $\gamma$  should be modified accordingly.) The operator  $Z \in \mathcal{B}(L^2(G_2 \times G_1))$  of Proposition 3.4 then becomes

$$\begin{aligned} (\mathcal{F}^{-1} \otimes 1)Z(\mathcal{F} \otimes 1)\xi(x, y; r) &= \int \xi(\tilde{x}, \tilde{y}; r)\bar{e}\left[p \cdot \tilde{x} + \left(q - r \sum_{i,k} J_{ik}p_i\mathbf{q}_k\right) \cdot \tilde{y}\right] e[p \cdot x + q \cdot y] d\tilde{x}d\tilde{y}dpdq \\ &= \int \xi(\tilde{x}, \tilde{y}; r)\bar{e}\left[p \cdot \left(\tilde{x} - x - r \sum_{i,k} J_{ik}\tilde{y}_k\mathbf{x}_i\right)\right] \bar{e}[q \cdot (\tilde{y} - y)] d\tilde{x}d\tilde{y}dpdq \\ &= \xi\left(x + r \sum_{i,k=1}^n J_{ik}\mathbf{y}_k\mathbf{x}_i, y; r\right). \end{aligned}$$

Here, in the second equality, we used the fact that  $(r \sum_{i,k} J_{ik}p_i\mathbf{q}_k) \cdot \tilde{y} = r \sum_{i,k} J_{ik}p_i\tilde{y}_k = p \cdot (r \sum_{i,k} J_{ik}\tilde{y}_k\mathbf{x}_i)$ . And, in the last equality, the Fourier inversion theorem was used. From now on, for convenience, we will regard the operator  $Z \in \mathcal{B}(\mathcal{H})$  to mean the operator  $(\mathcal{F}^{-1} \otimes 1)Z(\mathcal{F} \otimes 1)$  above.

As indicated in Section 8 of [19], the matched pair  $(G_1, G_2)$ , together with the actions  $\alpha$  and  $\gamma$ , determines a multiplicative unitary operator. This is shown in part (1) of the following proposition. However, this only comes from the group structure on  $G$ , and not its Poisson structure. So it will not suffice for our purposes. In our case, we actually need to go a little further, and introduce a certain cocycle term  $\Theta$ . The definition of  $\Theta$  comes directly from the Poisson bracket, given in Proposition 2.4 (see also Proposition 3.2). Our multiplicative unitary operator, incorporating both the matched pair and the cocycle, is obtained in part (2) of the following proposition. Proposition 3.5 below is none other than Propostion 3.12 in [13].



**Proposition 3.5.** (1) Define the unitary operator  $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) = \mathcal{B}(L^2(H/Z \times G_1 \times H/Z \times G_1))$ , by  $V = (Z_{12}X_{24}Z_{12}^*)Y_{13}$ , using the standard leg notation. It is multiplicative, and it determines the two  $C^*$ -algebras:

$$A_V \cong C_0(G_1) \rtimes_{\gamma} (H/Z) \quad \text{and} \quad \hat{A}_V \cong C_0(H/Z) \rtimes_{\alpha} G_1.$$

They are actually (mutually dual)  $C^*$ -bialgebras, whose comultiplications are given by  $\Delta_V(a) = V(a \otimes 1)V^*$  for  $a \in A_V$ , and  $\hat{\Delta}_V(b) = V^*(1 \otimes b)V$  for  $b \in \hat{A}_V$ .

(2) Let  $\Theta(x, y, r; x', y', r') := \bar{e}\left[r'\beta(x, y')\right]\bar{e}\left[\frac{r'^2}{2} \sum_{i,k} J_{ik}y_k y'_i\right]$ , considered as a unitary operator contained in  $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ . Then the function  $\Theta$  is a cocycle for  $V$ . In this way, we obtain a multiplicative unitary operator  $V_{\Theta} := V\Theta \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ . Specifically,

$$V_{\Theta}\xi(x, y, r; x', y', r') = e\left[\frac{r'^2}{2} \sum_{i,k} J_{ik}y_k (y'_i - y_i)\right] \bar{e}\left[r'\beta(x, y' - y)\right] \\ \times \xi\left(x - r' \sum_{i,k} J_{ik}y_k x_i, y, r + r'; x' - x + r' \sum_{i,k} J_{ik}y_k x_i, y' - y, r'\right).$$

The  $C^*$ -bialgebras associated with  $V_{\Theta}$  are

$$S \cong C_0(G_1) \rtimes_{\gamma}^{\sigma} (H/Z), \quad \text{and} \quad \hat{S} \cong C_0(H/Z) \rtimes_{\alpha} G_1,$$

together with the comultiplications  $\Delta(a) := V_{\Theta}(a \otimes 1)V_{\Theta}^*$  for  $a \in S$ , and  $\hat{\Delta}(b) := V_{\Theta}^*(1 \otimes b)V_{\Theta}$  for  $b \in \hat{S}$ . Here,  $\sigma : r \mapsto \sigma^r$  is a continuous field of cocycles such that  $\sigma^r((x, y), (x', y')) = \bar{e}\left[\frac{r^2}{2} \sum_{i,k} J_{ik}y_k y'_i\right] \bar{e}\left[r\beta(x, y')\right]$ .

**Proof.** (1) The choice of the operator  $V$ , arising from the matched pair  $(G_1, G_2)$ , is suggested by Section 8 of [19]. As noted above, the operators  $X$  and  $Y$  encode the groups  $G_1$  and  $G_2$ , while the actions  $\alpha$  and  $\gamma$  are encoded by the operator  $Z$ . The multiplicativity of  $V$  is just a simple consequence of the fact that  $G$  is a group. From the general theory of multiplicative unitary operators [19], we thus obtain the (mutually dual)  $C^*$ -bialgebras  $A_V$  and  $\hat{A}_V$  by considering the “left [and right] slices” of  $V$ . The proof for the characterizations of the two  $C^*$ -algebras is also straightforward, and will be skipped.

(2) The function  $\Theta$  is a cocycle for  $V$ , since  $V_{\Theta}$  is also multiplicative. The verification of the pentagon equation,  $W_{12}W_{13}W_{23} = W_{23}W_{12}$  for  $W = V_{\Theta}$ , is straightforward.

As usual, the  $C^*$ -bialgebras associated with  $V_{\Theta}$  are obtained by

$$S = \{(\omega \otimes \text{id}_{\mathcal{H}})(V_{\Theta}) : \omega \in \mathcal{B}(\mathcal{H})_*\} (\subseteq \mathcal{B}(\mathcal{H})), \\ \hat{S} = \{(\text{id}_{\mathcal{H}} \otimes \omega)(V_{\Theta}) : \omega \in \mathcal{B}(\mathcal{H})_*\} (\subseteq \mathcal{B}(\mathcal{H})).$$

Their comultiplications are defined in the standard way, via the multiplicative unitary operator. For the verification of the  $C^*$ -algebra realizations of  $S$  and  $\hat{S}$  as twisted crossed product  $C^*$ -algebras above, refer to the proof of Proposition 3.12 of [13]. Since the groups  $G_1$  and  $H/Z$  are amenable (being abelian), the notions of the reduced and full (twisted) crossed products coincide.  $\square$

Observe that the cocycle term for the twisted crossed product  $C^*$ -algebra follows directly from the underlying Poisson structure. In fact, the  $C^*$ -bialgebra  $(S, \Delta)$  is essentially a “quantized  $C^*(H)$ ” or a “quantized  $C_0(G)$ ”. For instance, if  $J \equiv 0$ , then it is not difficult to show that  $S \cong C^*(H)$  as an algebra. In addition, see Section 3.3 below for the clarification that  $(S, \Delta)$  is indeed a deformation quantization of  $C_0(G)$ , in the direction of its Poisson bracket.

