Quantization, Quantum Groups, and Operator Algebras

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Preliminaries

[Observation]: Given a locally compact, Hausdorff space X, its topological properties are encoded in the (commutative) function algebra $C_0(X)$. For X: compact, we consider $C(X)$.

$C_0(X)$ is an operator algebra, with each $f \in C_0(X)$ regarded as an operator M_f acting on $\mathcal{H} = L^2(X)$, such that $M_f \xi(x) = f(x)\xi(x)$.

$C_0(X)$ is actually a (commutative) C^*-algebra contained in $\mathcal{B}\big(L^2(X)\big)$, together with

$$f \cdot g(x) = f(x)g(x), \qquad f^*(x) = \overline{f(x)}, \qquad \|f\| = \|f\|_\infty = \sup\big|f(x)\big|.$$

i. e. It is closed under *, and $\|\ \|$ satisfies: $\|f^*f\| = \|f\|^2$.

Pretty much "all" the information about X can be described in terms of the C^*-algebra language.
[Example]: For X: compact, Hausdorff, and Y: open subset in X,

$$C_0(Y) = \big\{f \in C(X) : f(x) = 0, \text{ for } x \in X \setminus Y\big\}$$

is a closed ideal in $C(X)$.

Gelfand–Naimark Theorem (1940's): Actually, any commutative C^*-algebra A can be canonically realized as $A \cong C_0(X)$, for some locally compact Hausdorff space X. Here, $X = \mathrm{sp}(A)$ ("spectrum of A") is the set of all complex homomorphisms $\pi : A \to \mathbb{C}$. A very crucial fact is that $\|f^*f\| = \|f\|^2$.

This means that Category of locally compact Hausdorff spaces and Category of commutative C^*-algebras are equivalent. In this sense, working with general (possibly non-commutative) C^*-algebras will be the study of *non-commutative topology/geometry*.

In short, C^*-algebras are considered as "non-commutative spaces".

Similarly, we may consider a weak closure (i. e. von Neumann algebra), and talk about *non-commutative measure theory*.

Chapter 1

Deformation quantization and noncommutative manifolds

1.1 Quantum plane

Traditionally, a quantum mechanical system is modeled using a Hilbert space formalism, associating self-adjoint (possibly unbounded) operators to physical observables. And, the pure quantum states are "rays" in the Hilbert space. The "non-simultaneous observability" of two observables (or, Heisenberg's uncertainty principle) corresponds to the non-commutativity of the corresponding operators.

[Example] (Schrödinger representation): Let $\mathcal{H} = L^2(\mathbb{R})$. Define

- [The position operator]: $Q \in \mathcal{L}(\mathcal{H})$, where
$$Q\xi(x) = x\xi(x).$$
- [The momemtum operator]: $P \in \mathcal{L}(\mathcal{H})$, where
$$P\xi(x) = i\hbar\frac{d\xi(x)}{dx}. \qquad (\hbar\text{: Planck constant})$$
- [The Hamiltonian]: $H \in \mathcal{L}(\mathcal{H})$, where
$$H\xi(x) = -\frac{\hbar^2}{2m}\frac{d^2\xi(x)}{dx^2} + V(x)\xi(x). \qquad (V\text{: some potential function})$$

Note that $QP \neq PQ$. Since
$$PQ\xi(x) = i\hbar\frac{d}{dx}\big(x\xi(x)\big) = i\hbar\xi(x) + i\hbar x\frac{d\xi(x)}{dx} = (i\hbar I + QP)\xi(x),$$
we have: $PQ - QP = i\hbar I$. Or, $[P, Q] = i\hbar I$. This is called the *canonical commutation relation (C.C.R.)*.

Often, a quantum mechanical system is considered to be a quantum version of a certain classical system (with "phase space" M, observables being functions on M). A suitable process for associating operators (quantum observables) to functions (classical observables) is called a **quantization**.

[Our problem]: We wish to know the properties a suitable quantization should satisfy.

[Motivational Example] (Case of one free particle in an n-dimensional space):
Let $\mathbb{R}^n$: configuration space ("position"), with $q = (q_1, q_2, \dots, q_n) \in \mathbb{R}^n$.
Let $\mathbb{R}^{2n}$: state space (or phase space), with $(p;q) = (p_1, \dots, p_n; q_1, \dots, q_n) \in \mathbb{R}^{2n}$.
On $\mathbb{R}^{2n}$, there exists the *Poisson bracket*:

$$\{f, g\} := \sum_{k=1}^{n} \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k}, \qquad \text{for } f, g \in C^\infty(\mathbb{R}^{2n}).$$

In particular, for the coordinate functions p_j, q_k, we have: $\{p_j, q_k\} = \delta_{jk}1$.

A corresponding quantum system should have an operator associated to each function, and in particular, should have self-adjoint operators P_j and Q_k, acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. We require that

$$[P_j, P_k] = [Q_j, Q_k] = 0, \text{ and } [P_j, Q_k] = \delta_{jk} i\hbar I \text{ ("Heisenberg commutation relation").}$$

Hopefully, we can extend the correspondence $p_j \mapsto P_j$, $q_k \mapsto Q_k$ to $C^\infty(\mathbb{R}^{2n})$. But, it is not obvious how to do this!

An answer ...
Weyl quantization (1930's): For convenience, let $n = 1$. Recall that for a "reasonable" function f on $\mathbb{R}^2$, for instance $f \in \mathcal{S}(\mathbb{R}^2)$, a Schwartz function, its Fourier transform can be defined: $f \mapsto f^\wedge$, where

$$f^\wedge(r, s) = \int f(p, q) \exp\big[-i(r, s)\cdot(p, q)\big]\, dpdq.$$

And, with a suitable Plancherel measure, the inverse Fourier transform can be defined:

$$\phi^\vee(p, q) = \int \phi(r, s) \exp\big[i(r, s)\cdot(p, q)\big]\, drds,$$

so that the "Fourier inversion theorem" holds. Namely, $\mathcal{F}^{-1}\big(\mathcal{F}(f)\big) = f$. It also holds that

$$(f \cdot g)^\wedge(x, y) = \int f^\wedge(r, s) g^\wedge(x - r, y - s)\, drds.$$

Motivated by the Fourier inversion theorem, we may quantize the function f by sending it to the operator U_f on $\mathcal{H}$, defined by

$$U_f = \int f^\wedge(r, s) \exp\big[i(rP + sQ)\big]\, drds.$$

This is the "Weyl quantization". [There are other possible prescriptions for quantization.]

[Side remark]: A Schwartz function $f \in \mathcal{S}(\mathbb{R}^2)$ is a C^∞-function such that $m\partial^\alpha f$ is bounded for any polynomial m and a multi-index $\alpha = (\alpha_1, \alpha_2)$. Clearly, $f \in C_0$ and $f \in L^1$. The Schwartz function space is dense in C^∞ and in L^1. If $f \in \mathcal{S}$, then $f^\wedge \in \mathcal{S}$, and the Fourier inversion theorem holds. So the Fourier transform sends $\mathcal{S}$ onto itself.

[A Lie theory result]: By the Baker–Campbell–Hausdorff formula, it can be shown that

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big[X,[X,Y]\big] + \frac{1}{12}\big[Y,[Y,X]\big] + \cdots\right).$$

In our case, $[P,Q] = i\hbar I$ and I is central. So, $\exp(P)\exp(Q) = \exp\big(P + Q + \frac{i\hbar}{2}I\big)$. It follows that $\exp\big[i(rP+sQ)\big] = e^{\frac{i\hbar rs}{2}}\exp(irP)\exp(isQ)$.

For the case of the Schrödinger representation, we have, for $\xi \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\exp(isQ)\xi(x) &= e^{isx}\xi(x),\\
\exp(irP)\xi(x) &= \left(\sum_{k=0}^{\infty}\frac{\big[ir(i\hbar\frac{d}{dx})\big]^k\xi}{k!}\right)(x) = \sum_{k=0}^{\infty}\frac{(-\hbar r)^k}{k!}\frac{d^k\xi(x)}{dx^k}\\
&= \text{ (by Taylor ...) } = \xi(x-\hbar r).
\end{aligned}$$

So we have: $\exp\big[i(rP+sQ)\big]\xi(x) = e^{\frac{i\hbar rs}{2}}\exp(irP)\exp(isQ)\xi(x) = e^{i(x-\hbar r/2)s}\xi(x-\hbar r)$. And, $U_f\xi(x) = \int f^\wedge(r,s)e^{i(x-\hbar r/2)s}\xi(x-\hbar r)\,drds$.

Moyal product: We may interpret the Weyl quantization procedure as a deformation of the algebra $\mathcal{S}(\mathbb{R}^2) \subseteq C^\infty(\mathbb{R}^2)$, via the product $\times_\hbar$ on it given by $U_{f\times_\hbar g} = U_fU_g$. It turns out that the direction of the deformation (parameter $\hbar$) is given by the Poisson bracket. Note that

$$\begin{aligned}
U_{f\times_\hbar g} = U_fU_g &= \int f^\wedge(r,s)g^\wedge(r',s')\exp\big[i(rP+sQ)\big]\exp\big[i(r'P+s'Q)\big]\,drdsdr'ds'\\
&= \int f^\wedge(r,s)g^\wedge(r',s')e^{-\frac{i\hbar}{2}(rs'-r's)}\exp\Big[i\big((r+r')P+(s+s')Q\big)\Big]\,drdsdr'ds'\\
&= \int f^\wedge(r,s)g^\wedge(x-r,y-s)e^{-\frac{i\hbar}{2}\big[r(y-s)-(x-r)s\big]}\exp\big[i\big(xP+yQ\big)\big]\,drdsdxdy,
\end{aligned}$$

so $(f\times_\hbar g)(p,q) = \int f^\wedge(r,s)g^\wedge(x-r,y-s)e^{-\frac{i\hbar}{2}\big[r(y-s)-(x-r)s\big]}\exp\big[i\big(xp+yq\big)\big]\,drdsdxdy.$

Formally, we can write:

$$e^{-\frac{i\hbar}{2}\big[r(y-s)-(x-r)s\big]} = 1 + \frac{i\hbar}{2}\big[(ir)(i(y-s)) - (i(x-r))(is)\big] + (\hbar^2\dots).$$

This observation, together with the fact that $(f\cdot g)^\wedge(x,y) = \int f^\wedge(r,s)g^\wedge(x-r,y-s)\,drds$, and that $\left(\dfrac{\partial f}{\partial p}\right)^\wedge(r,s) = f^\wedge(r,s)(ir)$, ..., it follows that:

$$(f\times_\hbar g)(p,q) = (f\cdot g)(p,q) + \frac{i\hbar}{2}\{f,g\}(p,q) + (\hbar^2\dots).$$

where $\{\ ,\ \}$ is the Poisson bracket as defined earlier.

All this can be carried out even in the C^*-algebra setting. Note first that $\mathcal{S}(\mathbb{R}^2)$, a dense subspace of $C_0(\mathbb{R}^2)$, becomes a *-algebra $\mathcal{A}_\hbar$, together with the product $f \times_\hbar g$ and the involution $f^* = \overline{f}$. Further define a norm:

$$\|f\|_\hbar := \sup\bigl\{\|U_f\| : \ U \text{ is a representation of the C.C.R.)}\bigr\}.$$

It can be shown that $\| \ \|_\hbar$ is a C^*-norm: $\|f^* \times_\hbar f\|_\hbar = \|f\|_\hbar^2$. Also note that $\|f\|_\hbar \le \|f\|_{L^1}$. So, by the Lebesgue's dominated convergence theorem, we can show that

$$\lim_{\hbar \to 0} \left\| \frac{f \times_\hbar g - g \times_\hbar f}{\hbar} - i\{f, g\} \right\|_\hbar = 0.$$

Let us write $A_\hbar$ to be the C^*-algebra completion of $(\mathcal{A}_\hbar, \times_\hbar, {}^*)$, with respect to the norm $\| \ \|_\hbar$. It is easy to see that $A_{\hbar=0} = C_0(\mathbb{R}^2)$. Whereas the non-commutative C^*-algebra $A_\hbar$, for $\hbar \neq 0$, could be considered as a *quantum plane* (or a quantum phase space). It can be further shown that $A_\hbar \cong C_0(\mathbb{R}) \rtimes_\tau \mathbb{R} \cong \mathcal{K}\bigl(L^2(\mathbb{R})\bigr)$. Since $\mathcal{A}_\hbar = \mathcal{S}(\mathbb{R}^2)$ may be regarded as providing the smooth structure on $A_\hbar$, this example should be a *non-commutative manifold*.

[Remark]: Given a group G, a function $\omega : G \times G \to \mathbb{T}$ is called a "2-cocycle", if

$$\omega(xy, z)\omega(x, y) = \omega(x, yz)\omega(y, z), \qquad \text{for } x, y, z \in G.$$

It is a "normalized" cocycle, if $\omega(e_G, x) = \omega(x, e_G) = 1$. In our case, $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{T}$, given by $\omega\bigl((r, s), (r', s')\bigr) := \exp\bigl[i\hbar(r \cdot s' - r' \cdot s)/2\bigr]$, is a normalized 2-cocycle. And, $U : (r, s) \mapsto U_{r,s} = \exp\bigl[i(rP + sQ)\bigr]$ is a "projective unitary representation" of $G = \mathbb{R}^{2n}$ for cocycle ω, satisfying the condition that $U_{r,s}U_{r',s'} = \omega\bigl((r, s), (r', s')\bigr)U_{r+r',s+s'}$. With the machinery of the projective representations and the Fourier transform, we can actually generalize the Weyl quantization to the setting of any locally compact abelian group.

1.2 Deformation quantization of Poisson manifolds

A **Poisson manifold** is a manifold M, equipped with a (bilinear) *Poisson bracket*. That is, we have a map $\{ \ , \ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that for all $f, g, h \in C^\infty(M)$,

1. $\{f, g\} = -\{g, f\}$
2. $\bigl\{f, \{g, h\}\bigr\} + \bigl\{g, \{h, f\}\bigr\} + \bigl\{h, \{f, g\}\bigr\} = 0 \qquad$ (Jacobi identity)
3. $\{f, gh\} = \{f, g\}h + g\{f, h\} \qquad$ (Leibnitz rule)

A Poisson bracket may also be described via a "Poisson bivector field", $\omega_M = \sum X_i \wedge X_j$, where X_i, X_j are tangent vectors. Then we have: $\{f, g\} = \bigl\langle df \wedge dg, \omega_M \bigr\rangle$. Note that for $f \in C^\infty(M)$, there is a "hamiltonian vector field" $X_f : g \mapsto \{g, f\}$, $g \in C^\infty(M)$.

(*) Loosely speaking, a Poisson manifold is a possible candidate to try a quantization.

[Examples of Poisson manifolds]:

1. Trivial Poisson bracket on any M, for which $\{f, g\} \equiv 0$.
2. On $M = \mathbb{R}^{2n}$,
$$\{f, g\} = \sum_{i,j=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right).$$
It is *symplectic* (non-degenerate everywhere).
3. Let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra. Consider $M = \mathfrak{g}^*$. Then the tangent space of $M = \mathfrak{g}^*$ at any point can be canonically identified with $\mathfrak{g}^*$ itself. So for any $\xi \in \mathfrak{g}$ and any $f \in C^\infty(\mathfrak{g}^*)$, we have: $(df)_\xi \in \mathfrak{g}$. Considering this, define on $C^\infty(\mathfrak{g}^*)$ the following *Linear (or Lie–Poisson) P.B.*:
$$\{f_1, f_2\}_{\mathrm{lin}}(\xi) = \big\langle \big[(df_1)_\xi, (df_2)_\xi\big], \xi \big\rangle, \qquad \forall \xi \in \mathfrak{g}^*.$$
It is "linear", in the sense that the Poisson bracket of linear functions is a linear map. Clearly, this one is not symplectic. All linear Poisson brackets are obtained in this way.

Deformation quantization of $\big(C^\infty(M), \{\ ,\ \}\big)$ is to find on $C^\infty(M)$ a deformed product, $\times_\hbar$, such that the direction of the deformation is given by the Poisson bracket. That is,
$$\frac{f \times_\hbar g - g \times_\hbar f}{\hbar} \longrightarrow i\{f, g\}, \qquad \text{as } \hbar \to 0.$$
Deformation quantization is usually carried out in terms of formal power series in $\hbar$, via the "star product" (Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer).

If the Poisson bracket on M comes from a symplectic structure, a deformation quantization exists (De Wilde, Lecompte), while the case of a general Poisson manifold was considered by Kontsevich. Both are in the formal power series setting. However, it is not known if this can be generalized to the C^*-algebra framework.

Strict deformation quantization (Rieffel, 1990's): Given a Poisson manifold M, by a "strict deformation quantization" we mean a dense *-subalgebra $\mathcal{A}$ of $C_0(M)$ on which $\{\ ,\ \}$ is defined, together with a family $\big(\times_\hbar, {}^{*_\hbar}, \|\ \|_\hbar\big)$ of C^*-algebra structures for $\hbar \in$ (some interval of $\mathbb{R}$ containing 0), such that

1. The completed C^*-algebras $A_\hbar = \overline{(\mathcal{A}, \times_\hbar, {}^{*_\hbar})}^{\|\ \|_\hbar}$ form a continuous field of C^*-algebras;
2. For $\hbar = 0$, we have: $A_{\hbar=0} \cong C_0(M)$;
3. We have the "correspondence relation": $\lim_{\hbar \to 0} \left\| \dfrac{f \times_\hbar g - g \times_\hbar f}{\hbar} - i\{f, g\} \right\|_\hbar = 0$.

(*) Clearly, the quantum plane earlier is an example of a strict deformation quantization. There are other examples, some of which will appear in § 1.3 below. However, it turns out that this notion needs to be weakened a little, to allow for some other cases (More on this matter in Chapter 3.).

1.3 Some examples of non-commutative manifolds

(1). Quantum plane: Seen in § 1.1.

(2). Quantum torus: Let $\Theta = (\theta_{jk})$ be a fixed $n \times n$ skew matrix (i. e. $\theta_{jk} = -\theta_{kj}$). On $C^\infty(\mathbb{T}^n)$, define the Poisson bracket:

$$\{f, g\} := \frac{1}{\pi} \sum_{j,k} \theta_{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}.$$

Write $e(t) = e^{2\pi i t}$, and let $\lambda_{jk} = e(2\theta_{jk})$. We wish to consider the universal C^*-algebra A_Θ generated by the unitary operators $U_1, \dots, U_n$, such that $U_k U_j = \lambda_{jk} U_j U_k$, for $j < k$. So for any $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$, let $U_p = U_1^{p_1} \cdots U_n^{p_n}$. Define also $\omega(p, q) = e(\Theta p \cdot q)$. We will require that $U_p U_q = \omega(p, q) U_{p+q}$. Since $U_{e_j} = U_j$ and $U_{e_k} = U_k$, this is consistent with $U_k U_j = e(\theta_{jk}) U_{e_k + e_j} = e(\theta_{jk}) \bar{e}(\theta_{kj}) U_j U_k = \big(e(\theta_{jk})\big)^2 U_j U_k$, or $U_k U_j = \lambda_{jk} U_j U_k$. Using the terminology that appeared earlier, we see that U is a projective unitary representation of $G = \mathbb{Z}^n$, for the cocycle ω.