### 3.3. $(S, \Delta)$ as a deformation quantization of $(G, \{, \})$

We constructed above a  $C^*$ -bialgebra  $(S, \Delta)$ , by means of the multiplicative unitary operator  $V_{\Theta}$ . There are strong indications that  $(S, \Delta)$  should be an appropriate quantum counterpart to the Poisson–Lie group  $(G, \{, \})$ . In this subsection, we make this picture clearer, by showing that the  $C^*$ -algebra  $S$  is a (strict) deformation quantization of  $C_0(G)$ , in the sense of Rieffel [6,20].

Let us analyze the  $C^*$ -algebra  $S$  a bit. For  $f \in C_c(G)$ , we can carry it into a function of the  $(x, y, r)$  variables by the (partial) Fourier transform  $f \mapsto f^{\vee} \in C_0(H/Z \times G_1)$ , where  $f^{\vee}(x, y, r) = \int f(p, q, r)e[p \cdot x + q \cdot y] dpdq$ . Considering this, let us define the operator  $L_f \in \mathcal{B}(\mathcal{H})$  by

$$L_f \xi(x, y, r) := \int f^{\vee}(\tilde{x}, \tilde{y}, r) \sigma^r((\tilde{x}, \tilde{y}), (x - \tilde{x}, y - \tilde{y})) \xi(x - \tilde{x}, y - \tilde{y}, r) d\tilde{x}d\tilde{y}, \tag{3.2}$$

where  $\sigma$  is the cocycle given in Proposition 3.5(2).

**Remark.** If  $\sigma \equiv 1$ , the above representation  $L : C_c(G) \ni f \mapsto L_f \in \mathcal{B}(\mathcal{H})$  is equivalent (by the partial Fourier transform) to  $\lambda \otimes L : C_c(G_2 \times G_1) \mapsto \mathcal{B}(L^2(G_2 \times G_1)) = \mathcal{B}(L^2(G))$ , with the representations  $L$  and  $\lambda$  on  $C_0(G_1)$  and  $C_0(G_2)$  defined earlier. See also Theorem 3.6(1). Since there is no worry about confusion, we chose to use the same name  $L$  for our (extended) representation.

By the result of Proposition 3.5(2), it is clear that  $S \cong \overline{L(C_c(G))}^{\|\cdot\|}$ , as a  $C^*$ -algebra. What all this means is that we do have a (deformed)  $*$ -algebra structure at the level of the functions on  $G$ , inherited from the  $*$ -algebra structure on  $S$ . To be more precise, let  $\mathcal{A} = \mathcal{S}_{3c}(G) (\subseteq C_0(G))$ , the space of Schwartz functions having compact support in the  $r$ -variable. It is slightly larger than  $C_c^\infty(G)$ , and is the image under the partial Fourier transform,  $\wedge$ , of the space  $\mathcal{S}_{3c}(H/Z \times G_1) (\subseteq C_0(H/Z \times G_1))$ . On  $\mathcal{A}$ , we can define the deformed product,  $\times$ , by

$$\begin{aligned} (f \times g)(p, q, r) &= (f^\vee *_\sigma g^\vee)^\wedge(p, q, r) \\ &= \int \bar{e}[p \cdot x + q \cdot y] f^\vee(\tilde{x}, \tilde{y}, r) g^\vee(x - \tilde{x}, y - \tilde{y}, r) \\ &\quad \times \bar{e} \left[ \frac{r^2}{2} \sum_{i,k} J_{ik} \tilde{y}_k (y_i - \tilde{y}_i) \right] \bar{e}[r\beta(\tilde{x}, y - \tilde{y})] d\tilde{x}d\tilde{y}dx dy. \end{aligned} \tag{3.3}$$

Using the definitions of  $f^\vee$  and  $g^\vee$ , together with the Fourier inversion theorem, this expression becomes

$$(f \times g)(p, q, r) = \int \bar{e}[(q - \tilde{q}) \cdot y] f \left( p + ry, q + \frac{r^2}{2} \sum_{i,k} J_{ik} y_i \mathbf{q}_k, r \right) g(p, \tilde{q}, r) d\tilde{q} dy. \tag{3.4}$$

Similarly, the involution on  $\mathcal{A}$  is given by

$$\begin{aligned} f^*(p, q, r) &= ((f^\vee)^*)^\wedge(p, q, r) = \int \overline{f(\tilde{p}, \tilde{q}, r)} \bar{e}[(p - \tilde{p}) \cdot x + (q - \tilde{q}) \cdot y] \\ &\quad \times \bar{e} \left[ \frac{r^2}{2} \sum_{i,k} J_{ik} y_i y_k \right] \bar{e}[r\beta(x, y)] d\tilde{p}d\tilde{q}dx dy. \end{aligned} \tag{3.5}$$

Clearly, the  $*$ -algebra  $(\mathcal{A}, \times, *)$  is a pre- $C^*$ -algebra, together with the  $C^*$ -norm  $f \mapsto \|L_f\|$ . Here, the representation  $L$  is just as in Eq. (3.2), having been extended to  $\mathcal{A}$ . By construction, we know that  $S \cong \overline{L(\mathcal{A})}^{\|\cdot\|}$ .

To show that the  $C^*$ -algebra  $S$  is a deformation of  $(C_0(G), \{, \})$ , let us now introduce the deformation parameter  $\hbar$ . We will follow the general procedure given in Theorem 3.4 of [16]. In our case, with the group  $H/Z$  being abelian, it does not need to vary and we only need to incorporate the parameter  $\hbar$  to the cocycle  $\sigma$ . Namely, consider the cocycle  $\sigma_\hbar : r \mapsto \sigma_\hbar^r$ , given by

$$\sigma_\hbar^r((x, y), (x', y')) = \bar{e} \left[ \frac{\hbar r^2}{2} \sum_{i,k} J_{ik} y_k y'_i \right] \bar{e}[\hbar r \beta(x, y')].$$

Then in exactly the same way as in Eqs. (3.4) and (3.5), but by using the cocycle  $\sigma_\hbar$  instead, we can construct on the function space  $\mathcal{A}$  the deformed multiplication  $\times_\hbar$  and the involution  $*_\hbar$ . As before, each  $(\mathcal{A}, \times_\hbar, *_\hbar)$  is a pre- $C^*$ -algebra. Similarly to Eq. (3.2), the functions  $f \in \mathcal{A}$  can be regarded as operators, with the operator norm now denoted by  $\| \cdot \|_\hbar$ . Let us define  $S_\hbar$  as the  $C^*$ -completion of  $(\mathcal{A}, \times_\hbar, *_\hbar)$ , under  $\| \cdot \|_\hbar$ . Using these ingredients, we can now describe the deformation quantization picture.

**Theorem 3.6.** Recall the Poisson bracket  $\{, \}$  on  $G$ , from Proposition 2.4. Let  $\mathcal{A} = \mathcal{S}_{3c}(G)$  be the (dense) subspace of  $C_0(G)$ , as defined above. For each  $\hbar \in \mathbb{R}$ , define on  $\mathcal{A}$  the deformed multiplication,  $\times_\hbar$ , and the involution,  $*_\hbar$ , as in the previous paragraph, together with the corresponding  $C^*$ -norm  $\| \cdot \|_\hbar$ . Then we have the following.

- (1) For  $\hbar = 0$ , the operations  $\times_\hbar, *_\hbar$  are exactly the pointwise product and the complex conjugation on  $\mathcal{A} (\subseteq C_0(G))$ . Also  $S_{\hbar=0} \cong C_0(G)$ , as a  $C^*$ -algebra.
- (2) The  $C^*$ -algebras  $\{S_\hbar\}_{\hbar \in \mathbb{R}}$  form a continuous field of  $C^*$ -algebras. In particular, the map  $\hbar \mapsto \|f\|_\hbar$  is continuous for any  $f \in \mathcal{A}$ .
- (3) For any  $f, g \in \mathcal{A}$  and  $(p, q, r) \in G$ , we have the following pointwise convergence:

$$\frac{1}{\hbar} (f \times_\hbar g - g \times_\hbar f)(p, q, r) \longrightarrow \frac{i}{2\pi} \{f, g\}(p, q, r),$$

as  $\hbar \rightarrow 0$ .

(4) The convergence in (3) is actually stronger. In fact, for  $f, g \in \mathcal{A}$ , we have

$$\lim_{\hbar \rightarrow 0} \left\| \frac{f \times_{\hbar} g - g \times_{\hbar} f}{i\hbar} - \frac{1}{2\pi} \{f, g\} \right\|_{\hbar} = 0. \tag{3.6}$$

All this means that the  $\ast$ -algebras  $(\mathcal{A}, \times_{\hbar}, \ast_{\hbar})_{\hbar \in \mathbb{R}}$  provide a “strict deformation quantization” (in the sense of Rieffel [6,20]) of  $\mathcal{A} (\subseteq C_0(G))$ , in the direction of the Poisson bracket  $(1/2\pi)\{, \}$ .

**Proof.** This is a special case of Theorem 3.4 of [16]. But, we will still carry out some main aspects of the proof.

(1) If  $\hbar = 0$ , the cocycle term becomes  $\sigma \equiv 1$ , and can be ignored. By the Fourier inversion theorem, Eqs. (3.4) and (3.5) thus become

$$\begin{aligned} (f \times g)(p, q, r) &= \int \bar{e}[(q - q') \cdot \tilde{y}] f(p, q', r) g(p, q, r) dq' d\tilde{y} = f(p, q, r) g(p, q, r), \\ f^{\ast}(p, q, r) &= \int \overline{f(\tilde{p}, \tilde{q}, r)} \bar{e}[(p - \tilde{p}) \cdot x + (q - \tilde{q}) \cdot y] d\tilde{p} d\tilde{q} dx dy = \overline{f(p, q, r)}. \end{aligned}$$

It is also easy to see that  $S_{\hbar=0} \cong C_0(G)$ , with its sup-norm as the  $C^{\ast}$ -norm. When  $\hbar = 1$ , we would recover the  $C^{\ast}$ -algebra  $S$  of Proposition 3.5(2).

(2) As for the  $C^{\ast}$ -algebras  $\{S_{\hbar}\}_{\hbar \in \mathbb{R}}$  forming a continuous field of  $C^{\ast}$ -algebras, note that each  $S_{\hbar}$  is really a twisted crossed product  $C^{\ast}$ -algebra of an abelian group  $H/Z$ , namely  $S_{\hbar} \cong C_0(G_1) \rtimes^{\sigma_{\hbar}} (H/Z)$ , and only the cocycle  $\sigma_{\hbar}$  is being changed as the parameter  $\hbar$  varies. Therefore, for each  $\hbar$ , the “amenability condition” holds, meaning that the notions of the “full” and the “reduced” crossed product  $C^{\ast}$ -algebras coincide. In [25], using the universal property of the full  $C^{\ast}$ -algebras and also taking advantage of the property of the reduced  $C^{\ast}$ -algebras that one is able to work with their specific representations, Rieffel gave an answer to the problem of the continuity of certain field of crossed product  $C^{\ast}$ -algebras. In short, under suitable conditions, Rieffel has shown that the field of “full” crossed product  $C^{\ast}$ -algebras is upper semi-continuous, while the field of “reduced” crossed product  $C^{\ast}$ -algebras is lower semi-continuous. Our case is simpler than the general case, and, with the amenability at hand, it follows that our field of  $C^{\ast}$ -algebras  $\{S_{\hbar}\}_{\hbar \in \mathbb{R}}$  is in fact continuous.

(3) For  $f \in \mathcal{A}$ , by the Fourier inversion theorem, we can write it as

$$f(p, q, r) = \int (\mathcal{F}^{-1}f)(\tilde{x}, \tilde{y}, \tilde{z}) \bar{e}[p \cdot \tilde{x} + q \cdot \tilde{y} + r\tilde{z}] d\tilde{x} d\tilde{y} d\tilde{z}.$$

Since  $\bar{e}[t] = e^{-2\pi it}$ , we thus have

$$df(p, q, r) = (-2\pi i) \int (\mathcal{F}^{-1}f)(\tilde{x}, \tilde{y}, \tilde{z}) \bar{e}[p \cdot \tilde{x} + q \cdot \tilde{y} + r\tilde{z}] \mathbf{X} d\tilde{x} d\tilde{y} d\tilde{z},$$

where  $\mathbf{X} = (\tilde{x}, \tilde{y}, \tilde{z})$ . Therefore, for  $f, g \in \mathcal{A}$ , the Poisson bracket from Proposition 2.4 becomes

$$\begin{aligned} \{f, g\}(p, q, r) &= (-4\pi^2) \int (\mathcal{F}^{-1}f)(\tilde{x}, \tilde{y}, \tilde{z}) (\mathcal{F}^{-1}g)(\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{z}}) \\ &\quad \times \left[ r(\beta(\tilde{x}, \tilde{y}) - \beta(\tilde{\tilde{x}}, \tilde{\tilde{y}})) + \frac{r^2}{2} \sum_{i,k=1}^n J_{ik}(\tilde{y}_k \tilde{y}_i - \tilde{\tilde{y}}_i \tilde{\tilde{y}}_k) \right] \\ &\quad \times \bar{e}[p \cdot (\tilde{x} + \tilde{\tilde{x}}) + q \cdot (\tilde{y} + \tilde{\tilde{y}}) + r(\tilde{z} + \tilde{\tilde{z}})] d\tilde{x} d\tilde{y} d\tilde{z} d\tilde{\tilde{x}} d\tilde{\tilde{y}} d\tilde{\tilde{z}}. \end{aligned}$$

In the  $(x, y, r)$  variables, by using the partial Fourier transform, this can be rewritten as

$$\begin{aligned} \{f, g\}(p, q, r) &= (-4\pi^2) \int f^{\vee}(\tilde{x}, \tilde{y}, r) g^{\vee}(\tilde{\tilde{x}}, \tilde{\tilde{y}}, r) \bar{e}[p \cdot (\tilde{x} + \tilde{\tilde{x}}) + q \cdot (\tilde{y} + \tilde{\tilde{y}})] \\ &\quad \times \left[ r(\beta(\tilde{x}, \tilde{y}) - \beta(\tilde{\tilde{x}}, \tilde{\tilde{y}})) + \frac{r^2}{2} \sum_{i,k=1}^n J_{ik}(\tilde{y}_k \tilde{y}_i - \tilde{\tilde{y}}_i \tilde{\tilde{y}}_k) \right] d\tilde{x} d\tilde{y} d\tilde{\tilde{x}} d\tilde{\tilde{y}}. \end{aligned} \tag{3.7}$$