As before, we may define a product and an involution on a dense subspace of $C_0(\mathbb{Z}^n)$. That is, for $\phi, \psi \in \mathcal{S}(\mathbb{Z}^n)$, define:

$$(\phi *_\omega \psi)(p) = \sum_{q \in \mathbb{Z}^n} \phi(q) \psi(p - q) \omega(q, p - q), \qquad \text{and} \qquad \phi^*(p) = \overline{\phi(-p)}.$$

This makes $\mathcal{S}(\mathbb{Z}^n)$ a *-algebra. Noting that the abelian groups $\mathbb{Z}^n$ and $\mathbb{T}^n$ are in Pontryagin duality and using the Fourier transform, we know that $\mathcal{S}(\mathbb{Z}^n) \cong \mathcal{S}(\mathbb{T}^n)$. The *-algebra structure above can carry over to $\mathcal{S}(\mathbb{T}^n)$, by $f \times_\Theta g = (f^\wedge *_\omega g^\wedge)^\vee$.

Any ω-projective representation U of $\mathbb{Z}^n$ provides a *-representation of $\mathcal{S}(\mathbb{T}^n)$, via $f \mapsto U_f$, where $U_f := \sum_{p \in \mathbb{Z}^n} f^\wedge(p) U_p$. So we can define a C^*-norm on $\mathcal{S}(\mathbb{T}^n)$ as follows:

$$\|f\|_{C^*} = \sup\big\{\|U_f\| : \ U \text{ a } \omega\text{-projective representation of } \mathbb{Z}^n\big\}.$$

Clearly, $\|f\|_{C^*} \leq \|f^\wedge\|_1$. Define A_Θ as the C^*-completion $\overline{\big(\mathcal{S}(\mathbb{T}^n), \times_\Theta, {}^*\big)}^{\|\ \|_{C^*}}$. When $\Theta \equiv 0$ (so $\omega \equiv 1$), we have: $A_0 \cong C(\mathbb{T}^n)$. In this sense, we may regard A_Θ as a *quantum torus*.

Meanwhile, the C^*-algebras $A_{\hbar\Theta}$ forms a continuous field of C^*-algebras as $\hbar$ varies. Moreover, it is not difficult to show that

$$\lim_{\hbar \to 0} \left\| \frac{f \times_{\hbar\Theta} g - g \times_{\hbar\Theta} g}{\hbar} - i\{f, g\} \right\|_{\hbar\Theta} = 0.$$

To prove the result, first show the pointwise convergence, using the fact that $\left(\dfrac{\partial f}{\partial x_j}\right)^\wedge (p) = f^\wedge(p)(2\pi i p_j)$, $\dots$, and the definition of the Poisson bracket. Then we obtain the L^1-convergence by the Lebesgue's dominated convergence theorem. Since all the C^*-norms are bounded above by the L^1-norm, we obtain the result. What all this means is that the $A_{\hbar\Theta}$, as $\hbar$ varies, provides a strict deformation quantization of the Poisson manifold $\big(\mathbb{T}^n, \{\ ,\ \}\big)$.

[Remark]: The quantum torus is among the more successful examples of non-commutative manifolds. Together with a suitable "Dirac operator", one can form a "spectral triple", giving us an example in *noncommutative differential geometry* (in the sense of Connes). A metric, and the notions like Chern character, Yang–Mills action, ... can be considered.

(3). Quantization of the linear (Lie–Poisson type) Poisson bracket (Rieffel):

Recall the *linear Poisson bracket* on the dual vector space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. For our purposes, we will assume that $\mathfrak{g}$ is a nilpotent or exponential Lie algebra, so by the exponential map we can regard $\mathfrak{g} \cong G$, where G denotes the (connected and simply connected) Lie group corresponding to $\mathfrak{g}$.

For the deformation parameter $\hbar$, define a Lie bracket $[\ ,\ ]_\hbar$ on $\mathfrak{g}$ by $[X,Y]_\hbar := \frac{1}{\hbar}[\hbar X, \hbar Y]$. The Lie group corresponding to $\mathfrak{g}_\hbar := \big(\mathfrak{g}, [\ ,\ ]_\hbar\big)$ is denoted by $G_\hbar$, and clearly, $\mathfrak{g}_\hbar \cong G_\hbar$ as spaces. For $\hbar = 0$, we will have $G_{\hbar=0}$ abelian, and we may regard $G_{\hbar=0} = (\mathfrak{g}, +)$. Then at the level of Schwartz functions, we have:

$$\mathcal{S}(\mathfrak{g}^*) \cong \mathcal{S}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g}_\hbar) \cong \mathcal{S}(G_\hbar),$$

where the first $\cong$ is via the Fourier transform and the last one is due to $\mathfrak{g}_\hbar \cong G_\hbar$. On $\mathcal{S}(G_\hbar)$, we can define the convolution product, $*_\hbar$, inherited from $L^1(G_\hbar)$. Unless $G_\hbar$ is abelian, the convolution product is noncommutative.

This means that on the dense subspace $\mathcal{A} := \mathcal{S}(\mathfrak{g}^*)$ of $C_0(\mathfrak{g}^*)$, we can define a family of products $\times_\hbar$, by $f \times_\hbar g := (f^\wedge *_\hbar g^\wedge)^\vee$. Similarly, we can define a family of involutions ${}^{*_\hbar}$, also inherited from $L^1(G_\hbar)$. It is not difficult to see that when $\hbar = 0$, the product and the involution are just the pointwise multiplication and the complex conjugation on $\mathcal{A} \subseteq C_0(\mathfrak{g}^*)$.

Write $\mathcal{A}_\hbar := (\mathcal{A}, \times_\hbar, {}^{*_\hbar})$. For each *-algebra $\mathcal{A}_\hbar$, we can give it a C^*-norm $\|\ \|_\hbar$, inherited from the enveloping C^*-algebra of $L^1(G_\hbar)$. Denote by $A_\hbar$ the C^*-completion of $\mathcal{A}_\hbar$ with respect to $\|\ \|_\hbar$. In this way, we obtain a continuous field of C^*-algebras $\{A_\hbar\}$. Note that $A_{\hbar=0} \cong C_0(\mathfrak{g}^*)$, with its natural commutative C^*-algebra structure. And, $A_\hbar \cong C^*(G_\hbar)$. By using a similar reasoning as in cases (1) and (2), we can also show that

$$\lim_{\hbar\to 0} \left\| \frac{f \times_\hbar g - g \times_\hbar f}{\hbar} - \frac{i}{2\pi}\{f,g\}_{\text{lin}} \right\|_\hbar = 0,$$

for $f, g \in \mathcal{A}$. This means that the $\{A_\hbar\}_{\hbar\in\mathbb{R}}$ provides a strict deformation quantization of the Poisson manifold $(\mathfrak{g}^*, \frac{1}{2\pi}\{\ ,\ \}_{\text{lin}})$.

(4). Quantization of certain non-linear Poisson bracket (Kahng):

We may generalize the case (3) to a certain class of non-linear Poisson brackets, in the form of $\{\ ,\ \}_{\text{lin}}+$ (some cocycle term). See below.

Let $\mathfrak{h}$ be a Lie algebra with center $\mathfrak{z}$, and let $\mathfrak{h}^*$ denote the dual vector space of $\mathfrak{h}$. Consider the vector space $V = C^\infty(\mathfrak{h}^*/\mathfrak{z}^\perp)\ \big(\subseteq C^\infty(\mathfrak{h}^*)\big)$, and give it the trivial $U(\mathfrak{h})$-module structure. Let Ω be a Lie algebra 2-cocycle for $\mathfrak{h}$ having values in V, satisfying the centrality condition: That is, Ω is a skew-symmetric bilinear map from $\mathfrak{h} \times \mathfrak{h}$ into V such that

$$\Omega\big(X_1, [X_2, X_3]\big) + \Omega\big(X_2, [X_3, X_1]\big) + \Omega\big(X_3, [X_1, X_2]\big) = 0, \qquad X_1, X_2, X_3 \in \mathfrak{h},$$

and for $Z \in \mathfrak{z}$, we have $\Omega(Z, X) = \Omega(X, Z) = 0$ for any $X \in \mathfrak{h}$.

Then we may regard $\mathfrak{h}+V$ as a Lie algebra, with the bracket on it given by

$$[X+v, Y+w]_{\mathfrak{h}+V} = [X,Y] + \Omega(X,Y), \qquad X,Y \in \mathfrak{h},\ v,w \in V.$$

Motivated by this, define a Poisson bracket $\{\ ,\ \}_\Omega : C^\infty(\mathfrak{h}^*) \times C^\infty(\mathfrak{h}^*) \to C^\infty(\mathfrak{h}^*)$, as follows:

$$\{f,g\}_\Omega(\mu) := \big\langle \big[(df)_\mu, (dg)_\mu\big], \mu\big\rangle + \Omega\big((df)_\mu, (dg)_\mu; \mu\big), \qquad \text{for } \mu \in \mathfrak{h}^*.$$

Note that if we denote $(df)_\mu$ and $(dg)_\mu$ by X and Y, as considered as elements in $\mathfrak{h}$, the right hand side is just the evaluation, at $\mu \in \mathfrak{h}^*$, of $[X,Y]+\Omega(X,Y) \in \mathfrak{h}+V \subseteq C^\infty(\mathfrak{h}^*)$, which is none other than the Lie bracket on $\mathfrak{h}+V$ above. Using the property of the Lie bracket and the cocycle identity of Ω, as well as the fact that $(d\chi)_\mu \in \mathfrak{z}$ for $\chi \in V$, we can show that $\{\ ,\ \}_\Omega$ is indeed a Poisson bracket on on $\mathfrak{h}^*$. We may regard this (non-linear) Poisson bracket as a "cocycle perturbation" of $\{\ ,\ \}_{\text{lin}}$ on $\mathfrak{h}^*$. This definition generalizes the notion of an "affine Poisson bracket", where Ω is a scalar-valued cocycle.

It turns out that the Poisson bracket $\{\ ,\ \}_\Omega$ above can be viewed as a "central extension" of the linear Poisson bracket on the space $(\mathfrak{h}/\mathfrak{z})^*$. This actually follows from the fact that the Lie bracket $[\ ,\ ]_{\mathfrak{h}+V}$ earlier can be transferred to a Lie bracket on $\mathfrak{h}/\mathfrak{z} \oplus V$. We will skip the details.

Suppose, from now on, that $\mathfrak{h}$ is nilpotent, and consider the non-linear Poisson bracket $\{\ ,\ \}_\Omega$ on $\mathfrak{h}^*$. (Some generalization is possible, but needs a more relaxed version of the deformation quantization framework.) As noted above, we may as well regard $\{\ ,\ \}_\Omega$ as a central extension of the linear Poisson bracket on $(\mathfrak{h}/\mathfrak{z})^*$. Let $K = H/Z$ be the Lie group corresponding to $\mathfrak{h}/\mathfrak{z}$, and as before, consider groups $K_\hbar$, $\hbar \in \mathbb{R}$, corresponding to the Lie bracket $(x,y) \mapsto \frac{1}{\hbar}[\hbar x, \hbar y]$ on $\mathfrak{h}/\mathfrak{z}$.

Fix $\hbar \in \mathbb{R}$. From the Ω information, it is possible to construct a continuous field $\sigma_\hbar : \mathfrak{h}^*/\mathfrak{z}^\perp \ni r \mapsto \sigma_\hbar^r$, where each $\sigma_\hbar^r : K_\hbar \times K_\hbar \to \mathbb{T}$ is a group 2-cocycle. Define the twisted crossed product C^*-algebras $A_\hbar = C^*\big(K_\hbar, C_0(\mathfrak{h}^*/\mathfrak{z}^\perp), \sigma_\hbar\big)$. It can be shown that the $\{A_\hbar\}_{\hbar\in\mathbb{R}}$ provides a strict deformation quantization of the Poisson manifold $\big(\mathfrak{h}^*, \frac{1}{2\pi}\{\ ,\ \}_\Omega\big)$.

Sketch of Proof. For $f,g \in C_c^\infty(K_\hbar; \mathfrak{h}^*/\mathfrak{z}^\perp)$, define the twisted convolution product, $*_\hbar$, by

$$(f *_\hbar g)(x;r) = \int f(z;r) g(z^{-1} \cdot_\hbar x; r) \sigma_\hbar^r(z, z^{-1} \cdot_\hbar x)\, dx,$$

where $\cdot_\hbar$ denotes the multiplication on the group $K_\hbar$. Consider also the involution, given by $f^*(x;r) = \overline{f(x^{-1};r)\sigma_\hbar^r(x,x^{-1})}\Delta_{K_\hbar}(x^{-1})$. Viewing the *-algebra $C_c^\infty(K_\hbar; \mathfrak{h}^*/\mathfrak{z}^\perp)$ as a subalgebra of the L^1-algebra $L^1\big(K_\hbar, C_0(\mathfrak{h}^*/\mathfrak{z}^\perp)\big)$, and by considering its enveloping C^*-algebra, we obtain the C^*-algebra $A_\hbar$. Since the groups $K_\hbar$ here are amenable (being nilpotent), we have no distinction between the "full" and the "reduced" versions of the twisted crossed product C^*-algebras. Therefore, by a result of Rieffel, it turns out that the $\{A_\hbar\}_{\hbar\in\mathbb{R}}$ forms a continuous field of C^*-algebras.

By the partial Fourier transform, the twisted convolution products $*_\hbar$ can be transferred to the deformed products $\times_\hbar$ on a dense subspace $\mathcal{A}$ of $C_0(\mathfrak{h}^*)$. The C^*-algebras $A_\hbar$ may be regarded as the completions of the $(\mathcal{A}, \times_\hbar)$. When $\hbar = 0$, the group $K_{\hbar=0}$ is isomorphic to an additive abelian group $(\mathfrak{h}/\mathfrak{z}, +)$, and $\sigma_\hbar \equiv 1$, thereby giving us $A_{\hbar=0} \cong C^*\big(\mathfrak{h}/\mathfrak{z}, C_0(\mathfrak{h}^*/\mathfrak{z}^\perp)\big) \cong C^*(\mathfrak{h}, +) \cong C_0(\mathfrak{h}^*)$. Finally, we can also verify the correspondence relation, first by showing the pointwise convergence, then by utilizing the Lebesgue dominated convergence theorem and the fact that the C^*-norms are dominated by the L^1-norm. $\square$

[Remark]: This class of Poisson brackets is fairly general as is, but by developing a weaker notion of the deformation quantization framework, we may be able to further generalize the situation, for instance by incorporating some actions as well as the cocycles. Meanwhile, we may also introduce some groupoid algebra notions. See Chapter 3 for further discussion on these matters.

[Remark]: Using this deformation quantization framework for the case of certain Poisson–Lie groups having non-linear Poisson structure of the above type, the author could construct some examples of locally compact quantum groups, in the C^*-algebra framework.

1.4 Poisson–Lie groups

We are in particular interested in constructing *quantum groups* (more on quantum groups later, in Chapter 2). One possible approach is by deformation quantization of ordinary groups. For this, we need to begin with a *Poisson–Lie group*, which is a Lie group whose underlying manifold is a Poisson manifold equipped with a Poisson bracket compatible with the group structure.

[Some preliminaries]: (1). Given two Poisson manifolds M and N, a smooth map $\Phi : M \to N$ is called a "Poisson map", if

$$\{f_1, f_2\}_N \circ \Phi = \{f_1 \circ \Phi, f_2 \circ \Phi\}_M$$

for all $f_1, f_2 \in C^\infty(N)$.

(2). If M and N are Poisson manifolds, the product manifold $M \times N$ is a Poisson manifold, equipped with the "product Poisson structure", given by

$$\{f_1, f_2\}_{M\times N}(x, y) = \big\{f_1(x,\ ), f_2(x,\ )\big\}_N(y) + \big\{f_1(\ , y), f_2(\ , y)\big\}_M(x).$$

[Definition]: A Lie group G (itself a manifold) is a **Poisson–Lie group**, if it is equipped with a Poisson bracket $\{\ ,\ \}$ on $C^\infty(G)$ such that the multiplication map $\mu : G \times G \to G$, $\mu(x, y) = xy$, of G is a Poisson map, where $G \times G$ is given the product Poisson structure.

(*) However, the inversion map $\iota : x \mapsto x^{-1}$ is in general not a Poisson map. Neither for the left translations and right translations.

Remembering the product structure, the compatibility condition becomes:

$$\{f_1, f_2\}(xy) = \{f_1 \circ L_x, f_2 \circ L_x\}(y) + \{f_1 \circ R_y, f_2 \circ R_y\}(x),$$

for $f_1, f_2 \in C^\infty(G)$ and $x, y \in G$.

Let G be a Poisson–Lie group. Then it turns out that its Lie algebra $\mathfrak{g}(=T_eG)$ has a natural *Lie bialgebra* structure, via a certain cobracket $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$. It also turns out that $\mathfrak{g}^*$ is itself a Lie algebra. In fact, for $\xi_1,\xi_2\in\mathfrak{g}^*$, find $f_1,f_2\in C^\infty(G)$ such that $\xi_i=(df_i)_e$. Then:

$$[\xi_1,\xi_2]_{\mathfrak{g}^*}:=\big(d\{f_1,f_2\}\big)_e.$$

It is well-defined. That is, $[\xi_1,\xi_2]_{\mathfrak{g}^*}$ does not depend on the choice of f_1 and f_2. Actually,

$$[\xi_1,\xi_2]_{\mathfrak{g}^*}(X)=\big\langle\delta(X),\xi_1\otimes\xi_2\big\rangle=\big\langle X,\delta^*(\xi_1\otimes\xi_2)\big\rangle,\qquad\text{for } X\in\mathfrak{g}.$$

Let us try to be a little more precise ...

Given the Poisson bracket $\{\ ,\ \}$ on $C^\infty(G)$, recall that we have the (skew-symmetric) Poisson bivector $\omega:x\mapsto\omega_x\in T_xG\otimes T_xG$ such that

$$\{f_1,f_2\}(x)=\big\langle\omega_x,(df_1)_x\otimes(df_2)_x\big\rangle.$$

From the compatibility condition of the Poisson bracket, we have:

$$\big\langle\omega_{xy},(df_1)_{xy}\otimes(df_2)_{xy}\big\rangle=\big\langle\omega_y,d(f_1\circ L_x)_y\otimes d(f_2\circ L_x)_y\big\rangle+\big\langle\omega_x,d(f_1\circ R_y)_x\otimes d(f_2\circ R_y)_x\big\rangle.$$

By chain rule, $d(f_1\circ L_x)_y=(df_1)_{xy}\cdot(L_x)'_y$, etc., It follows that

$$\omega_{xy}=\big((L_x)'_y\otimes(L_x)'_y\big)(\omega_y)+\big((R_y)'_x\otimes(R_y)'_x\big)(\omega_x).$$

Meanwhile, we may define $\omega^R:G\to\mathfrak{g}\otimes\mathfrak{g}$, by bringing back (via the right translation) the Poisson bivector ω of G to $e\in G$. We then have:

$$\omega^R(xy)=(\mathrm{Ad}_x\otimes\mathrm{Ad}_x)\omega^R(y)+\omega^R(x).$$

This means that ω^R is a *1-cocycle* of G having values in $\mathfrak{g}\otimes\mathfrak{g}$, where G acts on $\mathfrak{g}\otimes\mathfrak{g}$ by Ad-representation. [c.f. $\Phi:G\to A$ such that $\Phi(xy)=\Phi(x)+x.\Phi(y)$.]