Meanwhile, let us rewrite the deformed product,  $f \times_{\hbar} g$  for  $f, g \in \mathcal{A}$ , in a more symmetric form. Basically, we start from the definition given in Eq. (3.3), together with the adjustment in the cocycle term incorporating the parameter  $\hbar$ . Perform the change of variables  $x - \tilde{x} \mapsto \tilde{\tilde{x}}$  and  $y - \tilde{y} \mapsto \tilde{\tilde{y}}$ . Then we would have

$$\begin{aligned} (f \times_{\hbar} g)(p, q, r) &= \int \bar{e}[p \cdot (\tilde{x} + \tilde{\tilde{x}}) + q \cdot (\tilde{y} + \tilde{\tilde{y}})] f^{\vee}(\tilde{x}, \tilde{y}, r) g^{\vee}(\tilde{\tilde{x}}, \tilde{\tilde{y}}, r) \\ &\quad \times \bar{e}[\hbar r \beta(\tilde{x}, \tilde{y})] \bar{e} \left[ \frac{\hbar r^2}{2} \sum_{i,k} J_{ik} \tilde{y}_k \tilde{y}_i \right] d\tilde{x} d\tilde{y} d\tilde{\tilde{x}} d\tilde{\tilde{y}}. \end{aligned}$$

It follows that we have

$$\begin{aligned} \frac{1}{\hbar}(f \times_{\hbar} g - g \times_{\hbar} f)(p, q, r) &= \frac{1}{\hbar} \int \bar{e}[p \cdot (\tilde{x} + \tilde{x}) + q \cdot (\tilde{y} + \tilde{y})] f^{\vee}(\tilde{x}, \tilde{y}, r) g^{\vee}(\tilde{x}, \tilde{y}, r) \\ &\times \left( \bar{e}[\hbar r \beta(\tilde{x}, \tilde{y})] \bar{e} \left[ \frac{\hbar r^2}{2} \sum_{i,k} J_{ik} \tilde{y}_k \tilde{y}_i \right] - \bar{e}[\hbar r \beta(\tilde{x}, \tilde{y})] \bar{e} \left[ \frac{\hbar r^2}{2} \sum_{i,k} J_{ik} \tilde{y}_k \tilde{y}_i \right] \right) d\tilde{x} d\tilde{y} d\tilde{z}. \end{aligned} \tag{3.8}$$

In the above, since  $\bar{e}[t] = e^{-2\pi i t} = 1 - 2\pi i t + \dots$ , we have

$$\begin{aligned} \frac{1}{\hbar} \left( \bar{e}[\hbar r \beta(\tilde{x}, \tilde{y})] \bar{e} \left[ \frac{\hbar r^2}{2} \sum_{i,k} J_{ik} \tilde{y}_k \tilde{y}_i \right] - \bar{e}[\hbar r \beta(\tilde{x}, \tilde{y})] \bar{e} \left[ \frac{\hbar r^2}{2} \sum_{i,k} J_{ik} \tilde{y}_k \tilde{y}_i \right] \right) \\ = (-2\pi i) \left( r \beta(\tilde{x}, \tilde{y}) + \frac{r^2}{2} \sum_{i,k} J_{ik} \tilde{y}_k \tilde{y}_i - r \beta(\tilde{x}, \tilde{y}) - \frac{r^2}{2} \sum_{i,k} J_{ik} \tilde{y}_k \tilde{y}_i \right) + \mathcal{O}(\hbar). \end{aligned}$$

Therefore, comparing with Eq. (3.7), we can readily observe the pointwise convergence:

$$\frac{1}{\hbar}(f \times_{\hbar} g - g \times_{\hbar} f)(p, q, r) \longrightarrow \frac{i}{2\pi} \{f, g\}(p, q, r),$$

as  $\hbar \rightarrow 0$ .

(4) In our case, each  $S_{\hbar}$  (for  $\hbar \neq 0$ ) is isomorphic to the (reduced) twisted crossed product  $C^*$ -algebra  $C_0(G_1) \rtimes^{\sigma_{\hbar}}(H/Z)$ , and, therefore, the  $C^*$ -norm  $\|\cdot\|_{\hbar}$  is dominated by the  $L^1$ -norm on  $L^1(H/Z, C_0(G_1))$ . By the partial Fourier transform in the  $r (\in G_1)$  variable, this  $L^1$ -norm is equivalent to the  $L^1$ -norm on  $L^1(H/Z \times Z) = L^1(H)$ . Even when  $\hbar = 0$ , for which we know  $S_{\hbar=0} \cong C_0(G) \cong C^*(H)$  by the Fourier transform, it holds that the  $C^*$ -norm  $\|\cdot\|_{\hbar=0}$  is also dominated by the  $L^1$ -norm on  $L^1(H)$ . All this means that, to show the norm convergence in Eq. (3.6), we just need to show the convergence with respect to the  $L^1$ -norm on  $L^1(H)$ , transferred to  $\mathcal{A} \subseteq L^1(G)$  by the Fourier transform.

This can be achieved by Lebesgue's dominated convergence theorem. We already know the pointwise convergence in  $\mathcal{A}$ , while, in  $\mathcal{A} = \mathcal{S}_{3c}(G)$ , we are able to find an  $L^1$ -bound for the expressions,  $(f \times_{\hbar} g - g \times_{\hbar} f)/\hbar - (i/2\pi)\{f, g\}$ , since the convergence involving the cocycle terms can be controlled in a compact set on which the convergence is uniform.  $\square$

**Remark.** In the proof of item (2) above, we were aided by the fact that  $H/Z$  is abelian. In general, however, the group may not be abelian, and may also need to vary (as the parameter value changes) in the definition of the  $C^*$ -algebras  $S_{\hbar}$ . This would make the proof of the continuity of  $\{S_{\hbar}\}_{\hbar \in \mathbb{R}}$  more difficult. Our current example does not have this problem, but refer to the proof of Theorem 3.4 in [16] for a more general situation.

Meanwhile, as for the proof of the correspondence relation in Eq. (3.6), note that a pointwise convergence result like (3) would usually be sufficient for most of the formal power series frameworks, like in the case of a QUE algebra. But, for our "strict deformation quantization" framework, we further needed to show the norm convergence, as in (4) above. See [6,20] for more general discussions. The idea for the proof of (4) was obtained from the one given in [26], with a small adjustment of restricting things to  $\mathcal{S}_{3c}(G)$ , instead of the space  $\mathcal{S}(G)$  of all Schwartz functions on  $G$ .

### 3.4. The comultiplication on $(S, \Delta)$

To further strengthen our case that  $(S, \Delta)$  is a "quantized  $C_0(G)$ ", let us look at the comultiplication  $\Delta$ , which will show that it reflects the group multiplication law on  $G$ .

**Proposition 3.7.** *With the representation  $f \mapsto L_f \in \mathcal{B}(\mathcal{H})$ ,  $f \in \mathcal{A}$ , defined in Eq. (3.2), the comultiplication  $\Delta$  from Proposition 3.5(2) becomes*

$$\Delta(L_f) = (L \otimes L)_{\Delta f},$$

where  $\Delta f \in C_b(G \times G)$  is the function defined by

$$(\Delta f)(p, q, r; p', q', r') = f \left( p + p', q + q' + r' \sum_{i,k} J_{ik} p_i q_k, r + r' \right).$$

**Proof.** Write  $L_f = \int (\mathcal{F}^{-1}f)(\tilde{x}, \tilde{y}, \tilde{z}) L_{\tilde{x}, \tilde{y}, \tilde{z}} d\tilde{x} d\tilde{y} d\tilde{z}$ , where  $\mathcal{F}^{-1}f \in C_c(H)$  is the (inverse) Fourier transform of  $f$ . Then  $L_{\tilde{x}, \tilde{y}, \tilde{z}} \in \mathcal{B}(\mathcal{H})$  is such that

$$L_{\tilde{x}, \tilde{y}, \tilde{z}} \xi(x, y, r) = \bar{e}[r\tilde{z}] \sigma^r((\tilde{x}, \tilde{y}), (x - \tilde{x}, y - \tilde{y})) \xi(x - \tilde{x}, y - \tilde{y}, r).$$

Comparing with the definition of  $L_f$  in Eq. (3.2), we may regard  $L_{\tilde{x}, \tilde{y}, \tilde{z}} = L_F$ , where the function  $F \in C_b(G)$  is such that  $F(p, q, r) = \bar{e}[p \cdot \tilde{x} + q \cdot \tilde{y} + r\tilde{z}]$ . Indeed,  $L_{\tilde{x}, \tilde{y}, \tilde{z}}$  is contained in the multiplier algebra  $M(S)$ . In a sense, the operators  $L_{\tilde{x}, \tilde{y}, \tilde{z}}$ , for  $(\tilde{x}, \tilde{y}, \tilde{z}) \in H$ , form the building blocks for the “regular representation”  $L$  (or, equivalently, for  $C^*$ -algebra  $S$ ).

For  $\zeta \in \mathcal{H}$ , we have

$$\begin{aligned} (\Delta(L_{\tilde{x}, \tilde{y}, \tilde{z}}))\zeta(x, y, r; x', y', r') &= V_{\Theta}(L_{\tilde{x}, \tilde{y}, \tilde{z}} \otimes 1)V_{\Theta}^*\zeta(x, y, r; x', y', r') \\ &= \bar{e}[(r + r')\tilde{z}]\bar{e} \left[ \frac{r^2}{2} \sum_{i,k} J_{ik}\tilde{y}_k(y_i - \tilde{y}_i) \right] \bar{e}[r\beta(\tilde{x}, y - \tilde{y})] \\ &\quad \times \bar{e} \left[ \frac{r'^2}{2} \sum_{i,k} J_{ik}\tilde{y}_k(y'_i - \tilde{y}_i) \right] \bar{e}[r'\beta(\tilde{x}, y' - \tilde{y})]\bar{e} \left[ rr' \sum_{i,k} J_{ik}\tilde{y}_k(y_i - \tilde{y}_i) \right] \\ &\quad \times \zeta \left( x - \tilde{x} - r' \sum_{i,k} J_{ik}\tilde{y}_k x_i, y - \tilde{y}, r; x' - \tilde{x}, y' - \tilde{y}, r' \right). \end{aligned}$$

Meanwhile, consider  $\Delta F \in C_b(G \times G)$ , given by

$$(\Delta F)(p, q, r; p', q', r') = \bar{e} \left[ (p + p') \cdot \tilde{x} + (q + q') \cdot \tilde{y} + r' \sum_{i,k} J_{ik}p_i\tilde{y}_k + (r + r')\tilde{z} \right].$$

Then, by a straightforward computation using the Fourier inversion theorem, we can see that, for  $\zeta \in \mathcal{H}$ ,

$$(L \otimes L)_{\Delta F}\zeta(x, y, r; x', y', r') = (\Delta(L_{\tilde{x}, \tilde{y}, \tilde{z}}))\zeta(x, y, r; x', y', r').$$

In other words,  $(L \otimes L)_{\Delta F} = \Delta(L_f)$ . Remembering the definitions, it follows easily that  $\Delta(L_f) = (L \otimes L)_{\Delta f}$  for any  $f \in C_c(G)$ , where  $\Delta f$  is as defined above.  $\square$

**Remark.** This proposition shows that, for  $f \in C_c(G)$ , the comultiplication sends it to  $\Delta f \in C_b(G \times G)$ , such that

$$(\Delta f)(p, q, r; p', q', r') = f((p, q, r)(p', q', r')),$$

preserving the group multiplication law on  $G$  as given in Eq. (2.1). Together with the result of Theorem 3.6, this result supports our assertion made earlier that  $(S, \Delta)$  is a “quantized  $C_0(G)$ ”.

At this moment, the  $C^*$ -bialgebra  $(S, \Delta)$  is just a quantum semi-group. For it to be properly considered as a locally compact quantum group, we need further discussions on maps like antipode and Haar weight. This is given in the following section.

Meanwhile, notice the similarity between our example  $(S, \Delta)$  above and the one constructed by Enock and Vainerman in Section 6 of [27]. Looking at the comultiplications and the cocycles involved, we see some resemblance. However, the methods of construction are rather different between the two. In addition, there is another crucial difference. Namely, in the example of [27], the underlying von Neumann algebra is isomorphic to the group von Neumann algebra  $\mathcal{L}(H) = C^*(H)''$  of  $H$ , while, in our case,  $S$  is isomorphic to a “twisted” crossed product algebra: unless  $J \equiv 0$ , the  $C^*$ -algebra  $S$  is not isomorphic to  $C^*(H)$ .

In the author’s opinion, the example  $(S, \Delta)$  given here has more merit, considering that (1) its Poisson–Lie group counterpart and its multiplicative unitary operator have all been obtained; (2) the relationship between the Poisson bracket and the cocycle bicrossed product construction of the multiplicative unitary operator have been manifested; and (3) that the underlying  $C^*$ -algebra is built on the framework of twisted crossed product algebras (more general than ordinary group  $C^*$ -algebras or group von Neumann algebras).

#### 4. The quantum group structure

##### 4.1. $(S, \Delta)$ is a locally compact quantum group

We now turn our attention to showing that the  $C^*$ -bialgebra  $(S, \Delta)$  we constructed above is indeed a *locally compact quantum group*, in the precise sense of Kustermans and Vaes [28,29], or that of Masuda et al. [30]. We could construct the Haar weight and other maps, along the lines of the general results by Van Daele [31,32]. However, since it can be shown that our example is a case of a “cocycle bicrossed product” (in the sense of [15]), it is not really necessary to be overly technical. See Lemma 4.1 and Theorem 4.2 below.