From $\omega^R:G\to\mathfrak{g}\otimes\mathfrak{g}$, obtain: $\delta=(d\omega^R)_e:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$. Since ω^R is a 1-cocycle for the Ad-representation of G on $\mathfrak{g}\otimes\mathfrak{g}$, it follows that δ is a (Lie algebra) 1-cocycle for the ad-representation of $\mathfrak{g}$ on $\mathfrak{g}\otimes\mathfrak{g}$. That is,

$$\delta\big([X,Y]\big)=X.\delta(Y)-Y.\delta(X)=(\mathrm{ad}_X\otimes 1+1\otimes\mathrm{ad}_X)\delta(Y)-(\mathrm{ad}_Y\otimes 1+1\otimes\mathrm{ad}_Y)\delta(X).$$

Moreover, by construction, we have:

$$[\xi_1,\xi_2]_{\mathfrak{g}^*}(X)=\big(d\{f_1,f_2\}\big)_e(X)=\big\langle(d\omega^R)_e(X),(df_1)_e\otimes(df_2)_e\big\rangle=\big\langle\delta(X),\xi_1\otimes\xi_2\big\rangle.$$

[Definition]: Let $\mathfrak{g}$ be a Lie algebra. A **Lie bialgebra** structure on $\mathfrak{g}$ is a skew-symmetric linear map (the "cobracket") $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ such that

- On $\mathfrak{g}^*$, the dual map $\delta^*:\mathfrak{g}^*\otimes\mathfrak{g}^*\to\mathfrak{g}^*$ determines a Lie bracket.
- δ is a (Lie algebra) 1-cocycle for the ad-representation of $\mathfrak{g}$ on $\mathfrak{g}\otimes\mathfrak{g}$.

[Fact]: If G is a Poisson–Lie group, then its Lie algebra $\mathfrak{g}$ has a natural Lie bialgebra structure. Conversely, if G is connected and simply connected, every Lie bialgebra structure on $\mathfrak{g}$ determines a unique Poisson structure on G, making it into a Poisson–Lie group.

(*) Therefore, to obtain a Poisson–Lie group, we may look for a Lie algebra equipped with a suitable cobracket!

[Remark]: If $(\mathfrak{g}, \delta)$ is a Lie bialebra, then $(\mathfrak{g}^*, \theta)$ is also a Lie bialgebra, where $\theta^* : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket on $\mathfrak{g}$. Similarly, at the group level, if $\bigl(G, \{\ ,\ \}_G\bigr)$ is a Poisson–Lie group, we can consider its *dual Poisson–Lie group* $\bigl(G^*, \{\ ,\ \}_{G^*}\bigr)$ corresponding to the Lie bialgebra $(\mathfrak{g}^*, \theta)$.

Among the 1-cocycles for $\mathfrak{g}$ are 1-coboundaries, in the form of

$$\delta(X) = X.(r) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)(r),$$

for some $r \in \mathfrak{g} \otimes \mathfrak{g}$. For such a δ to determine a Lie bialgebra structure on $\mathfrak{g}$ (obtaining a "coboundary Lie bialgebra"), we need:

- $r_{12} + r_{21}$ is $\mathfrak{g}$-invariant (i. e. $X.(r_{12} + r_{21}) = 0$, for all $X \in \mathfrak{g}$).
- $\bigl[r_{12}, r_{13}\bigr] + \bigl[r_{12}, r_{23}\bigr] + \bigl[r_{13}, r_{23}\bigr]$ is $\mathfrak{g}$-invariant.

Here, for $r = \sum_i (a_i \otimes b_i)$, we mean: $r_{21} = \sum_i b_i \otimes a_i$, and $\bigl[r_{13}, r_{23}\bigr] = \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j]$, ...

In particular, we would obtain a Lie bialgebra structure on $\mathfrak{g}$ if we can find $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $r_{12} + r_{21}$ is $\mathfrak{g}$-invariant and satisfies the so-called "Classical Yang–Baxter equation (CYBE)":

$$\bigl[r_{12}, r_{13}\bigr] + \bigl[r_{12}, r_{23}\bigr] + \bigl[r_{13}, r_{23}\bigr] = 0.$$

If so, the resulting Lie bialgebra is *quasitriangular*. If, in addition, $r_{12} + r_{21} = 0$ (i. e. r is skew-symmetric), then we have a *triangular* Lie bialgebra.

[Some examples]: Let $\mathfrak{h}$ be the $(2n+1)$-dimensional Heisenberg Lie algebra, with basis vectors $\mathbf{x}_i$, $\mathbf{y}_i$ $(i = 1, \dots, n)$, $\mathbf{z}$, satisfying the following relations:

$$[\mathbf{x}_i, \mathbf{y}_j] = \delta_{ij}\mathbf{z}, \qquad [\mathbf{x}_i, \mathbf{x}_j] = [\mathbf{y}_i, \mathbf{y}_j] = [\mathbf{z}, \mathbf{x}_i] = [\mathbf{z}, \mathbf{y}_i] = 0.$$

Writing $x = x_1\mathbf{x}_1 + \cdots + x_n\mathbf{x}_n$, etc., we have: $\bigl[(x, y, z), (x', y', z')\bigr]_{\mathfrak{h}} = (0, 0, x \cdot y' - x' \cdot y)$, for $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Consider also $\mathbf{d}$ such that

$$[\mathbf{d}, \mathbf{x}_i] = \mathbf{x}_i, \qquad [\mathbf{d}, \mathbf{y}_i] = -\mathbf{y}_i, \qquad [\mathbf{d}, \mathbf{z}] = 0.$$

Then $\tilde{\mathfrak{h}} = \mathrm{span}\bigl(\mathbf{x}_i, \mathbf{y}_i (i = 1, \dots, n), \mathbf{z}, \mathbf{d}\bigr)$ is the extended Heisenberg Lie algebra.

We have the following examples of classical r-matrices, giving us different Lie bialgebra structures on $\mathfrak{h}$.

1. Let $r = \lambda(\mathbf{z} \otimes \mathbf{d} - \mathbf{d} \otimes \mathbf{z})$, $\lambda \neq 0$. It determines a "triangular" Lie bialgebra structure on $\tilde{\mathfrak{h}}$. Restricted to $\mathfrak{h}$, we have the following cobracket $\delta_1 : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$.

$$\delta_1(\mathbf{x}_j) = \lambda \mathbf{x}_j \wedge \mathbf{z}, \qquad \delta_1(\mathbf{y}_j) = -\lambda \mathbf{y}_j \wedge \mathbf{z}, \qquad \delta_1(\mathbf{z}) = 0.$$

2. Let $r = 2\lambda\left(\sum_{i=1}^n (\mathbf{x}_i \otimes \mathbf{y}_i) + \frac{1}{2}(\mathbf{z}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{z})\right)$, $\lambda \neq 0$. It determines a "quasitriangular" Lie bialgebra structure on $\tilde{\mathfrak{h}}$. Restricted to $\mathfrak{h}$, we have the following cobracket $\delta_2 : \mathfrak{h} \to \mathfrak{h}\wedge\mathfrak{h}$.

$$\delta_2(\mathbf{x}_j) = \lambda\mathbf{x}_j \wedge \mathbf{z}, \qquad \delta_2(\mathbf{y}_j) = \lambda\mathbf{y}_j \wedge \mathbf{z}, \qquad \delta_2(\mathbf{z}) = 0.$$

3. Let $r = \sum_{i,j=1}^n J_{ij}\mathbf{x}_j \otimes \mathbf{x}_i$, where (J_{ij}) is skew, $n \times n$ matrix ($n \geq 2$). It determines a "triangular" Lie bialgebra structure on $\mathfrak{h}$, given by the following cobracket $\delta_3 : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$.

$$\delta_3(\mathbf{x}_j) = 0, \qquad \delta_3(\mathbf{y}_j) = \sum_{i=1}^n J_{ij}\mathbf{x}_i \wedge \mathbf{z}, \qquad \delta_3(\mathbf{z}) = 0.$$

[Remark]: Corresponding to each of these Lie bialgebra structures $(\mathfrak{h}, \delta)$, we can find the Poisson–Lie group H, the dual Lie bialgebra $(\mathfrak{g} = \mathfrak{h}^*, \theta)$, and the dual Poisson–Lie group G. It turns out that Case (1) gives a linear Poisson bracket on G, while Case (2) and (3) give rise to non-linear Poisson brackets on G, of our "cocycle perturbation" type. See below:

1. The Poisson bracket on $C^\infty(\mathfrak{h}^*)$ dual to $(\mathfrak{h}, \delta_1)$ is:

$$\{\phi, \psi\}(p, q, r) = r(x \cdot y' - x' \cdot y),$$

for $\phi, \psi \in C^\infty(G)$. Here $d\phi(p, q, r) = (x, y, z)$ and $d\psi(p, q, r) = (x', y', z')$, which are naturally considered as elements of $\mathfrak{h}$.

2. The Poisson bracket on $C^\infty(\mathfrak{h}^*)$ dual to $(\mathfrak{h}, \delta_2)$ is:

$$\{\phi, \psi\}(p, q, r) = \left(\frac{e^{2\lambda r} - 1}{2\lambda}\right)(x \cdot y' - x' \cdot y), \qquad \text{for } \phi, \psi \in C^\infty(G).$$

Again, $d\phi(p, q, r) = (x, y, z)$ and $d\psi(p, q, r) = (x', y', z')$, as elements of $\mathfrak{h}$.

3. The Poisson bracket on $C^\infty(\mathfrak{h}^*)$ dual to $(\mathfrak{h}, \delta_3)$ is:

$$\{\phi, \psi\}(p, q, r) = r(x \cdot y' - x' \cdot y) + \frac{r^2}{2}\sum_{k,j=1}^n J_{kj}(y_j y_k' - y_k y_j'), \qquad \text{for } \phi, \psi \in C^\infty(G).$$

Again, $d\phi(p, q, r) = (x, y, z)$ and $d\psi(p, q, r) = (x', y', z')$, as elements of $\mathfrak{h}$.

Therefore, as we have seen in §1.3 (3), (4), we can use the framework of twisted crossed product C^*-algebras to carry out the deformation quantization of these Poisson structures. Since the underlying Poisson manifold is actually a group, it is possible (though non-trivial) to construct suitable extra structure maps (including the comultiplication map) on the C^*-algebra, to obtain an example of a **(C^*-algebraic) locally compact quantum group**, which may be considered as a "quantized $C_0(G)$".

(*). The category of quantum groups is large, and it contains more examples than those obtained as deformations of Poisson–Lie groups. A more careful discussion on locally compact quantum groups is given in Chapter 2 below.

Chapter 2

Locally compact quantum groups

Loosely speaking, the category of *quantum groups* is a generalized notion that contains the ordinary groups as special cases. Properly defining it is difficult, but let us begin with the notion of a *Hopf algebra*.

2.1 Hopf algebras

The theory of Hopf algebras is purely algebraic, so the tensor products $\otimes$ in this section will be all algebraic. See standard references (Abe, Sweedler).

A **Hopf algebra** (over $\mathbb{C}$) is a pair (A, Δ), where A is an algebra over $\mathbb{C}$ with identity, and $\Delta : A \to A \otimes A$ is a unital homomorphism (called the *comultiplication*) satisfying the "coassociativity" condition: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, such that there exists a linear map $S : A \to A$ and a linear map $\epsilon : A \to \mathbb{C}$, satisfying

$$(\epsilon \otimes \mathrm{id})\Delta(x) = x = (\mathrm{id} \otimes \epsilon)\Delta(x), \quad \text{and} \quad m\big((S \otimes \mathrm{id})\Delta(x)\big) = \epsilon(x)1 = m\big((\mathrm{id} \otimes S)\Delta(x)\big),$$

or,

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & A \arrow[d, "\cong"] \\ A \otimes A \arrow[r, "\epsilon\otimes\mathrm{id}"] & \mathbb{C} \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & A \arrow[d, "\cong"] \\ A \otimes A \arrow[r, "\mathrm{id}\otimes\epsilon"] & A \otimes \mathbb{C} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}\circ\epsilon"] \arrow[d, "\Delta"'] & A & A \arrow[l, "\mathrm{id}\circ\epsilon"'] \arrow[d, "\Delta"] \\ A \otimes A \arrow[r, "S\otimes\mathrm{id}"] & A \otimes A \arrow[u, "m"] & A \otimes A \arrow[l, "\mathrm{id}\otimes S"] \end{tikzcd}$$

Here : $A\otimes A \to A$ is the multiplication map $m(a\otimes b) = ab$ for all $a, b \in A$. Whenever (A, Δ) is a Hopf algebra, the maps S and ϵ are uniquely determined. We call S the *antipode* and ϵ the *counit* of (A, Δ). We have that S is an anti-homomorphism and that ϵ is a homomorphism.

(EX 1.) To see that the Hopf algebras could be considered as generalizations of ordinary groups, consider a finite group G. Let $K(G)$ denote the set of all complex functions on G. Here, define $\Delta : K(G) \to K(G) \otimes K(G) = K(G \times G)$ by $(\Delta(f))(s, t) = f(st)$. Then we have the coassociativity condition: $\big((\Delta\otimes \mathrm{id})\Delta(f)\big)(r,s,t) = f\big((rs)t\big) = f\big(r(st)\big) = \big((\mathrm{id}\otimes\Delta)\Delta(f)\big)(r,s,t)$. We can see that $\big(K(G),\Delta\big)$ is a Hopf algebra, together with the maps $\big(S(f)\big)(t) := f(t^{-1})$ and $\epsilon(f) := f(e_G)$.

(EX 2.) Consider also the group algebra $\mathbb{C}G$, which is a linear span of the elements $\{\lambda_t : t\in G\}$ and has a multiplication given by $\lambda_s\lambda_t = \lambda_{st}$. On this algebra, we can define the comultiplcation map $\hat{\Delta}$ by $\hat{\Delta}(\sum_t c_t\lambda_t) = \sum_t c_t\lambda_t\otimes\lambda_t$. Then $\big(\mathbb{C}G,\hat{\Delta}\big)$ is a Hopf algebra, together with the maps $S(\sum_t c_t\lambda_t) = \sum_t c_t\lambda_{t^{-1}}$ and $\epsilon(\lambda_t) = 1$ for $t\in G$. In fact, these two examples are dual objects in the category of Hopf algebras.

[Remark]: The above examples suggest that Hopf algebras could be considered as quantum groups. There are some drawbacks, however. For one thing, there is no notion of topology nor the notion of the Haar measure. So we cannot do much of harmonic analysis. When we wish to consider analogues of non-compact groups or infinite-dimensional group representations, working with the Hopf algebras will be problematic. In addition, when a Hopf algebra is infinite dimensional, it is not possible in general to construct a dual Hopf algebra. The general theory of *locally compact quantum groups* was developed to resolve these issues.

2.2 Toward topological quantum groups

A topological group is a topological space that is a group. We further require that the group operations are continuous. In particular, if the underlying space is locally compact, we have a locally compact group (e. g. Lie group). We wish to find a suitable generalization, both algebraically and spatially. In this sense, our notion of a "locally compact quantum group" would be different, though related, from the more algebraic notions like the "quantized universal enveloping (QUE) algebras (by Drinfeld, Jimbo, ...).

Considering the paradigm of non-commutative geometry that the C^*-algebras are quantum spaces and the fact that the maps Δ, S, ϵ in a Hopf algebra are analogues of the group multiplication, inverse, identity, respectively, a natural approach would be to begin with the C^*-bialgebra (A,Δ), where A is a C^*-algebra and Δ is a (non-degenerate) *-homomorphism satisfying the coassociativity condition. One would then try to define a locally compact quantum group by giving an appropriate definition for S and ϵ. However, these approaches have all failed, for several reasons.

Among the most important reasons is that there have been some examples found (like the "quantum $SU(2)$ group" or the "quantum $E(2)$ group" by Woronowicz), in which the maps S and ϵ are unbounded. Then it is difficult to make sense of the expressions like $\epsilon\otimes\mathrm{id}$ or $S\otimes\mathrm{id}$ that appear in the definition of a Hopf algebra. In addition, since the C^*-algebra A can be infinite-dimensional, the multiplication map m can be unbounded. What all this means is that we have a difficulty defining expressions like $S\otimes\mathrm{id}$, and we are not able to define m on large enough domains. So we would have a serious difficulty to give a meaning to an axiom like $m\big((S\otimes\mathrm{id})\Delta(x)\big)=\epsilon(x)1$ for all $x\in A$.

In the finite-dimensional case, though, we do not have these technical issues. Consider as before a finite group G and $K(G)$ the set of all complex functions on G. Then $A=K(G)$ is a commutative Hopf algebra. It is also a *Hopf* **-algebra*, which means that A is a *-algebra (with the complex conjugation as the involution) and the map Δ is a *-homomorphism. As a consequence, the map ϵ is also a *-homomorphism. For the map S, while not necessarily a *-map, it is anti-multiplicative and $S\big(S(a^*)^*\big)=a$ for $a\in A$.

In general, if (A,Δ) is a Hopf *-algebra, we also have the two linear maps T_1 and T_2 from $A\otimes A$ into itself, defined by

$$T_1(a\otimes b)=(\Delta a)(1\otimes b),\qquad T_2(a\otimes b)=(a\otimes 1)(\Delta b).$$

They are in fact linear isomorphisms, with $T_1^{-1}(a\otimes b)=\big((\mathrm{id}\otimes S)(\Delta a)\big)(1\otimes b)$ and $T_2^{-1}(a\otimes b)=(a\otimes 1)\big((S\otimes\mathrm{id})(\Delta b)\big)$. For the case of $A=K(G)$, these maps correspond to the bijections $(s,t)\mapsto(st,t)$ and $(s,t)\mapsto(s,st)$, and therefore manifest the left/right cancellation properties of the group.

Meanwhile, together with the sup norm, $A=K(G)$ is actually a finite-dimensional C^*-algebra. In fact, every (finite-dimensional) Hopf *-algebra (A,Δ) with A a commutative C^*-algebra is isomorphic to $\big(K(G),\Delta\big)$ for some finite group G. The space $G=\mathrm{sp}(A)$ is obtained by the Gelfand-Naimark theorem, and the group structure on G is obtained by the Hopf algebra structure. For instance, for $\lambda,\mu\in G$, the characters on A, define: $\lambda\cdot\mu:=(\lambda\otimes\mu)\Delta$. This observation suggests the following definition:

[Definition]: A Hopf *-algebra (A,Δ) is a **finite quantum group**, if A is a finite-dimensional C^*-algebra. Or, equivalently, we have: $a^*a=0$ if and only if $a=0$.
(*) There are examples of (non-commutative and non-cocommutative) Hopf algebras which are not finite quantum groups (because no suitable *-structures can be given).

Let us return to developing the definition of general locally compact quantum groups. Here are two fundamental examples:

(1). $A=C_0(G)$, for G: locally compact group. It is a commutative C^*-algebra, together with $\Delta:A\to M(A\otimes A)$, a non-degenerate *-homomorphism

$$C_0(G)\ni f\mapsto\Delta f\in C_b(G\times G),\quad\text{where }\Delta f(s,t)=f(st).$$

Note that the "coassociativity" holds: $(\Delta\otimes\mathrm{id})\Delta=(\mathrm{id}\otimes\Delta)\Delta$.
$\varepsilon:A\to\mathbb{C}$, a *-homomorphism $\quad C_0(G)\ni f\mapsto f(1)\in\mathbb{C}$.
$S:A\to A$, an (anti)-automorphism $\quad\big(S(f)\big)(t)=f(t^{-1})$.

[Remark]: Consider $W\in\mathcal{B}\big(L^2(G\times G)\big)$, defined by $W\xi(s,t):=\xi(s,s^{-1}t)$. Then:

$$W^*(1\otimes M_f)W\xi(s,t)=(1\otimes M_f)W\xi(s,st)=f(st)W\xi(s,st)=f(st)\xi(s,t)$$

We can see that: $W^*(1\otimes M_f)W = (M\otimes M)_{\Delta f}$. This operator W is a so-called **"multiplicative unitary operator"**, in the sense of Baaj and Skandalis.

(2) $\hat{A} = C^*_{\mathrm{red}}(G)$, for G: locally compact group. It is a C^*-algebra defined as $C^*_{\mathrm{red}}(G) = \overline{L\big(C_c(G)\big)}^{\|\ \|_{\mathrm{op}}}$, where L is the (left regular) representation over $L^2(G)$ given by

$$L_f\xi(t) = \int f(z)L_z\xi(t)\,dz = \int f(z)\xi(z^{-1}t)\,dz.$$

It has

$\hat{\Delta}f \in M(\hat{A}\otimes\hat{A})$ given by $\hat{\Delta}f = \int f(z)L_z\otimes L_z\,dz$. (So it is "cocommutative".)

$\hat{\varepsilon}:\hat{A}\to\mathbb{C}$ given by $\hat{\varepsilon}(f) = \int f(s)\,ds$.

$\hat{S}:\hat{A}\to\hat{A}$ given by $\hat{S}(f) = \int f(z)L_{z^{-1}}\,dz$.

As before, consider $W\in\mathcal{B}\big(L^2(G\times G)\big)$, defined by $W\xi(s,t) := \xi(s,st)$. Then:

$$W(L_f\otimes 1)W^*\xi(s,t) = (L_f\otimes 1)W^*\xi(s,s^{-1}t) = \int f(z)W^*\xi(z^{-1}s,s^{-1}t)\,dz = \int f(z)\xi(z^{-1}s,z^{-1}t)\,dz.$$

We can see that: $W(L_f\otimes 1)W^* = \hat{\Delta}f$.

In addition to these fundamental examples, valid non-commutative, non-cocommutative examples do exist! However, as we noted earlier, some technical obstacles occur (involving ϵ, S, m, ...) when we try to formulate the definition of a "locally compact quantum group".

[Remedy]: A strategy is to work with the cancellation property of a group, and the property that there exists an (invariant) Haar measure on a locally compact group.

[A side remark]: If a set is equipped with only an associative multiplication, it is called a "semigroup". It is known that a semigoup G is in fact a group if and only if for any $s\in G$, the maps $x\mapsto sx$ and $y\mapsto ys$ are 1-1 and onto, as maps from G to G. (These are the left and right cancellation properties.)

The density conditions in the definition of a *compact quantum group* given below are motivated by the cancellation properties.

[Definition (Woronowicz)]: Let A be a unital C^*-algebra, and let $\Delta: A\to A\otimes A$ be a unital *-homomorphism saisfying $(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$. Then (A,Δ) is a **compact quantum group**, if $\Delta(A)(A\otimes 1)$ and $\Delta(A)(1\otimes A)$ are dense subspaces of $A\otimes A$.

[Remark]: We will skip the details, but it turns out that the definition given above is a valid one. Being a unital C^*-algebra implies compactness. For every compact group G, earlier example $\big(C(G),\Delta\big)$ is a compact quantum group. And for every discrete group Γ, the example $\big(C^*_{\mathrm{red}}(\Gamma),\hat{\Delta}\big)$ earlier is a compact quantum group, similar for $C^*(\Gamma)$.

From the axioms above, one can define the *antipode* map S. Also, the existence of a unique, (left/right invariant) *Haar state* φ can be proved (Woronowicz, Van Daele). Here, $\varphi : A \to \mathbb{C}$ is a state satisfying the following condition:

$$(\mathrm{id} \otimes \varphi)\Delta(a) = \varphi(a)1 = (\varphi \otimes \mathrm{id})\Delta(a).$$

We will give some examples and discuss a little about compact quantum groups in the next section (§ 2.3). Meanwhile, we point out that if we were to consider non-compact quantum groups (when A is non-unital), there are some further technical issues such that the existence of left/right invariant Haar weights needs to be required as part of the axiom. More on this later, in § 2.4.

2.3 Compact quantum groups

The first significant example of a compact quantum group was Woronowicz's quantum $SU(2)$ group, $SU_\mu(2)$. (There is also an approach given by Soibelman and Vaksman, viewed as the dual of the QUE algebra $U_q(\mathfrak{su}_2)$, but we will just consider here the C^*-algebra approach by Woronowicz.) It is constructed by the method of generators and relations, as follows.

(1). Compact quantum group $SU_\mu(2)$:

Recall that the compact Lie group $SU_2(\mathbb{C})$ consists of all matrices of the form $\begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$, where $\alpha, \gamma \in \mathbb{C}$ and $\alpha\bar{\alpha} + \bar{\gamma}\gamma = 1$. Consider the coordinate functions $a, c \in C\big(SU_2(\mathbb{C})\big)$ defined by $a : \begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} \mapsto \alpha$ and $c : \begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} \mapsto \gamma$. Then it is not difficult to see that as a C^*-algebra, $C\big(SU_2(\mathbb{C})\big)$ is isomorphic to the universal unital commutative C^*-algebra generated by elements a, c satisfying $a^*a + c^*c = 1$.

Now fix $\mu \in [-1, 1]$. Then denote by $SU_\mu(2)$ the universal unital C^*-algebra generated by elements a, c under the condition that the following matrix is unitary:

$$u := \begin{pmatrix} a & -\mu c^* \\ c & a^* \end{pmatrix}$$

Or, equivalently, the generators a and c satisfy

$$c^*c = cc^*, \qquad ac = \mu ca, \qquad ac^* = \mu c^* a, \qquad a^*a + c^*c = 1, \qquad aa^* + \mu^2 c^*c = 1.$$

Clearly, if $\mu = 1$, we have: $SU_{\mu=1}(2) \cong C\big(SU_2(\mathbb{C})\big)$.

For each $\mu \in (0, 1]$, define the *-homomorphism $\Delta : SU_\mu(2) \to SU_\mu(2) \otimes SU_\mu(2)$, by

$$\Delta(a) = a \otimes a - \mu c^* \otimes c, \qquad \Delta(c) = c \otimes a + a^* \otimes c.$$

It turns out that $\big(SU_\mu(2), \Delta\big)$ is a compact quantum group. The *-subalgebra $\mathcal{A}$ generated by a and c is dense in $SU_\mu(2)$, and more or less can be considered as set of "smooth functions".

The antipode map S is not defined everywhere on $SU_\mu(2)$, but is defined on the dense subalgebra $\mathcal{A}$, as follows:

$$S(a) = a^*, \quad S(a^*) = a, \quad S(c) = -\mu c, \quad S(c^*) = -\mu^{-1} c^*.$$

While $S^2 \not\equiv \mathrm{Id}$, we see that $S\big(S(x^*)^*\big) = x$ for all $x \in \mathcal{A}$. There also exists a unique Haar state on $\big(SU_\mu(2), \Delta\big)$, satisfying the left/right invariance condition.

(2). Compact matrix quantum groups (Woronowicz):

The compact quantum group $SU_\mu(2)$ appeared earlier, but it is a special case of a *compact matrix quantum group* (see below). But first, let us clarify some notation: If A is a C^*-algebra and if $u = (u_{ij}) \in M_n(A)$, we write: $\bar{u} = (u_{ij}^*)$ and $u^T = (u_{ji})$. So $u^* = \bar{u}^T$.

By definition, a **compact matrix quantum group (CMQG)** is a unital C^*-algebra A, together with a *-homomorphism $\Delta : A \to A \otimes A$ (minimal tensor product, always!), and a unitary $u = (u_{ij}) \in M_n(A)$ for some $n \in \mathbb{N}$, such that

- the elements u_{ij} $(1 \leq i, j \leq n)$ generate A
- $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, for all $1 \leq i, j \leq n$
- $\bar{u}$ is an invertible matrix

As the same suggests, (A, Δ) above is indeed a compact quantum group. The matrices u and $\bar{u}$ are "corepresentation matrices" of (A, Δ).

Many examples of compact quantum groups are in fact CMQGs. To see this, suppose G is a closed subgroup of $U_n(\mathbb{C})$, for some $n \in \mathbb{N}$ (so G is compact). For $i, j = 1, \dots, n$, define the function $u_{ij} \in C(G)$ by $u_{ij}(x) = x_{ij}$. Then the C^*-algebra $C(G)$ and $u = (u_{ij})$ determines a CMQG, which coincides with $\big(C(G), \Delta\big)$ earlier. Meanwhile, it is clear from the definition that $SU_\mu(2)$ earlier is also a CMQG.

On the other hand, not all compact quantum groups are CMQGs. A correct statement is that a general compact quantum group is an inverse limit of CMQGs.

(3). Quantum permutation group (Wang, Banica, ...):

For $n \in \mathbb{N}$. Denote by $A_s(n)$ the universal C^*-algebra generated by elements u_{ij} $(i, j = 1, \dots, n)$, such that $u = (u_{ij})$ is a *magic* unitary matrix. That is,

- each u_{ij} is a projection
- the projections $u_{i1}, \dots, u_{in}$ are orthogonal and $\sum_k u_{ik} = 1$
- the projections $u_{1j}, \dots, u_{nj}$ are orthogonal and $\sum_k u_{kj} = 1$

Note that $\bar{u} = u$ for every magic unitary matrix u. We can show that there exists a unital *-homomorphism $\Delta : A_s(n) \to A_s(n) \otimes A_s(n)$ such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. And, it turns out that $\big(A_s(n), \Delta, u\big)$ is a CMQG. This is the *quantum permutation group* on n letters.

The quantum permutation group is related to the group $S(n)$ of permutations on n letters, and is a special case of a *quantum automorphism group*.

(4). Universal quantum groups (Wang, Van Daele):

Let $Q \in GL_n(\mathbb{C})$, Q: positive, and define

$$A_u(Q) := C^*\big\{u_{ij} : \ u \text{ is unitary (so } u^* = u^{-1}) \text{ and } (u^T)^{-1} = Q\bar{u}Q^{-1}\big\},$$

where $u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$. There exists a unital *-homomorphism $\Delta_u : A_u(Q) \to A_u(Q) \otimes A_u(Q)$ such that $\Delta_u(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. It can be shown that $\big(A_u(Q), \Delta_u\big)$ is a compact quantum group.

The quantum groups $A_u(Q)$ are universal objects in the category of compact quantum groups. In fact, any arbitrary compact matrix quantum group can be realized as a triple (A, Δ, π), where $\pi : A_u(Q) \to A$ and $\Delta : A \to A \otimes A$ are C^*-morphisms such that

- $A = C^*\big\{\pi(u_{ij}) : i, j = 1, \dots, n\big\}$
- $\Delta\big(\ker(\pi) \subseteq \ker(\pi \otimes \pi)$

Similarly, suppose $n \geq 2$ and consider now $Q \in GL_n(\mathbb{C})$ such that $Q\bar{Q} \in \mathbb{R}I_n$. Define:

$$B_u(Q) = C^*\big\{u_{ij} : u^* = u^{-1}, (u^T)^{-1} = QuQ^{-1}\big\},$$

where $u = (u_{ij})$. It is also a compact quantum group, with the comultiplication map defined similarly as before. Again, the $B_u(Q)$ become universal objects.

[Remark]: We skip details, but loosely speaking, the $A_u(Q)$ play the role of the unitary group U_n, and the $B_u(Q)$ play the role of the orthogonal group O_n. For this reason, $B_u(Q)$ is often denoted by $A_o(Q)$. The $A_u(Q)$ are referred to as *free unitary quantum groups*, and the $B_u(Q)$ (or $A_o(Q)$) as *free orthogonal quantum groups*. Unlike some other examples including the quantum $SU_\mu(2)$, these examples turn out to be non-nuclear in general, and cannot be obtained as a deformation.

(5). Some recent developments on quantum automorphism groups and quantum isometry groups (Banica, Bichon, Goswami, Bhowmick, ...):

[Definition]: We say that a compact quantum group (A, Δ) acts on a (unital) C^*-algebra B, if there is a unital *-homomorphism $\alpha : B \to B \otimes A$ such that

$$(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$$

and that the linear span of $\alpha(B)(1 \otimes A)$ is dense in $B \otimes A$.

In particular, we are interested in the case when the C^*-algebra B is associated with a certain "spectral triple" (in the sense of Connes), together with a certain Dirac operator. The notion of a spectral triple is motivated by the classical examples of Riemannian spin manifolds, and encodes the information about the underlying topology and the Riemannian metric. Therefore, such a B may be loosely considered as a "quantum metric space" (Connes, Rieffel). Then a natural question arises: If we are given a spectral triple with the underlying C^*-algebra B, what are all compact quantum group actions on B, for which the given spectral triple is equivariant? A universal object in this setting will lead us to the notion of a *quantum isometry group*.

Most of the earlier work were about some quantum automorphism groups of a "finite" structure, including the quantum permutation group mentioned earlier. See works by Wang, Banica, Bichon, Collins, ... Then recently, Goswami and Bhowmick developed various notions of quantum isometry groups on certain spectral triples. They are defined as universal objects in some subcategories of compact quantum groups acting on B, which satisfy some technical conditions.

(6). Miscellany

One can consider *free products* and *tensor products* of compact quantum groups. One can define the notions of quantum subgroups and normal quantum subgroups of compact quantum groups. Recently, Wang has introduced a definition of a *simple* compact quantum group. Using these notions, one may begin studying the problem of classifying compact quantum groups. But this is not easy, even just for simple compact quantum groups.

One can consider the corepresentation theory of compact quantum groups, which turns out to be more or less analogous to the representation theory of compact groups. By a *unitary corepresentation* of a compact quantum group(A, Δ) on a Hilbert space $\mathcal{H}$, we mean a map $\delta : \mathcal{H} \to \mathcal{H} \otimes A$ such that

- $\big\langle \delta(\eta), \delta(\xi) \big\rangle = \langle \eta, \xi \rangle \cdot 1_A$ for all $\eta, \xi \in \mathcal{H}$
- the set $\delta(\mathcal{H})A$ is linearly dense in $\mathcal{H} \otimes A$
- $(\mathrm{id}_{\mathcal{H}} \otimes \Delta) \circ \delta = (\delta \otimes \mathrm{id}_A) \circ \delta$

If δ is a unitary corepresentation of (A, Δ) on $\mathcal{H}$, then the map $X : \mathcal{H} \odot A \to \mathcal{H} \otimes A$, $\eta \odot a \mapsto \delta(\eta)a$, extends to a unitary operator $X \in \mathcal{B}(\mathcal{H} \otimes A)$, and $X_{12}X_{13} = (\mathrm{id} \otimes \Delta)(X)$. Going backwards, if X is such an operator, then the map $\delta : \eta \mapsto X(\eta \otimes 1)$ is a unitary corepresentation. One studies the corepresentation theory in terms of these "unitary corepresentation operators". Theory of multiplicative unitary operators (in the sense of Baaj and Skandalis) is useful here. Meanwhile, using C^*-categories, Woronowicz proved a generalization of the Tannaka–Krein duality theorem: A compact quantum group is effectively determined by its tensor category of finite-dimensional unitary corepresentations.

The notion of multiplicative unitary operators is also useful in general locally compact quantum group theory and duality theory. We will discuss these later. For the case of a compact quantum group, the "dual" object should be a *discrete quantum group*. The underlying algebra of a discrete quantum group would be a direct sum of some matrix algebras. Van Daele provided a general framework for unifying the theories of both compact and discrete quantum groups, by introducing a purely algebraic theory of *multiplier Hopf algebras*. This category includes far more than just compact and discrete quantum groups, and has a satisfactory duality theory. The general theory of *locally compact quantum groups* includes this, but it is of note that the multiplier Hopf algebras provided a strong motivation for the development of the general theory.

2.4 Locally compact quantum groups

To develop a general theory of locally compact quantum groups, where the underlying C^*-algebra is possibly non-unital, we have to overcome some technical difficulties. Some of these issues are not too serious, and can be managed by the usual operator algebra techniques. For instance, instead of defining the comultiplication map as a unital *-homomorphism $\Delta : A \to A \otimes A$ satisfying the coassociativity, we may require that Δ is a non-degenerate *-homomorphism from A to $M(A \otimes A)$, where $M(B)$ denotes the "multiplier algebra" of B. By the non-degeracy of Δ, the condition $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ makes sense.

However, there are more serious issues: It turns out that if we just use a definition similar to the one given for the compact case, the existence of the Haar weights cannot be proved out of the axioms. Note here that we no longer expect a Haar state, because the situation is in general non-compact. We have to replace the notion of a state or a linear functional to that of a weight, but then it is already not easy to make sense of the left or right invariance. It turns out that the correct approach (by Kustermans, Vaes, Masuda, Nakagami, Woronowicz) is to assume the existence of suitable Haar weights as part of the axioms.

There are difficulties also in constructing examples. Recall that most of the known examples of compact quantum groups (see previous section) are described in terms of generators and relations, where the generators are sometimes coordinate functions. However, in the non-compact case, the generators are usually unbounded, and this causes serious obstacles. For the "quantum $E(2)$ group" (see below), Woronowicz introduced the notion of unbounded elements "affiliated" to C^*-algebras and managed the problem, but in general, working with the generators and relations approach in the non-compact case is highly non-preferable.

Having said these, let us introduce the general theory of locally compact quantum groups. We will first begin with some preliminaries.

2.4.1 Preliminaries: Weights and Tomita–Takesaki theory

[**Weights**] Let M be a von Neumann algebra. A *weight* on M is a map $\varphi : M_+ \to [0, \infty]$ such that $\varphi(a + \lambda b) = \varphi(a) + \lambda\varphi(b)$ for $a, b \in M_+$ and $\lambda \in [0, \infty)$, with the convention that $\lambda + \infty = \infty$, $\lambda \cdot \infty = \infty$ if $\lambda > 0$, and $0 \cdot \infty = 0$. Given a weight we consider the following sets:

$$\mathfrak{N}_\varphi = \{a \in M : \varphi(a^*a) < \infty\}, \qquad \mathfrak{M}_\varphi^+ = \{a \in M_+ : \varphi(a) < \infty\}, \qquad \mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi.$$

The weights we usually consider are "f.n.s. (faithful normal semi-finite) weights". This means that $\varphi(a) = 0$, $a \in M_+$, means $a = 0$; $\mathfrak{M}_\varphi$ is σ-weakly dense in M (it is "finite", if $\mathfrak{M}_\varphi = M$); and φ is (σ-weakly) lower semicontinuous.

We can also consider the notion of a weight on a C^*-algebra A, and the sets $\mathfrak{N}_\varphi$, $\mathfrak{M}_\varphi^+$, $\mathfrak{M}_\varphi$. In the setting of C^*-algebras, we usually work with the "approximately KMS weights".

Basically, they are the ones that naturally extend to f.n.s. weights on its enveloping von Neumann algebra.