First, recall the matched pair  $(G_1, G_2)$  we considered in Definition 3.3. Our formulation at the time was motivated by the Poisson geometric data. But this time, to make things to fit the algebraic framework given in [15], let us work with the pair  $(G_1, H/Z)$ , where  $H/Z$  is the dual of  $G_2$ . To be more precise, consider

$$G_1 = \{(0, 0, r) : r \in \mathbb{R}\} \quad \text{and} \quad H/Z = \{(x, y, 0) : x, y \in \mathbb{R}^n\}.$$



We may use the (partial) Fourier transform to move between the functions on  $G_2$  and those on  $H/Z$ . It is not difficult to see that  $(G_1, H/Z)$  forms a matched pair. By abuse of notation, we again denote the actions by  $\alpha : G_1 \times H/Z \rightarrow H/Z$  and  $\gamma : H/Z \times G_1 \rightarrow G_1$ . We then have

$$\alpha_r(x, y) := \left( x + r \sum_{i,k=1}^n J_{ik} y_k \mathbf{x}_i, y \right), \quad \gamma_{(x,y)}(r) := r.$$

At the algebra level, we obtain the \*-isomorphism  $\tau : L^\infty(G_1) \otimes L^\infty(H/Z) \rightarrow L^\infty(G_1) \otimes L^\infty(H/Z)$ , given by

$$(\tau(f))(r; (x, y)) = f(\gamma_{(x,y)}(r); \alpha_r(x, y)) = f\left(r; x + r \sum_{i,k=1}^n J_{ik} y_k \mathbf{x}_i, y\right).$$

In fact, these computations were carried out earlier, though implicitly, in our discussion following Proposition 3.4 leading up to Proposition 3.5.

**Lemma 4.1.** *As above, consider the matched pair  $(G_1, H/Z)$ , together with the corresponding actions  $\alpha$  and  $\gamma$ . Define  $\mathcal{U} : G_1 \times G_1 \times H/Z \rightarrow \mathbb{T}$  and  $\mathcal{V} : G_1 \times H/Z \times H/Z \rightarrow \mathbb{T}$ , given by*

$$\mathcal{U} \equiv \text{Id}, \quad \text{and} \quad \mathcal{V}(r; (x, y), (x', y')) = \bar{e} \left[ \frac{r^2}{2} \sum_{i,k} J_{ik} y'_k y_i \right] \bar{e}[r\beta(x', y)].$$

Then  $(\tau, \mathcal{U}, \mathcal{V})$  is a “cocycle matching” of  $L^\infty(G_1)$  and  $L^\infty(H/Z)$ , with their natural quantum group structures.

**Remark.** Observe that  $\mathcal{V}$  is such that  $\mathcal{V}(r; (x, y), (x', y')) = \sigma^r((x', y'), (x, y))$ , where  $\sigma$  is the cocycle function given in Proposition 3.5(2).

**Proof.** Using the definition, we can verify the cocycle conditions given in Eq. (4.2) of [15]. Namely, the maps  $\mathcal{U}$  and  $\mathcal{V}$  satisfy

- $\mathcal{U}(r, r'; \alpha_{r'}(x, y)) \mathcal{U}(r + r', r''; (x, y)) = \mathcal{U}(r', r''; (x, y)) \mathcal{U}(r, r' + r''; (x, y))$ ,
- $\mathcal{V}(\gamma_{(x,y)}(r); (x', y'), (x'', y'')) \mathcal{V}(r; (x, y), (x' + x, y' + y)) = \mathcal{V}(r; (x, y), (x', y')) \mathcal{V}(r; (x' + x, y' + y), (x'', y''))$ ,
- $\mathcal{V}(r + r'; (x, y), (x', y')) \mathcal{U}(r, r'; (x' + x, y' + y)) = \mathcal{U}(r, r'; (x, y)) \mathcal{U}(\gamma_{\alpha_{r'}(x,y)}(r), \gamma_{(x,y)}(r'); (x', y'))$   
 $\times \mathcal{V}(r; \alpha_{r'}(x, y), \alpha_{\gamma_{(x,y)}(r')}(x', y')) \mathcal{V}(r'; (x, y), (x', y'))$ .

This is to be expected, considering that  $\mathcal{V}$  comes from the cocycle function  $\sigma$ . Thus by Lemma 4.11 of [15], we prove the result.  $\square$

Therefore, by Definition 2.2 in [15] and Theorem 2.13 in [15], we obtain the “cocycle bicrossed product”  $M = L^\infty(G_1)_{\alpha, \mathcal{U}} \times L^\infty(H/Z)$ , which is a locally compact quantum group. The associated dual locally compact quantum group is denoted by  $\hat{M}$  (see again Theorem 2.13 of [15]). By construction, it turns out that our  $(S, \Delta)$  obtained in the previous section is really the  $C^*$ -algebra counterpart to  $\hat{M}$ . The result is as follows.

**Theorem 4.2.** *Our  $C^*$ -bialgebra  $(S, \Delta)$  is none other than the dual of the cocycle bicrossed product obtained by the matched pair  $(G_1, H/Z)$  and the cocycle maps  $\mathcal{U}$  and  $\mathcal{V}$ . Therefore, we conclude that  $(S, \Delta)$  is itself a locally compact quantum group.*

**Proof.** An efficient way is to work with the multiplicative unitary operators. So consider  $W_1 \in \mathcal{B}(L^2(G_1 \times G_1))$  and  $W_2 \in \mathcal{B}(L^2(H/Z \times H/Z))$  such that

$$W_1 \xi(r; r') = \xi(r; r + r'), \quad \hat{W}_2 \zeta(x, y; x', y') = \zeta(x, y; x' + x, y' + y),$$

for  $\xi \in L^2(G_1 \times G_1)$  and  $\zeta \in L^2(H/Z \times H/Z)$ . They determine the natural quantum group structures on  $L^\infty(G_1)$  and  $L^\infty(H/Z)$ . By Definition 2.2 of [15], as well as the discussion in Section 4.4 of the same paper, the quantum groups  $M$  and  $\hat{M}$  are determined by the multiplicative unitary operator  $\hat{W} \in \mathcal{B}(L^2(G_1 \times H/Z \times G_1 \times H/Z))$ , defined by

$$\hat{W} = (\gamma \otimes \text{id} \otimes \text{id})((W_1 \otimes 1)\mathcal{U}^*)(\text{id} \otimes \text{id} \otimes \alpha)(\mathcal{V}(1 \otimes \hat{W}_2)).$$

In our case, this becomes

$$\begin{aligned} \hat{W} \xi(r; x, y; r'; x', y') &= \overline{\mathcal{U}(\gamma_{(x,y)}(r); -\gamma_{(x,y)}(r) + r'; (x', y'))} \mathcal{V}(r; (x, y), \alpha_{[-\gamma_{(x,y)}(r)+r']}(x', y')) \\ &\quad \times \xi(r; \alpha_{[-\gamma_{(x,y)}(r)+r']}(x', y') + (x, y); -\gamma_{(x,y)}(r) + r'; (x', y')) \\ &= \mathcal{V}(r; (x, y); \alpha_{[r'-r]}(x', y')) \xi(r; (x, y) + \alpha_{[r'-r]}(x', y'); r' - r; x', y') \end{aligned}$$

$$\begin{aligned}
 &= \bar{e} \left[ \frac{r^2}{2} \sum_{i,k} J_{ik} y'_k y_i \right] \bar{e} \left[ r\beta \left( x' + (r' - r) \sum_{i,k} J_{ik} y'_k \mathbf{x}_i, y \right) \right] \\
 &\quad \times \xi \left( r; x + x' + (r' - r) \sum_{i,k} J_{ik} y'_k \mathbf{x}_i, y + y'; r' - r, x', y' \right) \\
 &= e \left[ \frac{r^2}{2} \sum_{i,k} J_{ik} y'_k y_i \right] \bar{e} \left[ r r' \sum_{i,k} J_{ik} y'_k y_i \right] \bar{e} [r\beta(x', y)] \\
 &\quad \times \xi \left( r; x + x' + (r' - r) \sum_{i,k} J_{ik} y'_k \mathbf{x}_i, y + y'; r' - r, x', y' \right).
 \end{aligned}$$

By general theory, it is known to be multiplicative, so  $\hat{W} \in \hat{M} \otimes M$ . The right slices of  $\hat{W}$  generate  $\hat{M}$  while the left slices of  $\hat{W}$  generate  $M$ .