By standard theory, for an f.n.s. weight φ on M, one can associate to it a "GNS-construction" $(\mathcal{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$. Here, $\mathcal{H}_\varphi$ is a Hilbert space, $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathcal{H}_\varphi$ is a linear map such that $\Lambda_\varphi(\mathfrak{N}_\varphi)$ is dense in $\mathcal{H}_\varphi$ and $\langle \Lambda_\varphi(a), \Lambda_\varphi(b) \rangle = \varphi(b^*a)$ for $a, b \in \mathfrak{N}_\varphi$, and π_φ is a *-representation of M on $\mathcal{H}_\varphi$ defined by $\pi_\varphi(c)\Lambda_\varphi(b) = \Lambda_\varphi(cb)$ for $c \in M$, $b \in \mathfrak{N}_\varphi$. The GNS-construction is unique up to a unitary transformation. It turns out that π_φ is isometric and is a σ-weak homomorphism of M onto $\pi_\varphi(M)$. Therefore, it is customary to identify M with $\pi_\varphi(M)$ and regard $M \subseteq \mathcal{B}(\mathcal{H}_\varphi)$, $\pi_\varphi(a) = a$.

[**Generalized Hilbert algebras**] For a fixed f.n.s. weight φ on M, consider the set $\mathcal{U}_\varphi = \Lambda_\varphi(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*)$. It is an involutive algebra, with multiplication: $\Lambda_\varphi(a) \cdot \Lambda_\varphi(b) := \Lambda_\varphi(ab)$, and the involution $^\sharp$ defined by $\Lambda_\varphi(a)^\sharp := \Lambda_\varphi(a^*)$, for $a, b \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$. It is dense in $\mathcal{H}_\varphi$, and with respect to the inner product, we have: $\langle \xi\eta, \zeta \rangle = \langle \eta, \xi^\sharp\zeta \rangle$, for all $\xi, \eta, \zeta \in \mathcal{U}_\varphi$. In addition, its involution is antilinear preclosed mapping. That is, the map $\xi \mapsto \xi^\sharp$, $\xi \in \mathcal{U}_\varphi$, extends to a closed operator S.

In general, we call *left Hilbert algebra* an involutive algebra $(\mathcal{U}, ^\sharp)$ equipped with an inner product, such that the involution is an antilinear preclosed mapping in the Hilbert space $\mathcal{H}$ associated, and such that the left-multiplication representation π of $\mathcal{U}$ is non-degenerate, bounded and involutive. We write S the closure of $^\sharp$, and F the adjoint S^* of S. The domains of S and F are denoted by $\mathcal{D}^\sharp$ and $\mathcal{D}^\flat$.

Consider now the class of "right bounded" elements, defined by

$$\mathcal{U}' = \{\xi \in \mathcal{D}^\flat : \text{ the map } \eta \mapsto \pi(\eta)\xi,\ \eta \in \mathcal{U}, \text{ is bounded } \}.$$

Then each $\xi \in \mathcal{U}'$ gives rise to a bounded operator $\pi'(\xi) \in \mathcal{B}(\mathcal{H})$, such that $\pi'(\xi)\eta = \pi(\eta)\xi$. Note that if $\xi \in \mathcal{U}'$, then $\xi^\flat \in \mathcal{U}'$ and $\pi'(\xi^\flat) = \pi'(\xi)^*$. In fact. the set $\mathcal{U}'$, together with the multiplication: $\xi_1 \cdot \xi_2 := \pi'(\xi_2)\xi_1$ and the involution: $\xi \mapsto \xi^\flat = F\xi$, is a *right Hilbert algebra* (with the obvious definition). So by repeating a similar argument and considering the "left bounded" elements, we would obtain another left Hilbert algebra $\mathcal{U}''$. Clearly, the algebra $\mathcal{U}''$ contains $\mathcal{U}$ as an involutive subalgebra, with the operator S again being the closure of the involution of $\mathcal{U}''$.

A left Hilbert algebra $\mathcal{U}$ is called "achieved", if $\mathcal{U}'' = \mathcal{U}$. For an f.n.s. weight φ, the left Hilbert algebra $\mathcal{U}_\varphi$ constructed above is achieved. Conversely, any achieved left Hilbert algebra $\mathcal{U}$ is of the form $\mathcal{U}_\varphi = \Lambda_\varphi(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*)$, where $M = \pi(\mathcal{U})''$ and φ is defined by

$$\varphi(a) = \begin{cases} \|\xi\|^2, & \text{if } a = \pi(\xi^\sharp\xi) \text{ for some } \xi \in \mathcal{U} \\ \infty, & \text{otherwise} \end{cases}$$

which turns out to be an f.n.s. weight on M.

[**Tomita–Takesaki theory**] Let φ be a f.n.s. weight on a von Neumann algebra M, with GNS triple $(\mathcal{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$. We identify M with $\pi_\varphi(M)$. Then there exists a self-adjoint antiunitary operator J and an invertible (possibly unbounded) positive self-adjoint operator Δ in $\mathcal{H} = \mathcal{H}_\varphi$ such that

- $S = J\Delta^{1/2}$ and $F = J\Delta^{-1/2}$ are the polar decompositions of the operators S and F
- $J\Delta J = \Delta^{-1}$, and consequently, $Jf(\Delta)J = \bar{f}(\Delta^{-1})$ for any Borel function f on $(0,\infty)$
- $\Delta^{it}M\Delta^{-it} = M$ for all $t \in \mathbb{R}$, and $JMJ = M'$

Note that we can see from above that $\mathcal{D}^\sharp = \mathrm{Dom}(\Delta^{1/2})$ and $\mathcal{D}^\flat = \mathrm{Dom}(\Delta^{-1/2})$.

This result leads us to the notion of the *modular automorphism group* (σ_t^φ), which is a σ-weakly continuous one-parameter group defined by

$$\sigma_t^\varphi(a) = \Delta^{it}a\Delta^{-it}, \qquad \text{for } a \in M,\ t \in \mathbb{R}.$$

It can be shown that $\varphi = \varphi \circ \sigma_t^\varphi$, and that the modular automorphism group satisfies the "KMS condition": That is, for every $a, b \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, there exists a bounded function F on the strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq 1\}$, holomorphic in its interior, such that for all $t \in \mathbb{R}$, we have: $F(t) = \varphi\big(\sigma_t^\varphi(b)a\big)$ and $F(t+i) = \varphi\big(a\sigma_t^\varphi(b)\big)$. In fact, these two properties characterize the modular automorphism group.

For technical reasons, it is useful to know that there exists a maximal subalgebra $\mathcal{U}_0$ of $\mathcal{U}'' \cap \mathcal{U}'$, which is both a left and right Hilbert algebra, with $\mathcal{U}_0' = \mathcal{U}'$ and $\mathcal{U}_0'' = \mathcal{U}''$, and which is globally invariant under the closed linear operators Δ^z, for all $z \in \mathbb{C}$. The algebra $\mathcal{U}_0$ is called the "maximal modular subalgebra of $\mathcal{U}''$ (or the "Tomita algebra").

If φ is a *trace*, so $\varphi(a^*a) = \varphi(aa^*)$ for all $a \in M$, then $\mathfrak{N}_\varphi = \mathfrak{N}_\varphi^*$, $S_\varphi = J_\varphi$, $\Delta_\varphi = 1$, and $\sigma_t^\varphi \equiv 1$. In particular, if $\mathrm{tr}_\mathcal{H}$ is the canonical trace on $\mathcal{B}(\mathcal{H})$, the $\mathfrak{N}_{\mathrm{tr}_\mathcal{H}}$ is the algebra of Hilbert–Schmidt operators on $\mathcal{H}$.

In addition to these basics weight theory, one can also consider the notions of "tensor product", "Radon–Nikodym derivative", etc. Please refer to the textbooks.

2.4.2 Definition of a locally compact quantum group

[Definition] (Kustermans, Vaes): Lat A be C^*-algebra, and consider a non-degenerate *-homomorphism $\Delta : A \to M(A \otimes A)$ such that

1. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
2. $\big[\{(\omega \otimes \mathrm{id})(\Delta a) : \omega \in A^*, a \in A\}\big] = A$ and $\big[\{(\mathrm{id} \otimes \omega)(\Delta a) : \omega \in A^*, a \in A\}\big] = A$, where $[X]$ denotes the closed linear span of X.
3. We further assume that there exist weights φ and ψ such that φ is a faithful, *left invariant* approximate KMS weight on (A, Δ). That is,

   $$\varphi\big((\omega \otimes \mathrm{id})(\Delta a)\big) = \omega(1)\varphi(a),$$

   for all $a \in \mathfrak{M}_\varphi{}^+$ and $\omega \in A_+^*$. By $\omega(1)$, we mean $\|\omega\|$. While, ψ is a *right invariant* approximate KMS weight on (A, Δ). Or,

   $$\psi\big((\mathrm{id} \otimes \omega)(\Delta a)\big) = \omega(1)\psi(a).$$

Then we say that (A, Δ) is a **(reduced) C^*-algebraic quantum group**.

[Remarks]: First condition is the "coassociativity" condition for the *comultiplication* Δ. By the non-degeneracy, it can be naturally extended to $M(A)$ [we can also extend $(\Delta \otimes \mathrm{id})$ and $(\mathrm{id} \otimes \Delta)$], thereby making the expression valid. The two density conditions more or less correspond to the cancellation property in the case of ordinary groups. The last axiom corresponds to the existence of Haar measure. As we noted earlier, the approximate KMS weights on a C^*-algebra are more or less the ones corresponding to the f.n.s. weights on the von Neumann algebra level. In fact, the weights φ and ψ actually turn out to be faithful KMS weights.

In the definition above, the left and right invariance conditions of the Haar weights are required to hold only for $a \in \mathfrak{M}_\varphi{}^+$. This is a very weak form of left/right invariance. In the case of locally compact quantum groups, it turns out that the results can be extended and much stronger left/right invariance conditions can be proved. The proof is non-trivial. This was one of the important contributions made by Kustermans and Vaes. It can be also shown that the Haar weights unique, up to a scalar multiplication.

The fact that the existence of the Haar weights need to be included in the axioms is a drawback. Nevertheless, the definition is relatively simple, and is a huge improvement over various attempts which have been made earlier. From the definition, one can build other structure maps, especially the antipode. We will talk a little about this later. An alternative (but essentially equivalent) formulation was given by Masuda, Nakagami, Woronowicz.

This definition gives us a "quantized $C_0(G)$", where G is a locally compact group. In fact, when (A, Δ) is a C^*-algebraic quantum group and A is commutative, then we can find a locally compact group G such that (A, Δ) is exactly the Example (1) given earlier in § 2.2. Meanwhile, there is the notion of a "von Neumann algebraic locally compact quantum group", which would correspond to a "quantized $L^\infty(G)$". See below.

[Definition]: Let M be a von Neumann algebra, together with a unital normal *-homomorphism $\Delta : M \to M \otimes M$ such that the "coassociativity condition" holds: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. Furthermore, we assume the existence of a left invariant weight and a right invariant weight, as follows:

- φ is an f.n.s. weight on M that is left invariant:

$$\varphi\big((\omega \otimes \mathrm{id})(\Delta x)\big) = \varphi(x)\omega(1), \quad \text{for all } \omega \in M_*^+,\ x \in \mathfrak{M}_\varphi^+.$$

- ψ is an f.n.s. weight on M that is right invariant:

$$\psi\big((\mathrm{id} \otimes \omega)(\Delta x)\big) = \psi(x)\omega(1), \quad \text{for all } \omega \in M_*^+,\ x \in \mathfrak{M}_\psi^+.$$

Then we call (M, Δ) a **von Neumann algebraic quantum group**. It can be shown that the Haar weights are unique, up to scalar multiplication.

[Remark]: It turns out that this definition is known to be equivalent to the definition in the C^*-algebra setting. For instance, the enveloping von Neumann algebra of a C^*-algebraic quantum group, together with the natural extensions of the map Δ and the weights φ, ψ, will give us a von Neumann algebraic quantum group. This equivalence result is surprising but not unexpected, because a similar result is known to hold in the ordinary group case. The noticeable difference between the two approaches is the absence of the density conditions in the von Neumann algebra setting: They follow automatically from the other conditions. Since the two approaches are completely equivalent, and since the vN-algebra approach seems simpler, we will from now on work mostly with the von Neumann algebraic definition.

In the next a few subsections, we will discuss some basic results on locally compact quantum groups, that follow immediately from the definition.

2.4.3 Multiplicative unitary operator

Let (M, Δ) be a von Neumann algebraic quantum group, and let us fix φ, the left invariant Haar weight. By means of the GNS-construction $(\mathcal{H}, \iota, \Lambda)$ for φ, we view M as a subalgebra of the operator algebra $\mathcal{B}(\mathcal{H})$, such as $M = \iota(M) \subseteq \mathcal{B}(\mathcal{H})$. So we will have: $\langle \Lambda(x), \Lambda(y) \rangle = \varphi(y^*x)$ for $x, y \in \mathfrak{N}_\varphi$, and $x\Lambda(y) = \Lambda(xy)$ for $y \in \mathfrak{N}_\varphi$, $x \in M$. As in standard weight theory, we can consider the modular conjugation and the modular automorphism group (σ_t^φ).

Meanwhile, there exists a unitary operator $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, called the *multiplicative unitary operator* for (M, Δ). It is defined by $W^*\big(\Lambda(a) \otimes \Lambda(b)\big) = (\Lambda \otimes \Lambda)\big((\Delta b)(a \otimes 1)\big)$, for $a, b \in \mathfrak{N}_\varphi$. It satisfies the pentagon equation: $W_{12}W_{13}W_{23} = W_{23}W_{12}$, and one can check that $\Delta a = W^*(1 \otimes a)W$, for $a \in M$. It is essentially the "left regular representation" (associated with φ), and it gives the following useful characterization of M:

$$M = \overline{\mathcal{A}(W)}^w = \overline{\big\{(\mathrm{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\big\}}^w \ \big(\subseteq \mathcal{B}(\mathcal{H})\big),$$

where $-^w$ denotes the von Neumann algebra closure (the closure under σ-weak topology). If we consider instead the norm closure $\overline{\mathcal{A}(W)}^{\|\ \|}$, we would obtain a characterization of the associated C^*-algebraic quantum group (A, Δ).

[Some words on multiplicative unitaries]: A unitary operator $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is said to be a **multiplicative unitary**, if it satisfies the pentagon equation: $V_{12}V_{13}V_{23} = V_{23}V_{12}$. First examples were around for some time, but Baaj and Skandalis introduced the general definition and developed a systematic theory.

Multiplicative unitaries are fundamental to the theory of locally compact quantum groups, and to generalizations of Pontrjagin duality. Loosely speaking, a multiplicative unitary is a map that simultaneously encodes *all* structure maps of a quantum group and of its dual object. Out of every multiplicative unitary, one can construct a dual pair of von Neumann bialgebras. So multiplicative unitaries are useful in constructing examples of quantum groups and its dual (they are useful not just in constructing bialgebras but sometimes in obtaining the antipodes or the Haar weights). This is especially important in the case of non-compact quantum groups, because there are not as many construction methods available as in the compact case.

Given a multiplicative unitary operator $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, one considers two subspaces, $\mathcal{A}(V)$ and $\hat{\mathcal{A}}(V)$, of $\mathcal{B}(\mathcal{H})$, as follows:

$$\mathcal{A}(V) = \big\{(\mathrm{id} \otimes \omega)(V) : \omega \in \mathcal{B}(\mathcal{H})_*\big\} \qquad \text{and} \qquad \hat{\mathcal{A}}(V) = \big\{(\omega \otimes \mathrm{id})(V) : \omega \in \mathcal{B}(\mathcal{H})_*\big\}.$$

They are actually subalgebras. For instance, for elements $a = (\mathrm{id} \otimes \omega)(V)$ and $b = (\mathrm{id} \otimes \omega')(V)$ in $\mathcal{A}(V)$, for $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$, we have:

$$ab = (\mathrm{id} \otimes \omega \otimes \omega')(V_{12}V_{13}) = (\mathrm{id} \otimes \omega \otimes \omega')(V_{23}V_{12}V_{23}^*) = (\mathrm{id} \otimes \theta)(V) \in \mathcal{A}(V),$$

where $\theta \in \mathcal{B}(\mathcal{H})_*$ is such that $\theta(T) := (\omega \otimes \omega')(V(T \otimes \mathrm{id})V^*)$. In addition, each of the sets $\mathcal{A}(V)\mathcal{H}$, $\mathcal{A}(V)^*\mathcal{H}$, $\hat{\mathcal{A}}(V)\mathcal{H}$, $\hat{\mathcal{A}}(V)^*\mathcal{H}$ is linearly dense in $\mathcal{H}$.

On the other hand, the subalgebras $\mathcal{A}(V)$ and $\hat{\mathcal{A}}(V)$ are not self-adjoint in general. There are some criteria that ensure the self-adjointness. To be a little more precise, consider the space $\mathcal{C}(V) := \overline{\big\{(\omega \otimes \mathrm{id})(V\Sigma) : \omega \in \mathcal{B}(\mathcal{H})_*\big\}}^{\|\ \|}$ $\big(\subseteq \mathcal{B}(\mathcal{H})\big)$, where $\Sigma \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ denotes the flip map $\xi \otimes \eta \mapsto \eta \otimes \xi$. A multiplicative unitary operator V is said to be "regular", if $\mathcal{K}(\mathcal{H}) = \mathcal{C}(V)$, and "semi-regular", if $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{C}(V)$ (Baaj, Skandalis). There are also the notion of "manageability" (Woronowicz) and a more general notion of "modularity" (Soltan, Woronowicz). For these well-behaved multiplicative unitaries, we are able to obtain C^*-bialgebras $A(V) = \overline{\mathcal{A}(V)}^{\|\ \|}$ and $\hat{A}(V) = \overline{\hat{\mathcal{A}}(V)}^{\|\ \|}$.

It should be noted that a multiplicative unitary operator (even a well-behaved one) only determines a pair of C^*-bialgebras, not a quantum group. For example, the operator $\mathrm{id}_{\mathcal{H}\otimes\mathcal{H}}$ is a regular multiplicative unitary, but it does not give us a quantum group. On the other hand, many examples of quantum groups are known to be regular, including $C_0(G)$, the Kac–von Neumann algebras, and the compact quantum groups. However, the quantum group $E_\mu(2)$ was shown to be semi-regular but not regular (Baaj), and there are some examples of locally compact quantum groups that are not even semi-regular (Baaj, Skandalis, Vaes). For these examples, it turns out that the C^*-algebra $[A\hat{A}]$ can be highly non-trivial.

The manageability condition and the modularity condition are somewhat more technical, and we do not describe them here. But, they are particularly well adapted to unitaries associated to quantum groups. It is known that the multiplicative unitary of every locally compact quantum group is manageable.

2.4.4 The antipode

Let (M, Δ) be a locally compact quantum group with left Haar weight φ, right Haar weight ψ, and multiplicative unitary W. Then there exists a unique closed, densely defined linear map $S : \mathrm{Dom}(S) \subseteq M \to M$ such that

- $\mathrm{span}\big\{(\mathrm{id} \otimes \varphi)(\Delta(b^*)(1 \otimes a)) : a, b \in \mathfrak{N}_\varphi\big\}$ $(\subseteq M)$ is a core for S, and

$$S\big((\mathrm{id} \otimes \varphi)(\Delta(b^*)(1 \otimes a))\big) = (\mathrm{id} \otimes \varphi)((1 \otimes b^*)\Delta(a)), \qquad \text{for all } a, b \in \mathfrak{N}_\varphi.$$

- $\operatorname{span}\{(\psi\otimes\mathrm{id})((b^*\otimes 1)\Delta(a)) : a,b\in\mathfrak{N}_\psi\}$ $(\subseteq M)$ is a core for S, and
$$S\big((\psi\otimes\mathrm{id})((b^*\otimes 1)\Delta(a))\big) = (\psi\otimes\mathrm{id})(\Delta(b^*)(a\otimes 1)), \qquad \text{for all } a,b\in\mathfrak{N}_\psi.$$
- $\mathcal{A}(W)$ $(\subseteq M)$ is a core for S, and $S\big((\mathrm{id}\otimes\omega)(W)\big) = (\mathrm{id}\otimes\omega)(W^*)$, for all $\omega\in\mathcal{B}(\mathcal{H})_*$.

In addition, there exist a unique σ-strongly* continuous one-parameter group (τ_t) and a unique *-anti-automorphism R on M, such that $S = R\tau_{-i/2}$ is the *polar decomposition* of S.

The details are quite technical. In short, one first constructs a certain closed, densely defined operator G on $\mathcal{H}$, which essentially corresponds to the map $a\mapsto S(a^*)$. The operator $N = G^*G$ is strictly positive, and there exists a unique anti-unitary I on $\mathcal{H}$ such that $G = IN^{1/2}$. It is known that $I = I^*$, $I^2 = 1$, and $INI = N^{-1}$. Then we define $R(a) = Ia^*I$ and $\tau_t(a) = N^{-it}aN^{it}$, for all $a\in M$ and $t\in\mathbb{R}$, and let $S := R\tau_{-i/2}$. Both the right Haar weight and the left Haar weight are needed to show that S is well-defined and densely defined. We call S the *antipode*, R the *unitary antipode*, and τ the *scaling group*.

In the below are some results concerning the antipode and its polar decomposition. See the main papers (Kustermans, Vaes) for proofs of these, as well as some other results.

- S, R, τ commute, i.e. $S\circ R = R\circ S$, $\tau_t\circ R = R\circ\tau_t$, $\tau_t\circ S = S\circ\tau_t$ for all $t\in\mathbb{R}$.
- $R^2 = \mathrm{id}_M$, $S^2 = \tau_{-i}$. Though $S^2\neq\mathrm{id}_M$ in general, it is injective, and $S^{-1} = \tau_{i/2}\circ R$.
- For all $a,b\in\mathrm{Dom}(S)$, we have $S(ab) = S(b)S(a)$ and $S\big(S(a)^*\big)^* = a$.
- $(R\otimes R)\circ\Delta = \Delta^{\mathrm{cop}}\circ R$, where Δ^{cop} is the co-opposite comultiplication.
- $\varphi\circ R$ is a right Haar weight and $\psi\circ R$ is a left Haar weight.
- There exists a constant $\nu > 0$ (called the *scaling constant*) such that $\varphi\circ\tau_t = \nu^{-t}\varphi$, $\psi\circ\tau_t = \nu^{-t}\psi$, $\varphi\circ\sigma_t^\psi = \nu^t\varphi$, $\psi\circ\sigma_t^\varphi = \nu^{-t}\psi$.

Considering the fifth item, it is customary to fix the right Haar weight as $\psi = \varphi\circ R$, with the corresponding GNS map written as Γ. The existence of the scaling constant (sixth item) is a purely quantum phenomenon that does not appear in the ordinary group case.

2.4.5 The dual quantum group

The scaling group (τ_t) is used to show that the multiplicative unitary operator W associated to the locally compact quantum group (M,Δ) is manageable. Therefore, by the theory of multiplicative unitaries, we obtain the von Neumann bialgebra $(\hat{M},\hat{\Delta})$, by
$$\hat{M} = \overline{\hat{\mathcal{A}}(W)}^{w} = \overline{\{(\omega\otimes\mathrm{id})(W) : \omega\in\mathcal{B}(\mathcal{H})_*\}}^{w},$$
together with the comultiplication $\hat{\Delta}b := \Sigma W(b\otimes 1)W^*\Sigma$, for $b\in\hat{M}$. By considering the norm completion, we would also obtain the C^*-bialgebra $(\hat{A},\hat{\Delta})$.

There is a natural way of defining the appropriate Haar weights $\hat{\varphi}$ and $\hat{\psi}$, and it can be shown that $(\hat{M}, \hat{\Delta})$ is also a locally compact quantum group, which is the *dual quantum group*. The operator $\hat{W} := \Sigma W^* \Sigma$ is the multiplicative unitary for $(\hat{M}, \hat{\Delta})$. It can be shown that $W \in M \otimes \hat{M}$ and $\Sigma W^* \Sigma \in \hat{M} \otimes M$.

The left Haar weight $\hat{\varphi}$ on $(\hat{M}, \hat{\Delta})$ is uniquely characterized by the GNS data $(\mathcal{H}, \iota, \hat{\Lambda})$, where the GNS map $\hat{\Lambda} : \mathfrak{N}_{\hat{\varphi}} \to \mathcal{H}$ is given by the following formulas:

$$\hat{\Lambda}\big((\omega \otimes \mathrm{id})(W)\big) = \xi(\omega) \quad \text{and} \quad \big\langle \xi(\omega), \Lambda(x) \big\rangle = \omega(x^*).$$

To be a little more precise, consider:

$$\mathcal{I} = \big\{ \omega \in \mathcal{B}(\mathcal{H})_* : \exists L \geq 0 \text{ such that } |\omega(x^*)| \leq L \|\Lambda(x)\| \text{ for all } x \in \mathfrak{N}_\varphi \big\}.$$

Then for every $\omega \in \mathcal{I}$, we can find $\xi(\omega) \in \mathcal{H}$ such that $\omega(x^*) = \big\langle \xi(\omega), \Lambda(x) \big\rangle$ for all $x \in \mathfrak{N}_\varphi$ (by Riesz theorem). The equation above is understood as saying that the elements $(\omega \otimes \mathrm{id})(W)$, $\omega \in \mathcal{I}$, form a core for $\hat{\Lambda}$ and that $\hat{\Lambda}\big((\omega \otimes \mathrm{id})(W)\big) = \xi(\omega)$.

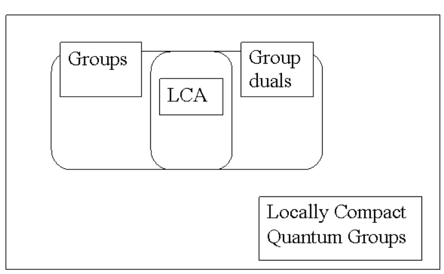
Meanwhile, analogously as before, with $\hat{W} = \Sigma W^* \Sigma$ now being the multiplicative unitary, the (dense) subspace of the elements $(\omega \otimes \mathrm{id})(W^*)$, for $\omega \in \mathcal{B}(\mathcal{H})_*$, forms a core for the antipode $\hat{S}$, and $\hat{S}$ is characterized by $\hat{S}\big((\omega \otimes \mathrm{id})(W^*)\big) = (\omega \otimes \mathrm{id})(W)$. The unitary antipode and the scaling group can be also found, giving us the polar decomposition of $\hat{S} = \hat{R}\hat{\tau}_{-\frac{i}{2}}$.

Repeating the whole process beginning with $(\mathcal{H}, \iota, \hat{\Lambda})$, we can further construct the dual $(\hat{\hat{M}}, \hat{\hat{\Delta}})$ of $(\hat{M}, \hat{\Delta})$. The generalized Pontryagin duality result says: $(\hat{\hat{M}}, \hat{\hat{\Delta}}) = (M, \Delta)$, with $\hat{\hat{\varphi}} = \varphi$ and $\hat{\hat{\Lambda}} = \Lambda$. The other structure maps for $(\hat{\hat{M}}, \hat{\hat{\Delta}})$ are also identified with those of (M, Δ): For instance, $\hat{\hat{S}} = S$. One useful result is the following, similar to an earlier equation, with π now considered as the embedding map ι and $\hat{\hat{\Lambda}} = \Lambda$:

$$\big\langle \Lambda\big((\mathrm{id} \otimes \omega)(W^*)\big), \hat{\Lambda}(y) \big\rangle = \omega(y^*).$$

Here again, we actually need to consider a similar set $\hat{\mathcal{I}}$, and the equation is accepted with the understanding that the elements $(\mathrm{id} \otimes \omega)(W^*)$, $\omega \in \hat{\mathcal{I}}$, form a core for Λ.

[Remark]: The results here show that the quantum group theory has a satisfactory duality picture. In general, the "dual" object of an arbitrary locally compact group is no longer a group. However, by going to a larger category of quantum groups, one can consider both groups and their dual objects using the unified language of quantum groups. This point of view will be explored a little further in Chapter 3.



2.5 Examples of locally compact quantum groups

Due to the highly technical nature of the locally compact quantum group theory, there have been only a handful of examples constructed until recently. But the situation is slowly improving. The following is a small sample of examples.

(1). Compact quantum groups: Of course, all compact quantum groups are locally compact quantum groups. In the compact case, the Haar weights are always bounded, and there is no distinction between left and right Haar weights. So it is customary to consider the unique *Haar state*, by requiring that $\varphi(1) = 1$.

(2). Quantum $E(2)$ group: To construct a C^*-algebra that can be considered as a "quantized $C_0(G)$", a possible method of approach is to work in terms of generators and relations, where the generators are "(quantum) coordinate functions". Unlike in the compact case, however, these generators will be in general unbounded, and in particular, they cannot belong to the C^*-algebra itself. One needs to discuss unbounded elements that are *affiliated* with the C^*-algebra and clarify in which sense such elements can generate the C^*-algebra. Baaj and Woronowicz introduced the notion of affiliation relation, and used it to construct the quantum group $E_\mu(2)$.

Briefly speaking, an element T should be affiliated with a C^*-algebra A if bounded continuous functions of T belong to $M(A)$. In particular, we expect $z(T) \in M(A)$, where $z : \mathbb{C} \to \mathbb{C}$ is defined by $z(\lambda) = \frac{\lambda}{\sqrt{1+|\lambda|^2}}$. Since $1 - |z(\lambda)|^2 = (1 + |\lambda|^2)^{-1}$ for all $\lambda \in \mathbb{C}$, we should have: $z(T) = T\bigl(1 - |z(T)|^2\bigr)^{\frac{1}{2}} \in M(A)$. This observation suggests the following:

[Definition]: Let A be a C^*-algebra and $T : \mathrm{Dom}(T) \subseteq A \to A$ a densely defined linear map. We say T is *affiliated* with A, written $T\eta A$, if there exists a $z_T \in M(A)$ such that $\|z_T\| \leq 1$ and for all $x, y \in A$,

$$x \in \mathrm{Dom}(T) \text{ and } y = Tx \quad \Longleftrightarrow \quad \text{there exists } a \in A \text{ with } x = (1 - z_T^* z_T)^{\frac{1}{2}} a \text{ and } y = z_T a.$$

Here, the element z_T is called the "z-transform" of T, and is uniquely determined by T.

[Remark]: There exists the notion of an affiliated element of a von Neumann algebra, but it is different from the notion introduced above. If M is a von Neumann algebra and T is an affiliated element of M in the C^*-algebraic sense, then $T \in M$. Meanwhile, there are various technical issues to consider, some of which are not trivial, like making sense of the tensor products of affiliated elements, etc. Refer to the papers by Woronowicz.

The quantum group $E_\mu(2)$ was constructed by Woronowicz. First consider the group $E(2) \subset GL_2(\mathbb{C})$, consisting of all matrices of the form $g_{(v,n)} = \begin{pmatrix} v & n \\ 0 & \bar{v} \end{pmatrix}$, where $v \in \mathbb{T}, n \in \mathbb{C}$. The group acts on $\mathbb{C}$ by $g_{(v,n)}\lambda = v^2\lambda + vn$, and $\{I_2, -I_2\}$ is the kernel of this action. So $E(2)$ is the unique connected double cover of the group of rotations and dilations of the Euclidean plane. The *-algebra $\mathcal{A}$ of polynomial functions contained in $C\bigl(E(2)\bigr)$ is generated by the coordinate functions $v : g_{(v,n)} \mapsto v$ and $n : g_{(v,n)} \mapsto n$. As a deformation, define $\mathcal{A}_\mu$, the universal unital *-algebra generated by elements v and n, with the relations that: v is unitary ($v^*v = 1 = vv^*$), n is normal ($n^*n = nn^*$), and $vn = \mu nv$.

The C^*-algebra $E_\mu(2)$, constructed as a certain crossed product algebra, turns out to be a universal C^*-algebra with affiliated elements v and n satisfying the algebraic relations above. There also some additional spectral conditions on the generators, so that the comultiplication is well defined. In this way, one obtains a C^*-bialogebra $\big(E_\mu(2), \Delta\big)$. One can also consider the dual C^*-bialgebra $\hat{E}_\mu(2)$, as well as the corepresentation theories of $E_\mu(2)$ and $\hat{E}_\mu(2)$.

Later, a left- and right-invariant Haar weight on $\big(E_\mu(2), \Delta\big)$ was found by Baaj, which means that $\big(E_\mu(2), \Delta\big)$ is in fact a (unimodular) locally compact quantum group. He also studied the multiplicative unitary associated to $E_\mu(2)$ and showed that it is not regular but only semi-regular. More recently, Jacobs gave a different approach of constructing $E_\mu(2)$.

Meanwhile, it is known that the quantum groups $E_\mu(2)$ and $SU_\mu(2)$ are related via a certain contraction procedure.

(3). Quantum group $\widetilde{SU}_\mu(1.1)$: Recall that $SU(1,1)$, which is isomorphic to $SL(2,\mathbb{R})$, is the linear Lie group $\big\{X \in SL(2,\mathbb{C}) : X^*UX = U\big\}$, where $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Various attempts have been made since the early days of quantum groups to construct a "quantum $SU(1,1)$ group", but ultimately failed. Woronowicz showed that quantum $SU(1,1)$ does not exist as a locally compact quantum group! Not surprisingly, this was considered to be quite a setback for the theory of locally compact quantum groups in the operator algebra setting.

Later works gave strong indications that the normalizer $N = N_{SL(2,\mathbb{C})}\big(SU(1,1)\big)$ of $SU(1,1)$ in $SL(2,\mathbb{C})$ is a better quantization candidate than $SU(1,1)$ itself. By using the method of generators and relations, Koelink and Kustermans constructed N_q, a locally compact quantum group that is a deformation of N. A heavy dose of special function theory is needed, using a certain class of q-hypergeometric functions and their orthogonality relations, to construct the coassociative comultiplication. The Haar weight, antipode, and its polar decomposition were all obtained.

(4). Quantum $az + b$-group: The quantum $az + b$-group was originally constructed by Woronowicz, and further studied later by Van Daele and Soltan. First, one starts from the classical $az + b$-group of affine transformations of the plane $\mathbb{C}$. It can be identified with the subgroup G of $GL(2,\mathbb{C})$, consisting of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a, b \in \mathbb{C}$, $a \neq 0$. As before, consider the coordinate functions a and b, which are the generators of the *-subalgebra $\mathcal{A} \subset C(G)$. The subalgebra $\mathcal{A}$ can be characterized as the universal unital commutative *-algebra generated by elements a, b, where a is invertible. It becomes a Hopf *-algebra, together with $\Delta(a) = a \otimes a$, $\Delta(b) = a \otimes b + b \otimes 1$, and the maps ϵ and S given by $\epsilon(a) = 1$, $\epsilon(b) = 0$, $S(a) = a^{-1}$, $S(b) = -a^{-1}b$, $S(a^*) = (a^*)^{-1}$, $S(b^*) = -(a^*)^{-1}b^*$.

For a deformation, fix $\mu \in \mathbb{C}$, $\mu \neq 0$, and define $\mathcal{A}_\mu$, the unital *-algebra generated by elements a, b such that: a is normal and invertible, b is normal, $ab = \mu^2 ba$, $ab^* = b^*a$. It is again a Hopf *-algebra, with Δ, ϵ, S as before. And, there exists a dual pairing $\langle\ ,\ \rangle$ of $\mathcal{A}_\mu$ with itself such that $\langle a, a\rangle = \mu^2$, $\langle a, b\rangle = \langle b, a\rangle = 0$, $\langle b, b\rangle = t$, where $t \in \mathbb{C}$ is arbitrary.

When we try to move on to the C^*-algebra setting, the generators a and b should be normal operators on a Hilbert space. Let $a = u|a|$ and $b = v|b|$ be the polar decompositions. Then it is appropriate to replace the earlier algebraic relations by the following: a, b are normal, $\mathrm{Ker}(a) = \{0\}$, $ubu^* = \mu b$, $|a|^{it} b |a|^{-it} = \mu^{it} b$ for all $t \in \mathbb{R}$. The real issue is in constructing the comultiplication, because the comultiplication for the Hopf *-algebra $\mathcal{A}_\mu$ does not carry over. So similarly as in the case of the quantum group $E_\mu(2)$, we would need some extra spectral conditions on the generators, and construct a universal C^*-algebra as a certain crossed product algebra.

It turns out that it is actually easier to construct a good multiplicative unitary W. This can be done using certain "quantum exponential function", and also using the dual pairing above. The antipode map and the Haar weights follow from W, though technical. Using a similar approach as above, one can also construct the "quantum $ax + b$-group".

One interesting feature to point out is that in this example, we have: $\psi \circ \tau_t = |\mu^{-4it}|\psi$. This means that the scaling constant ν is in general not 1. The quantum $az + b$-group was the first example of a locally compact quantum group with scaling constant not equal to 1.

(5). The bicrossed product construction: Given an action of a group G on a C^*-algebra A, one obtains a C^*-dynamical system (G, A, α), from which one can construct the crossed product C^*-algebra $A \rtimes_\alpha G$. Regarding A as a quantum space or a quantum manifold, this framework would encode (non-commutative) geometric information about the system, including the orbits. A natural question to ask is: What if G acts (α) on a space X, which is itself a group and also acts (γ) on G, with the two actions "compatible" with each other? From this, we would be able to form two crossed products $C_0(X) \rtimes_\alpha G$ and $C_0(G) \rtimes_\gamma X$. With the "compatibility", they will become quantum groups. This suggests us the notion of a "matched pair" of groups (Kac, Majid, Baaj, Skandalis, Vaes, Vainerman, ...). See below.

Let G and H be locally compact groups, and suppose that the maps

$$\alpha : G \times H \to H, (g, s) \mapsto \alpha_g(s) \qquad \text{and} \qquad \gamma : G \times H \to G, (g, s) \mapsto \gamma_s(g)$$

are defined nearly everywhere and measurable. Suppose further that

$$\begin{aligned} &\gamma_{st}(g) = \gamma_s\big(\gamma_t(g)\big), && \alpha_g(st) = \alpha_{\gamma_t(g)}(s)\alpha_g(t), && \text{for nearly all } (s, t, g), \\ &\alpha_{gh}(s) = \alpha_g\big(\alpha_h(s)\big), && \gamma_s(gh) = \gamma_{\alpha_h(s)}(s)\gamma_s(h), && \text{for nearly all } (g, h, s). \end{aligned}$$

Then we say (G, H) is a *matched pair* of groups. Earlier definitions had the $\alpha_g(s)$ and the $\gamma_s(g)$ to be everywhere defined and continuous. But the conditions have been loosened, to allow for more interesting examples.