Now, consider an involutive operator  $K \in \mathcal{B}(L^2(G_1 \times H/Z))$ , defined by

$$K\xi(r; x, y) = \overline{\xi(-r; x + r \sum_{i,k} J_{ik} y_k \mathbf{x}_i, y)}. \tag{4.1}$$

We will postpone the discussion of the nature of the operator  $K$  for the time being (it has to do with the “antipode” map on our quantum group: see Proposition 4.3). Using this, define the operator  $\hat{V}$  by

$$\hat{V} = (K \otimes K) \Sigma \hat{W}^* \Sigma (K \otimes K).$$

Here,  $\Sigma$  denotes the flip. Then  $\hat{V}$  is also multiplicative, and the general theory shows that  $\hat{V} \in M' \otimes \hat{M}$ , where  $M'$  is the commutant of  $M$ . See Propostion 2.15 of [29], with the understanding that their  $J$  operator is  $K$  here, so that we do not cause any confusion with the skew-symmetric matrix  $J$  in our case. The left slices of  $\hat{V}$  generate  $\hat{M}$  while the right slices of  $\hat{V}$  generate  $M'$ .

After a straightforward computation using the formulas obtained above, we have

$$\begin{aligned}
 \hat{V}\xi(r; x, y; r'; x', y') &= e \left[ \frac{r'^2}{2} \sum_{i,k} J_{ik} y_k (y'_i - y_i) \right] \bar{e} [r'\beta(x, y' - y)] \\
 &\quad \times \xi \left( r + r'; x - r' \sum_{i,k} J_{ik} y_k \mathbf{x}_i, y; r'; x' - x + r' \sum_{i,k} J_{ik} y_k \mathbf{x}_i, y' - y \right).
 \end{aligned}$$

Compare this result with the definition of the multiplicative unitary operator  $V_\Theta$  we constructed in Proposition 3.5(2), which is exactly the same! (To be really precise, we need to flip the  $(x, y)$  and the  $r$ .) The multiplicative unitary operators being the same means that the  $C^*$ -algebras (or von Neumann algebras) they generate must agree. In particular, considering the left slices of  $\hat{V} = V_\Theta$ , we conclude that, at the  $C^*$ -algebra level,  $\hat{M}$  and  $S$  must coincide. It follows that our  $(S, \Delta)$  is actually a  $C^*$ -algebraic locally compact quantum group, whose von Neumann algebra envelope is  $\hat{M}$ .  $\square$

#### 4.2. Other structure maps: antipode and Haar weight

While the proof that  $(S, \Delta)$  is a quantum group is done, it will be still useful to know its other quantum group structure maps, namely, the antipode map and the Haar weight. We will try to be brief here (skipping some details), but we wish to point out some nice correspondence relations between the classical (Poisson) data and the quantum level, strengthening our case that  $(S, \Delta)$  is essentially a “quantized  $C_0(G)$ ”.

Correctly constructing the antipode map from the definitions is rather technical. See the main papers [28,29], and also a new treatment given in [33], which uses the Tomita–Takesaki theory. For our purposes, though, we will just use the following characterization of the antipode, denoted here by  $\kappa$ , given in terms of the multiplicative unitary operator:

$$\kappa((\omega \otimes \text{id})(V_\Theta)) = (\omega \otimes \text{id})(V_\Theta^*). \tag{4.2}$$

The subspace consisting of the elements  $(\omega \otimes \text{id})(V_\Theta)$ , for  $\omega \in \mathcal{B}(\mathcal{H})_*$ , is dense in  $S$ , and forms a core for  $\kappa$ .

At the level of the dense subspace of functions in  $C_c(H/Z \times G_1)$ , in the  $(x, y; r)$  variables, the antipode  $\kappa$  in our case takes the following form.

**Proposition 4.3.** Let  $\kappa : C_c(H/Z \times G_1) \rightarrow C_c(H/Z \times G_1)$  be defined by

$$(\kappa(f))(x, y, r) = \bar{e} \left[ \frac{r^2}{2} \sum_{i,k} J_{ik} y_i y_k \right] \bar{e} [r\beta(x, y)] f \left( -x - r \sum_{i,k} J_{ik} y_k \mathbf{x}_i, -y, -r \right).$$

This map corresponds to the definition of  $\kappa$  given in Eq. (4.2), and turns out to be a bounded map. By general theory, its extension to the  $C^*$ -algebra  $S$ , still denoted by  $\kappa$ , is the antipode map on  $(S, \Delta)$ . Moreover, the antipode map  $\kappa$  is related with the operator  $K$  in Eq. (4.1) by

$$K(L_f)^*K = L_{\kappa(f)}, \quad f \in C_c(H/Z \times G_1),$$

where  $L_f \in S$  denotes the operator realization of the function  $f$ . It follows that  $\kappa^2 \equiv \text{Id}$ .

**Proof.** For  $\eta, \zeta \in C_c(H/Z \times G_1)$ , consider  $\omega_{\eta, \zeta} \in \mathcal{B}(\mathcal{H})_*$ , defined by  $\omega_{\eta, \zeta}(T) := \langle T\eta, \zeta \rangle$ . Since  $C_c(H/Z \times G_1)$  is dense in  $\mathcal{H}$ , it is clear that the  $\omega_{\eta, \zeta}$  are dense in  $\mathcal{B}(\mathcal{H})_*$ . Meanwhile, by a straightforward calculation, we can show that the operator  $(\omega_{\eta, \zeta} \otimes \text{id})(V_\Theta)$  can be realized as  $L_f$ , where  $f$  is a function contained in  $C_c(H/Z \times G_1)$  defined by

$$f(x, y, r) = \int \eta(x, y, r + \tilde{r}) \zeta \left( x + r \sum_{i,k} J_{ik} y_k \mathbf{x}_i, y, \tilde{r} \right) d\tilde{r}.$$

Similarly,  $(\omega_{\eta, \zeta} \otimes \text{id})(V_\Theta^*)$  can be realized as  $L_g$ , where

$$g(x, y, r) = \int \bar{e} \left[ \frac{r^2}{2} \sum_{i,k} J_{ik} y_k y_i \right] \bar{e} [r\beta(x, y)] \eta \left( -x - r \sum_{i,k} J_{ik} y_k \mathbf{x}_i, -y, -r + \tilde{r} \right) \overline{\zeta(-x, -y, \tilde{r})} d\tilde{r}.$$

By Eq. (4.2), the function  $g$  is none other than  $\kappa(f)$ . Comparing it with the expression for  $f$  above, we obtain the result of the proposition. Since the  $\omega_{\eta, \zeta}$  are dense in  $\mathcal{B}(\mathcal{H})_*$ , this characterization of the  $\kappa$  map is sufficient.