Let (G, H) be a matched pair. Define $\alpha : L^\infty(H) \to L^\infty(G \times H)$ by $\big(\alpha(f)\big)(g, s) := f\big(\alpha_g(s)\big)$. Then we will have: $(\mathrm{id} \otimes \alpha)\alpha = (\Delta_G \otimes \mathrm{id})\alpha$, where Δ_G is the usual comultiplication on $L^\infty(G)$. From this data, we can define the crossed product von Neumann algebra M, as the von Neumann algebra generated by $\alpha\big(L^\infty(H)\big)$ and $\mathcal{L}(G) \otimes 1$, where $\mathcal{L}(G)$ is the group von Neumann algebra. We see that M acts naturally on the Hilbert space $L^2(G \times H)$. Now define the unitary operator $W \in \mathcal{B}\big(L^2(G \times H \times G \times H)\big)$, by

$$W\xi(g, s; h, t) := \xi\big(\gamma_{\alpha_g(s)^{-1}t}(h)g, s; h, \alpha_g(s)^{-1}t\big).$$

This becomes a multiplicative unitary, and we can define a comultiplication on M by the formula: $\Delta(z) = W^*(1 \otimes z)W$, for $z \in M$. It turns out that (M, Δ) is a locally compact quantum group, with its (left) Haar weight obtained as the dual weight on M of the (left) invariant weight on $L^\infty(H)$.

In addition to the matched pair data, we may incorporate cocycles satisfying certain equivariance conditions. In this way, we obtain a "cocycle matched pair", and as above, we can construct a locally compact quantum group by first constructing a suitable multiplicative unitary operator. These constructions are special cases of the *bicrossed product construction* and the *cocycle bicrossed production* frameworks, where we consider matched pairs or cocycle matched pairs of quantum groups. "Kac systems" by Baaj and Skandalis, and Drinfeld's celebrated "quantum double construction" can be covered under these frameworks.

The best aspect about the bicrossed product framework is that it does not rely on the (possibly unbounded) generators and relations, and therefore technically simpler. Still, the framework is sufficiently general to include some interesting examples. For instance, this was the method used by Baaj, Skandalis, Vaes, when they constructed a non-semiregular quantum group. However, as is the case for any general method, having the framework is not enough to construct actual examples: One would need a supply of specific matched pairs together with compatible cocycles. A possible suggestion is given below.

[Combining a geometric approach with the bicrossed product framework] (Kahng): We indicated earlier that Poisson–Lie groups are natural candidates to perform quantization, hopefully obtaining quantum groups. On the other hand, deformation quantization of a Poisson structure usually gives us only the underlying algebra. Construction of the comultiplication and other structure maps for the quantum group would be carried out separately, often guided by the Poisson–Lie group data.

In some cases, when enough information is known at the level of the Poisson–Lie group and if the Poisson bracket is of a certain type, we may be able to streamline the process a little by incorporating the bicrossed product framework.

For instance, consider the three examples of Poisson–Lie group structures given towards the end of § 1.4. Each of them is a *cocycle perturbation* of a linear Poisson bracket on $\mathfrak{h}^*$, and we know from §1.3 (4) that it can be also considered as a central extension of the linear Poisson bracket on $(\mathfrak{h}/\mathfrak{z})^*$. This means that there is a good chance of finding a suitable cocycle matched pair. Indeed, in our given example, we achieved a success by working with $G_1 \cong \mathfrak{z}^\perp$, $G_2 \cong \mathfrak{h}/\mathfrak{z}$, and a group cocycle corresponding to the Lie algebra cocycle built into the Poisson bracket. As quantum groups, these examples are relatively simple. By considering more complicated types of Poisson structures, we may be able to construct more interesting examples of quantum groups.

An advantage of having a Poisson geometric perspective on quantum groups is that the Poisson–Lie group data and the relevant "dressing actions" could shed some lights on the representation theory. For our specific quantum groups, some modest results on representation theory were obtained using the dressing orbits, the symplectic leaves, and Kirillov-type orbit analysis.

Chapter 3

Beyond quantum groups—Duality, Quantum groupoids

In a certain sense, the two most prominent motivations for studying quantum groups are: (1). Providing a framework for quantum symmetry; and (2). Providing a generalized duality framework.

In principle, a quantum group is a natural generalization of a group within the setting of non-commutative (or quantum) geometry. Just as the groups naturally arise as the objects encoding the symmetry of various physical systems, one may expect that the quantum groups should play similar roles as the "quantum symmetry" objects: There may be a situation in quantum physics in which the symmetry in the classical sense is broken, but by working with a more general framework of quantum groups we could encode the (quantum) symmetry of the system.

In fact, the very first notions of quantum groups were introduced by Faddeev, Drinfeld, et. al., as they were developing the "quantum inverse scattering method (QISM)", with the hope that they could construct and solve "integrable" quantum systems. Since then, there have been numerous examples constructed and whose representation theory studied, especially in the setting of quantized universal enveloping algebras. Even in the operator algebraic setting, the recent developments in quantum isometry groups (Bhowmick, Goswami) are in this direction.

In another direction, consider an abelian locally compact group G. Then it is known that the set of continuous characters on G can be also given an abelian locally compact group structure, written $\hat{G}$. The famous Pontryagin duality says that $\hat{\hat{G}}$, the "bidual" of G, is canonically isomorphic to G. This means the the category of abelian locally compact groups is "self-dual". However, when one tries to generalize the duality to non-abelian groups, one runs into serious difficulties. It is of note that there have been some duality results obtained over the years (for instance, the Tannaka–Krein duality, where one considers the set of equivalence classes of irreducible representations as the dual object), but none of them were quite satisfactory because any reasonable notion of the dual object of a group is no longer a group.

So it has been an interesting problem ever since to look for a *self-dual category* that includes all locally compact groups. A nice answer was given by Kac, Vainerman and Enock, Schwartz, which is now called the theory of *Kac algebras*. In can be said that the Kac algebra theory was the first topological theory of quantum groups. However, with the discovery of the quantum $SU(2)$ group, which cannot be obtained as a Kac algebra, and the development of compact quantum groups by Woronowicz, it became clear that we needed an even larger category that would unify compact quantum groups and Kac algebras (including all locally compact groups). With the generalized Pontryagin duality result proved by Kustermans and Vaes, the category of *locally compact quantum groups* satisfied this endeavor.

In this chapter, we will discuss a little about this duality aspect of the quantum groups. We do not plan to say much about the quantum symmetry aspect. Instead, we wish to introduce the recent (still being developed) notion of *quantum groupoids*. Groupoid category is known to contain groups, group actions, equivalence relations, etc., and the hope is that the category of quantum groupoids would provide a nice perspective on quantum groups as well as the actions of the quantum groups.

3.1 Generalized Pontryagin duality

Let us first review some known cases ...

(1). [Pontryagin duality theorem for LCA groups]: Suppose G is a locally compact abelian (LCA) group, with Haar measure. Then let

$$\begin{aligned}\hat{G} &= (\text{all irred. unitary rep.s}) = (\text{all continuous, } \mathbb{T}\text{-valued "characters"}) \\ &= \bigl\{\xi : G \to \mathbb{T}, \xi(st) = \xi(s)\xi(t)\bigr\}.\end{aligned}$$

By Gelfand theory, it is known that

$$\hat{G} \cong \mathfrak{M}\bigl(L^1(G), *\bigr) = \bigl(\text{multiplicative linear functionals on } L^1(G)\bigr) \subseteq L^\infty(G),$$

with the weak-* topology, and with the convolution product. In fact, for $\xi \in \hat{G}$, we have: $\chi_\xi \in \mathfrak{M}\bigl(L^1(G), *\bigr)$, such that

$$\chi_\xi : L^1 \ni f \mapsto \chi_\xi(f) = \int f(x)\xi(x)\,dx, \qquad \chi_\xi(f * g) = \chi_\xi(f)\chi_\xi(g).$$

Meanwhile, $\hat{G}$ with its topology becomes also a LCA group, and the *Pontryagin duality* holds: $\hat{\hat{G}} \cong G$. This means that the category of LCA groups is a "self-dual" category".
[Examples]: $\hat{\mathbb{T}} \cong \mathbb{Z}$, $\hat{\mathbb{Z}} \cong \mathbb{T}$, $\hat{\mathbb{R}} \cong \mathbb{R}$.

(*). However, as noted earlier, given an arbitrary locally compact group G, its dual object, $\hat{G} =$ (all irred. unitary rep.s), is in general not a group!

(2). [The Fourier transform]: One can define the *Fourier transform*, which is none other than the Gelfand map $\mathcal{F} : L^1(G) \ni f \mapsto \hat{f} \in C_0(\hat{G})$, where

$$\hat{f}(\xi) = \int_G f(s)\overline{\langle \xi, s\rangle}\, ds.$$

Here $\langle \xi, s\rangle = \xi(s)$ denotes the dual pairing, and $\lambda \mapsto \bar{\lambda}$ is the complex conjugation.

Following general results are known ...

- [Inverse F.T.]: There is a suitable Plancherel measure on $\hat{G}$ so that we can define:

$$h^{\vee}(s) = \int_{\hat{G}} h(\xi)\langle \xi, s\rangle\, d\xi.$$

- [Fourier inversion theorem]: $(\hat{f})^{\vee} = f$.
- [Plancherel theorem]: $\|\mathcal{F}(f)\|_2 = \|f\|_2 \qquad \dots \quad L^2(G) \cong L^2(\hat{G})$.
- At the algebra level, $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g) \quad \dots \quad C^*(G) \cong C_0(\hat{G})$.

Clearly, the Fourier transform is naturally tied with the duality theory.

(3). [Duality of the (finite-dimensional) Hopf algebras]: Recall the definition of a Hopf algebra, generalizing $K(G)$, the space of functions on a finite group G. Given a Hopf algebra (B, Δ, ϵ, S), one can consider $B^* =$ (dual v.s. of B), and give it a Hopf algebra structure $(B^*, \hat{\Delta}, \hat{\epsilon}, \hat{S})$. For instance, via

$$\langle a \,|\, f \cdot g\rangle = \langle \hat{\Delta}a \,|\, f \otimes g\rangle, \qquad \langle a \otimes b \,|\, \Delta f\rangle = \langle a \cdot b \,|\, f\rangle, \quad \dots$$

It is easy to see that $B^{**} \cong B$.

(*) Duality picture is all right in this setting. But, we know that Hopf algebras are not really satisfactory generalizations for groups!

[Remark]: Through (1), (2), (3), we saw that for a general (non-abelian) locally compact group G, the dual $\hat{G}$ is no longer a group. So there is *no* Pontryagin duality and *no* Fourier transform. Hopf algebras are useful and has nice duality theory, but *not* quite satisfactory. Later, the notion of Kac algebras was introduced to combine groups and Hopf algebras, but examples have been found that can be considered as reasonable generalizations of groups ("quantum groups") but not Kac algebras. All these indicated that we need a better framework to develop the duality theory. This is among the main motivations for the theory of *locally compact quantum groups*, where generalized Pontryagin duality holds.

In § 2.4.5, we saw that the generalized Pontryagin duality holds at the level of locally compact quantum groups, and the duality is encoded by the multiplicative unitary operator W. If

$W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is the multiplicative unitary associated with the (mutually dual) pair of quantum groups (M, Δ) and $(\hat{M}, \hat{\Delta})$, we have:

$$M = \overline{\mathcal{A}(W)}^{w} = \overline{\{(\mathrm{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{w},$$

together with the comultiplication $\Delta a = W^*(1 \otimes a)W$. And,

$$\hat{M} = \overline{\hat{\mathcal{A}}(W)}^{w} = \overline{\{(\theta \otimes \mathrm{id})(W) : \theta \in \mathcal{B}(\mathcal{H})_*\}}^{w},$$

with the comultiplication $\hat{\Delta} b := \Sigma W(b \otimes 1)W^*\Sigma$.

[Dual pairing]: Obviously, $\hat{M} \neq M^*$. But we still have a "dual pairing" at the level of the dense subalgebras $\mathcal{A}(W) \subseteq M$ and $\hat{\mathcal{A}}(W) \subseteq \hat{M}$. Define, for $a = (\mathrm{id} \otimes \omega)(W) \in \mathcal{A}(W)$ and $b = (\theta \otimes \mathrm{id})(W) \in \hat{\mathcal{A}}(W)$, let

$$\langle b \,|\, a \rangle := (\theta \otimes \omega)(W) = \theta(a) = \omega(b).$$

Then: $\langle b_1 b_2 \,|\, a \rangle = \big\langle b_1 \otimes b_2 \,|\, \Delta(a) \big\rangle$ and $\langle b \,|\, a_1 a_2 \rangle = \big\langle \hat{\Delta}^{\mathrm{cop}}(b) \,|\, a_1 \otimes a_2 \big\rangle$, where $a, a_1, a_2 \in \mathcal{A}(W)$ and $b, b_1, b_2 \in \hat{\mathcal{A}}(W)$.

[Problems with the duality picture]: With the generalized Pontryagin duality at the LCQG level and the (densely defined) dual pairing map, the duality picture is more or less complete. But, there are still some improvements to be made. Among others, we point out here two issues:

1. [Issue of the Fourier transform]: The dual pair of quantum groups, M and $\hat{M}$, are both defined on the same Hilbert space $\mathcal{H}$. Because of this, the Fourier transform is sort of "hidden". (Basically, we would be doing $L^2(G) \cong L^2(G)$, which makes the F.T. the identity map.) Nevertheless, the F.T. should be there. We just need to approach it properly.

2. [Issue with the dual pairing]: Unlike in the Hopf algebra case, there is no way of constructing a dual object of a W^*-bialgebra (M, Δ), without resorting to the existence of the Haar weight and the multiplicative unitary operator.

We will discuss Issue (1) in § 3.2 below. We will not discuss Issue (2) in much detail, but let us consider the following example.

[A problem concerning Issue (2)]: Consider a locally compact quantum group (A, Δ), together with the Haar weights. In some cases, we can find a certain "cocycle" σ such that we can "twist" (or deform) the comultiplication (e.g. $\delta = \sigma \circ \Delta$). A natural question is to see if we can construct a "twisted $\hat{A}$", via

$$\langle b \times_\sigma d \,|\, a \rangle = \big\langle b \otimes d \,|\, \delta(a) \big\rangle,$$

where $a \in A$ and $b, d \in \hat{A}$. For finite-dimensional Hopf algebras or quantum groups in the algebraic framework, this is do-able. But, in our "topological" case, things are not so easy.

Among the obstacles is that while the new map δ is still coassociative (so a valid comultiplication), it may as well happen that δ never determines a quantum group. That is, no Haar weights, and therefore no practical way of working with the dual pairing.

[Example] (Twisting of the quantum double): For $A = C^*(G)$, consider $A_D = \hat{A}^{\mathrm{op}} \bowtie A = C_0(G) \rtimes_\alpha G$, where $\alpha : G \times G \to G$ is the conjugation: $\alpha_z(s) = zsz^{-1}$. To see its structure, consider $F \in C_c(G \times G)$ and $\xi \in L^2(G \times G)$, define:

$$\begin{aligned} L_F\xi(s,t) &= \int F(s,z)\xi(z^{-1}sz, z^{-1}t)\,dz \qquad\qquad \text{“}\alpha_z(s) = zsz^{-1}\text{”} \\ F^*(s,t) &= \overline{F(t^{-1}st, t^{-1})} \end{aligned}$$

As a C^*-algebra, we have: $A_D = \overline{L\big(C_c(G \times G)\big)}^{\|\ \|_{\mathrm{op}}}$. It has the Hopf structure, and the Haar weight:

$$\begin{aligned} (\Delta_D F)\xi(s_1,t_1;s_2,t_2) &= \int F(s_1s_2,z)\xi(z^{-1}s_1z, z^{-1}t_1; z^{-1}s_2z, z^{-1}t_2)\,dz. \\ \big(S_D(F)\big)(s,t) &= F(t^{-1}st, t^{-1}) \\ \varphi_D(F) &= \int F(s, 1_G)\,ds \end{aligned}$$

In fact, $A_D = C_0(G) \rtimes_\alpha G$ is a C^*-algebraic locally compact quantum group, which is non-commutative and non-cocommutative. This is a version of Drinfeld's quantum double construction. Its dual is $\widehat{A_D} = A \otimes \hat{A}$.

Using a certain element $\mathcal{R} \in M(A_D \otimes A_D)$ (satisfying the "quantum Yang–Baxter equation": $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$), we can deform the comultiplication: $\delta = \mathcal{R} \circ \Delta_D$. Consider the following question (Q.): Can we construct a "twisted $\widehat{A_D}$"?

Since (A_D, δ) is no longer a quantum group, we do not have any multiplicative unitary operator to work with, and no general method is known. On the other hand, since the situation comes from an ordinary group, we do have an answer in this particular case, and we can actually construct a C^*-algebra that would be considered as having a product dual to δ, giving us the "twisted $\widehat{A_D}$". In turns out that

$$(\text{“twisted } \widehat{A_D}\text{”}) \cong \big[(1 \otimes \hat{A}^{\mathrm{op}})\Delta(A)\big] \cong C_0(G) \rtimes_\tau G \cong \mathcal{K}\big(L^2(G)\big),$$

where τ is the translation. Conceptually, it is based on the strategy that $\langle b \,|\, a\rangle = \langle b \otimes a | W_{\mathcal{R}}\rangle$, but $W_{\mathcal{R}}$ is not a multiplicative unitary operator.

(*) This result was not surprising, because similar results have been found in the Hopf algebra case (Lu, obtaining $\cong \mathrm{End}(H)$), and in the algebraic framework of multiplier Hopf algebras (Delvaux, Van Daele). However, the result "$\big[(1 \otimes \hat{A}^{\mathrm{op}})\Delta(A)\big]$" in the general case is not known whether it is true. If this result were indeed true, we may obtain in this way some highly-nontrivial C^*-algebras.

3.2 Fourier transform on locally compact quantum groups

Note that From the Fourier inversion theorem, $(\hat{f})^{\vee} = f$, we have:

$$f(x) = (\hat{f})^{\vee}(x) = \int\int f(z)\overline{\langle \xi, z\rangle}\langle \xi, x\rangle \, dz d\xi.$$

This implies that: $\int \overline{\langle \xi, z\rangle}\langle \xi, x\rangle \, d\xi = \delta_0(z - x)$. Therefore, for $a \in C_c(G)$ and $k \in C_c(G)$, we may consider:

$$\begin{aligned}\big\langle \hat{a} \,|\, k \big\rangle &:= \int \hat{a}(\xi)\chi_\xi(k)\, d\xi = \int \hat{a}(\xi)k(x)\langle \xi, x\rangle \, d\xi dx \\ &= \int k(x)a(z)\overline{\langle \xi, z\rangle}\langle \xi, x\rangle \, dz d\xi dx = \int k(x)a(x)\, dx.\end{aligned}$$

The computation suggests that $\big\langle \mathcal{F}(a) \,|\, c \big\rangle = \varphi(ca)$ or "$\mathcal{F}(a) = \varphi(\cdot\, a)$".

The question is whether we can still push further and re-formulate "$\mathcal{F}(a) = \varphi(\cdot\, a)$", in the setting of locally compact quantum groups. In the algebraic setting ("multiplier Hopf algebras"), Van Daele has achieved results in this direction. Our plan is to re-formulate this approach at the level of operator algebras.