Meanwhile, we also have  $K(L_f)^*K = L_{\kappa(f)}$ , where  $f \in C_c(H/Z \times G_1)$  and  $\kappa(f)$  is as above. Calculation is straightforward. Since  $K$  is a bounded operator and involutive, this implies that  $\kappa : L_f \mapsto L_{\kappa(f)}$  can be extended to a bounded map on all of the  $C^*$ -algebra  $S$ , with  $\kappa^2 \equiv \text{Id}$ .  $\square$

**Remark.** Note that, when  $\sigma \equiv 1$ , we have

$$(\kappa(f))(x, y, r) = f \left( -x - r \sum_{i,k} J_{ik} y_k \mathbf{x}_i, -y, -r \right).$$

If we express this in the  $(p, q, r)$ -variables, by the partial Fourier transform, it becomes

$$(\kappa(f))(p, q, r) = f \left( -p, -q + r \sum_{i,k=1}^n J_{ik} p_i \mathbf{q}_k, -r \right) = f((p, q, r)^{-1}).$$

What all this means is that in the commutative case (when  $\sigma \equiv 1$ ), the antipode map is just taking the inverse in the group  $G$ . This again strengthens our point that  $(S, \Delta)$  is a “quantized  $C_0(G)$ ”.

Finally, let us turn our attention to the Haar weight on our quantum group  $(S, \Delta)$ . At the classical level, recall that the group structure on  $G$  was chosen in Eq. (2.1) so that an ordinary Lebesgue measure on  $G = \mathbb{R}^{2n+1}$  becomes its (left invariant) Haar measure. This suggests building the Haar weight on  $(S, \Delta)$  from the Lebesgue measure on  $G$ . At the level of the functions in  $\mathcal{A} = \mathcal{S}_{3c}(G)$ , this suggestion is manifested in Definition 4.4 below.

**Definition 4.4.** (1) On  $\mathcal{A}$ , define a linear functional  $\varphi$  by

$$\varphi(f) = \int f(p, q, r) dpdqdr.$$

(2) At the level of the functions in  $\mathcal{S}_{3c}(H/Z \times G_1)$ , in the  $(x, y, r)$  variables, this is equivalent to the linear functional  $\varphi_S$  below:

$$\varphi_S(f) = \int f(0, 0; r) dr.$$

**Lemma 4.5.** Let  $\varphi_S$  be the linear functional given in Definition 4.4. It satisfies the following “left invariance property”:

$$(\text{id} \otimes \varphi_S)((1 \otimes f)(\Delta g)) = \kappa((\text{id} \otimes \varphi_S)((\Delta f)(1 \otimes g))), \tag{4.3}$$

for  $f, g \in \mathcal{S}_{3c}(H/Z \times G_1)$ .

**Proof.** Here, the expression  $(1 \otimes f)(\Delta g)$  means the function  $F$  such that  $(L \otimes L)_F = (1 \otimes L_f)(\Delta(L_g))$ . By using the definitions of the comultiplication, the product on  $S$ , and the definition of the functional  $\varphi_S$ , we have

$$\begin{aligned}
 (\text{id} \otimes \varphi_S)((1 \otimes f)(\Delta g))(x, y; r) &= \int f \left( -x + \tilde{r} \sum_{i,k} J_{ik} \mathcal{Y}_k \mathbf{x}_i, -y, \tilde{r} \right) g \left( x - \tilde{r} \sum_{i,k} J_{ik} \mathcal{Y}_k \mathbf{x}_i, y, r + \tilde{r} \right) \\
 &\quad \times e[\tilde{r} \beta(x, y)] \bar{e} \left[ \frac{\tilde{r}^2}{2} \sum_{i,k} J_{ik} \mathcal{Y}_i \mathcal{Y}_k \right] d\tilde{r}.
 \end{aligned}$$

By a similar computation, we also have

$$\begin{aligned}
 (\text{id} \otimes \varphi_S)((\Delta f)(1 \otimes f))(x, y; r) &= \int f \left( x - \tilde{r} \sum_{i,k} J_{ik} \mathcal{Y}_k \mathbf{x}_i, y, r + \tilde{r} \right) g \left( -x + \tilde{r} \sum_{i,k} J_{ik} \mathcal{Y}_k \mathbf{x}_i, -y, \tilde{r} \right) \\
 &\quad \times e[\tilde{r} \beta(x, y)] \bar{e} \left[ \frac{\tilde{r}^2}{2} \sum_{i,k} J_{ik} \mathcal{Y}_i \mathcal{Y}_k \right] d\tilde{r}.
 \end{aligned}$$

Therefore, by using the definition of the antipode map  $\kappa$ , as obtained in Proposition 4.3, we can show the following:

$$\kappa((\text{id} \otimes \varphi_S)((\Delta f)(1 \otimes g)))(x, y; r) = (\text{id} \otimes \varphi_S)((1 \otimes f)(\Delta g))(x, y; r). \quad \square$$

In Kac algebra theory, Eq. (4.3) has been used to define the left invariance of the Haar weight. Our proof was given only at the function level, but it nevertheless provides some justification to our choice of  $\varphi_S$ .

In general, jumping up from the linear functional at the level of the functions to the weight at the operator level can be quite technical. See papers on the Haar weights on general locally compact quantum groups, such as [31,32]. While we can actually proceed using a similar approach as in [34], we made a decision above to take advantage of the fact that our example is a case of a cocycle bicrossed product. The precise construction of the Haar weight can be found in Proposition 2.9 of [15] (see the remark below).

**Remark (Some Technical Remarks. See [15]).** We noted in Theorem 4.2 that  $(S, \Delta)$  is the  $C^*$ -algebra counterpart to the locally compact quantum group  $\hat{M}$ , obtained from the matched pair  $(L^\infty(G_1), L^\infty(H/Z))$  and the cocycle maps  $\mathcal{U}$  and  $\mathcal{V}$ . (To be really precise, we need to flip the  $(x, y)$  and the  $r$ .) Recall also the actions  $\alpha$  and  $\gamma$  from Section 4.1. By general theory on the cocycle bicrossed products,  $\hat{M}$  is generated by  $\gamma(M_1)$  and  $\{(\text{id} \otimes \text{id} \otimes \omega)(\mathcal{V}(1 \otimes \hat{W}_2)) : \omega \in (M_2)_*\}$ , where  $M_1 = L^\infty(G_1)$  and  $M_2 = L^\infty(H/Z)$ , in our case. It turns out that  $\gamma(M_1)$  is the fixed-point algebra of the dual action,  $\hat{\gamma}$ , of  $(\hat{M}_2, \hat{\Delta}_2^{\text{cop}})$  on  $\hat{M}$ . Then  $T = (\text{id} \otimes \text{id} \otimes \hat{\varphi}_2)\hat{\gamma}$ , where  $\hat{\varphi}_2$  is the Haar weight on  $\hat{M}_2$ , defines a normal, faithful operator-valued weight from  $\hat{M}$  to  $\gamma(M_1)$ . From the Haar weight  $\varphi_1$  on  $M_1$ , we can then define the normal, semi-finite weight  $\varphi_{\hat{M}}$  on  $\hat{M}$ , by  $\varphi_{\hat{M}} = \varphi_1 \circ \gamma^{-1} \circ T$ . From  $\varphi_{\hat{M}}$ , by restriction and the flip  $(r; x, y) \mapsto (x, y; r)$ , we would obtain the Haar weight on  $S$ .

Having given these remarks and Lemma 4.5, together with the knowledge that Haar weight is unique (up to a scalar multiplication), we will accept that the linear functional  $\varphi_S$  above does indeed extend to the correct Haar weight on  $(S, \Delta)$ .

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