Consider quantum groups $(M, \Delta, \varphi, \psi)$ and $(\hat{M}, \hat{\Delta}, \hat{\varphi}, \hat{\psi})$ on $\mathcal{H}$, together with the multiplicative unitary operator W. In addition, let Λ denote the GNS map for the weight φ on $\mathcal{H}$. And, let $\hat{\Lambda}$ be the GNS map for the weight $\hat{\varphi}$ on $\mathcal{H}$. Recall also the duality provided by W, at the level of the dense subalgebras $\mathcal{A}(W)\,(\subseteq M)$ and $\hat{\mathcal{A}}(W)\,(\subseteq \hat{M})$.
[Definition] (Kahng): Taking motivation from "$\mathcal{F}(a) = \varphi(\cdot\, a)$" or "$\big\langle \mathcal{F}(a) \,|\, c \big\rangle = \varphi(ca)$", we will define the **Fourier transform** of a, by

$$\mathcal{F}(a) := (\varphi \otimes \mathrm{id})\big(W(a \otimes 1)\big).$$

(*) Note that formally, we have: $\mathcal{F}(a) = (\theta \otimes \mathrm{id})(W) \in \hat{\mathcal{A}}$, where $\theta = \varphi(\cdot\, a)$. But we still need to clarify where $\mathcal{F}$ is defined ...

Consider the (dense) subset $\mathcal{J}\,(\subseteq M)$, where

$$\mathcal{J} = \big\{(\mathrm{id} \otimes \omega)(W^*), \omega \in \hat{M}_* : \exists L \geq 0 \text{ s.t. } |\omega(y^*)| \leq L\|\hat{\Lambda}(y)\|, \forall y \in \mathfrak{N}_{\hat{\varphi}}\big\}.$$

It is known that $\mathcal{J}$ forms a core for the GNS map Λ. Similarly, consider the (dense) subset $\hat{\mathcal{J}}\,(\subseteq \hat{M})$, where

$$\hat{\mathcal{J}} = \big\{(\theta \otimes \mathrm{id})(W), \theta \in M_* : \exists L \geq 0 \text{ s.t. } |\theta(x^*)| \leq L\|\Lambda(x)\|, \forall x \in \mathfrak{N}_{\varphi}\big\}.$$

As in the above, it is known that $\hat{\mathcal{J}}$ forms a core for the GNS map $\hat{\Lambda}$.

[Remark]: Loosely speaking, these are the spaces where the Fourier transform (and the inverse Fourier transform) would be defined. Note also that using the notation given earlier in § 2.4.5, we have $(\mathrm{id}\otimes\omega)(W^*)\in\mathcal{J}$ if and only if $\omega\in\hat{\mathcal{I}}$, and similarly, we have $(\theta\otimes\mathrm{id})(W)\in\hat{\mathcal{J}}$ if and only if $\theta\in\mathcal{I}$.

[Some technical comments]: Let $a\in\mathcal{J}$. Then for any $x\in\mathcal{N}_\varphi$, we have:

$$\big|\varphi(x^*a)\big| = \big|\langle\Lambda(a),\Lambda(x)\rangle\big| \le L\big\|\Lambda(x)\big\|,$$

where $L=\big\|\Lambda(a)\big\|$. There is still the issue of whether the map $\theta: x\mapsto\varphi(xa)$ is normal ($\theta\in M_*$), but this does work for elements contained in a certain dense subset of $\mathcal{J}$. [Consider the elements in the "Tomita algebra".] In that case, we would have: $(\theta\otimes\mathrm{id})(W)\in\hat{\mathcal{J}}$. In other words,

$$a\mapsto\mathcal{F}(a)=(\varphi\otimes\mathrm{id})\big(W(a\otimes 1)\big)=(\theta\otimes\mathrm{id})(W)\in\hat{\mathcal{J}}\,(\subseteq\hat{M}).$$

As a by-product, we also have: $\big\langle\hat{\Lambda}\big((\mathcal{F}(a)\big),\Lambda(x)\big\rangle=\theta(x^*)=\varphi(x^*a)=\big\langle\Lambda(a),\Lambda(x)\big\rangle$, true for any $x\in\mathfrak{N}_\varphi$. Since the vectors of the form $\Lambda(x)$, $x\in\mathfrak{N}_\varphi$, are dense in $\mathcal{H}$, it follows that $\hat{\Lambda}\big(\mathcal{F}(a)\big)=\Lambda(a)$ in the Hilbert space $\mathcal{H}$.

Similarly, there is a dense subset of $\hat{\mathcal{J}}\,(\subseteq\hat{M})$ so that for an element b contained in the set, we can define $\mathcal{F}^{-1}(b)\in\mathcal{J}\,(\subseteq M)$, by

$$\mathcal{F}^{-1}(b):=(\mathrm{id}\otimes\hat{\varphi})\big(W^*(1\otimes b)\big),$$

which is the **inverse Fourier transform**. As before, we have: $\big\langle\Lambda\big(\mathcal{F}^{-1}(b)\big),\hat{\Lambda}(y)\big\rangle=\big\langle\hat{\Lambda}(b),\hat{\Lambda}(y)\big\rangle$, for any $y\in\mathfrak{N}_{\hat{\varphi}}$. Since the vectors of the form $\hat{\Lambda}(y)$, $y\in\mathfrak{N}_{\hat{\varphi}}$, are dense in $\mathcal{H}$, this also means that $\Lambda\big(\mathcal{F}^{-1}(b)\big)=\hat{\Lambda}(b)$ in the Hilbert space $\mathcal{H}$.

[Side remark]: The dense subset of $\mathcal{J}$ that is mentioned above is slightly smaller than (but dense in) the set $L_{(-1/2)}$ considered by Caspers. He recently formulated a definition of a "L^p-Fourier transform" (for $1\le p\le 2$) on locally compact quantum groups. Our case would be, of course, the L^2-Fourier transform.

[The Fourier Inversion Theorem]:

- For $a\in\mathcal{J}$, we have: $\mathcal{F}^{-1}\big(\mathcal{F}(a)\big)=a$.
- For $b\in\hat{\mathcal{J}}$, we have: $\mathcal{F}\big(\mathcal{F}^{-1}(b)\big)=b$.

[Proof]: Use $\Lambda\big(\mathcal{F}^{-1}(\mathcal{F}(a))\big)=\hat{\Lambda}\big(\mathcal{F}(a)\big)=\Lambda(a)$.

[Plancherel Theorem]:

- For $a\in\mathcal{J}\,(\subseteq M)$, we have: $\hat{\varphi}\big(\mathcal{F}(a)^*\mathcal{F}(a)\big)=\varphi(a^*a)$.
- For $b\in\hat{\mathcal{J}}\,(\subseteq\hat{M})$, we have: $\varphi\big(\mathcal{F}^{-1}(b)^*\mathcal{F}^{-1}(b)\big)=\hat{\varphi}(b^*b)$.

[Proof]: Note that $\hat{\varphi}\big(\mathcal{F}(a)^*\mathcal{F}(a)\big)=\big\langle\hat{\Lambda}(\mathcal{F}(a)),\hat{\Lambda}(\mathcal{F}(a))\big\rangle=\big\langle\Lambda(a),\Lambda(a)\big\rangle=\varphi(a^*a)$.

[Convolution products on M and $\hat{M}$]: For $a, c \in \mathcal{J}\,(\subseteq M)$, we may define their "convolution product", as follows:

$$a * c := \mathcal{F}^{-1}\big(\mathcal{F}(a)\mathcal{F}(c)\big).$$

Then we have: $a * c = (\varphi \otimes \mathrm{id})\big([(S^{-1} \otimes \mathrm{id})(\Delta c)](a \otimes 1)\big)$. Similarly, for $b, d \in \hat{\mathcal{J}}\,(\subseteq \hat{M})$, we define their "convolution product", as follows:

$$b * d := \mathcal{F}\big(\mathcal{F}^{-1}(b)\mathcal{F}^{-1}(d)\big).$$

Then we have: $b * d = (\hat{\varphi} \otimes \mathrm{id})\big([(\hat{S}^{-1} \otimes \mathrm{id})(\hat{\Delta} d)](b \otimes 1)\big)$.

[The dual pairing]: The dual pairing map $\langle\ |\ \rangle : \hat{\mathcal{A}}(W) \times \mathcal{A}(W) \to \mathbb{C}$, as given earlier, has the following alternative description:

$$\langle b \,|\, a\rangle = \varphi\big(a\mathcal{F}^{-1}(b)\big) = \hat{\varphi}\big(\mathcal{F}(a^*)^* b\big) = (\varphi \otimes \hat{\varphi})\big[(a \otimes 1)W^*(1 \otimes b)\big].$$

For this to be valid, we need to restrict $\langle\ |\ \rangle$ to an appropriate domain. Even so, this provides a more direct way of evaluating the dual pairing map, unlike the earlier definition.

[Case of an ordinary group G]: Let us see how all these are manifested in the case of an ordinary locally compact group G, with a fixed left Haar measure, dx. Let $\mathcal{H}$ be the Hilbert space $L^2(G)$. Consider two well-known von Neumann algebras:

- $M = L^\infty(G)$, where $a \in L^\infty(G)$ is viewed as the multiplication operator π_a on $\mathcal{H} = L^2(G)$, by $\pi_a\xi(x) = a(x)\xi(x)$.
- $\hat{M} = \mathcal{L}(G)$, given by the left regular representation. That is, for $b \in C_c(G)$, let $L_b \in \mathcal{B}(\mathcal{H})$ be such that $L_b\xi(x) = \int_G b(z)\xi(z^{-1}x)\,dz$. We take $\mathcal{L}(G)$ as the W^*-closure of $L\big(C_c(G)\big)$.

As we have seen earlier, these two von Neumann algebras can be given (mutually dual) quantum group structures, whose comultiplication maps are determined by the multiplicative unitary operator $W \in \mathcal{B}\big(L^2(G \times G)\big)$, where $W\xi(s,t) = \xi(s, s^{-1}t)$.

In this case, our Fourier transform takes the following form:

- For $a \in C_c(G)$, we have: $\pi_a \in M$ and $\mathcal{F}(\pi_a) = L_a \in \hat{M}$.
- For $b \in C_c(G)$, we have: $L_b \in \hat{M}$ and $\mathcal{F}^{-1}(L_b) = \pi_b \in M$.

[Remark]: At the level of functions, the Fourier transform would give us $\mathcal{F} : a(x) \longrightarrow a(x)$, and so the Fourier transform would look trivial. This is because we viewed both M and $\hat{M}$ as defined on the same Hilbert space: $L^2(G) = L^2(G)$. For a better perspective, consider an abelian group G and $\hat{G}$ its dual group. By classical Fourier theory, we actually have the spatial isomorphism $F : L^2(G) \cong L^2(\hat{G})$. Then we can show easily that $L_a = F^{-1}\pi_{\hat{a}}F$, for $a \in C_c(G)$. [Here, $\hat{a} \in C_0(\hat{G})$ is the (classical) Fourier transform of a, and $\pi_{\hat{a}} \in \mathcal{B}\big(L^2(\hat{G})\big)$.]

We thus have: $\mathcal{F}(\pi_a) \cong \pi_{\hat{a}}$. Or, at the function level: $C_c(G) \ni a(x) \mapsto \hat{a}(p) \in C_0(\hat{G})$. In addition, we have:

$$(a * c)(x) = \int a(s)c(s^{-1}x)\, ds$$

and

$$\langle L_b \,|\, \pi_a \rangle = \int a(x)b(x)\, dx,$$

as expected. These observations show that our definition of the Fourier transform at the level of quantum groups is a reasonable generalization of the classical notion.

[Final remarks]: At this stage, however, the Fourier transform theory is rather primitive. The definition is now clarified, but not much else, unfortunately. The hope is that with more development, it could be useful in enhanced understanding of the general duality picture at the quantum group level.

From a different point of view, there is a sense that the Fourier transform should provide a certain generalization of the direct integral of the irreducible representations. By considering the "matrix coefficients" of the representations, we may gain some insights on q-special function theory. Meanwhile, having the alternative description of the dual pairing should be also useful when working with specific examples.

3.3 Quantum groupoids

An important aspect of groups is that they are often considered in terms of symmetries of a set. But then, these symmetries are defined on the whole of the set. In many situations, however, the important "symmetry" is not the one associated with globally defined bijections. In general, the algebraic structure characterizing symmetry (even including non-homogeneous symmetry) is provided by *groupoids*. The category of groupoids include as special cases groups, group actions, equivalence classes, and more. Therefore, it is natural to ask if we can consider a notion of a "quantum groupoid", similar to the notion of a "quantum group" to a group.

There are other motivations for studying quantum groupoids. One such is from the theory of subfactors, where one wishes to describe the symmetries of a given subfactor. To see this, suppose a group G acts on a factor N. We then obtain two inclusions of factors $N^G \subset N \subset N \rtimes G$, where N^G is the fixed point algebra, and $N \rtimes G$ is the crossed product. The subfactor $N^G \subset N$ is irreducible and of depth 2. Then it is a natural question to ask whether all irreducible, depth 2 subfactors are of this form. The answer is no, but Enock and Nest showed that for every irreducible depth 2 inclusion $N_0 \subset N_1$, there exists a locally compact quantum group M and an action of M on N_1 such that N_0 is equal to the fixed point algebra N_1^M, and the inclusion $N_1 \subset N_2$ given by the "basic construction" is isomorphic to the inclusion $N_1 \subset N_1 \rtimes M$. If the irreducibility assumption is dropped, then M is no longer a quantum group but shown to be a quantum groupoid (Enock, Vallin). We can see that quantum groupoids arise naturally in the study of subfactors.

3.3.1 Definition of a groupoid

Before trying to talk about quantum groupoids, we should review a little about ordinary groupoids. By definition, a groupoid is a small category with inverses. In terms of algebra, a groupoid can be regarded intuitively as a set with a partially defined multiplications for which the usual properties of a group hold whenever they make sense. See below:

[Definition]: A set $\mathcal{G}$ is called a **groupoid over a set** X, if it is equipped with the following structure maps:

1. A pair of maps $r : \mathcal{G} \to X$ and $s : \mathcal{G} \to X$, called the *range* and the *source* map. In a sense, an element $g \in \mathcal{G}$ may be thought of as an arrow (morphism) from $s(g)$ to $r(g)$.

2. A *(partial) multiplication* $m : \mathcal{G}^{(2)} \to \mathcal{G}$, where $\mathcal{G}^{(2)}$ is the set of composable pairs: $\mathcal{G}^{(2)} = \big\{(g, h) \in \mathcal{G} \times \mathcal{G} : s(g) = r(h)\big\}$. it is customary to write $m(g, h)$ as gh. The multiplication map must have the associativity property. That is, $(gh)k = g(hk)$, when $s(g) = r(h)$, $s(h) = r(k)$.

3. An embedding map $\epsilon : X \to \mathcal{G}$, called the *identity section*, such that $\epsilon\big(r(g)\big)g = g = g\epsilon\big(s(g)\big)$. Note that $r\big(\epsilon(x)\big) = x$ and $s\big(\epsilon(x)\big) = x$, for any $x \in X$.

4. There is also an *inversion* map $\iota : \mathcal{G} \to \mathcal{G}$, usually denoted by $\iota(g) = g^{-1}$, such that for all $g \in \mathcal{G}$, we have: $g^{-1}g = \epsilon\big(s(g)\big)$ and $gg^{-1} = \epsilon\big(r(g)\big)$.

Often, we write the base space X as $X = \mathcal{G}^{(0)}$. Among the more important examples of groupoids include: equivalence relations, "transformation groupoid" (coming from a group acting on a set), "fundamental groupoid", etc.

By a *topological groupoid*, we mean a groupoid equipped with a topology on its set of arrows, for which the inversion and the multiplication are continuous. Then also the range and source map are continuous. Incorporating the smooth structure, one can also consider *Lie groupoids*. Here, the maps r and s are further required to be surjective submersions, and the map ϵ and the partial multiplication are smooth.

Let $\mathcal{G}$ be a topological groupoid that is locally compact, Hausdorff, and second countable. To perform translation-invariant integration on $\mathcal{G}$, we need an analogue of the Haar measure of a locally compact group. The precise definition involves the fibers of the range and the source map, and we can define a *Haar system*. Using this, one can construct a *groupoid C^*-algebra*, based on convolution product (Renault).

3.3.2 Deformation quantization revisited

In noncommutative geometry, groupoids play a significant role as candidates for noncommutative locally compact spaces and manifolds, and produce the C^*-algebras normally regarded as "noncommutative spaces". Examples include Connes' foliation algebras. Through groupoids, the noncommutative geometric context provides the algebra of observables of a quantum mechanical system. Similarly, deformation quantization is often naturally interpreted in terms of the "gauge groupoid" or the "tangent groupoid" of Connes (see below). In addition, Lie groupoids and Lie algebroids play a useful role in the study of Poisson manifolds.

[Definition (Connes' tangent groupoid)]: Let M be a manifold. From $N = M \times M$ and an immersion $M \hookrightarrow M \times M$, $x \mapsto (x, x)$, one can define: $\mathcal{G}_M := \big(\{0\} \times TM\big) \cup \big((0,1] \times (M \times M)\big)$, which is a smooth manifold with boundary. We can give $\mathcal{G}_M$ a Lie groupoid structure over the base space $\mathcal{G}_M^{(0)} = [0,1] \times M$, in the following way:

- The fiber at $\hbar = 0$ is TM, the tangent bundle, which is a groupoid over M under the canonical bundle projection and addition in each $T_x M$.
- The fiber at any $\hbar \in (0,1]$ is the "pair groupoid" $M \times M$ over M.

The total space $\mathcal{G}_M$ is a groupoid with respect to fiberwise operations. Then $\mathcal{G}_M$ is called the *tangent groupoid* of M. There is a natural way to associate a groupoid C^*-algebra $C^*(\mathcal{G}_M)$, and it turns out that it is the section algebra of a continuous field of C^*-algebras over $[0,1]$. The fiber algebras are: $A_{\hbar=0} = C_0\big(T^*M\big)$ at $\hbar = 0$, and $A_\hbar = \mathcal{K}\big(L^2(M)\big)$, for $\hbar \in (0,1]$.
(*). The definition of a tangent groupoid is actually more general. What we gave above is a special case corresponding to the immersion $M \hookrightarrow M \times M$.

[Tangent groupoid as providing a deformation quantization framework]: It is known that the cotangent bundle T^*M over a manifold M is equipped with a canonical Poisson structure, called a "linear Poisson structure". By using the framework of Connes' tangent groupoid, Landsman and Ramazan could give a deformation quantization of this Poisson structure. In addition to observing that $A_{\hbar=0} = C_0\big(T^*M\big)$ and $A_\hbar = \mathcal{K}\big(L^2(M)\big)$, they also needed to show an appropriate "correspondence relation" at the level of a dense subspace of functions in $C_0(T^*M)$. This example does not belong to Rieffel's strict deformation quantization framework, so they modified Rieffel's definition a little to introduce the notion of "strict quanization".

This result suggests that for certain types of Poisson manifolds, one may be able to use some version of Connes' tangent groupoid and a relaxed version of Rieffel's framework to carry out deformation quantization in the C^*-algebras setting. Not much work has been done in this direction, however.

[A possible future project]: In the work of Landsman and Ramazan, a key point is that the manifold T^*M can be viewed as a dual of certain Lie bialgebroid, thereby obtaining a linear Poisson structure. This is more or less the same in principle to case (3) considered in § 1.3. It will be interesting to see if we can introduce a suitable cocycle perturbation to the linear Poisson structure such that a strategy similar to case (4) of § 1.3 could be applied. It is likely that one needs a quantum groupoid framework to work this out.

3.3.3 Weak Hopf algebras and quantum groupoids

Now that we reviewed the notion of a groupoid and gave some motivations for studying "quantum groupoids", we will give some report on the definitions being proposed and some results. Considering the technical difficulties, we will be brief, mentioning an algebraic framework (*weak multiplier Hopf algebras*) and a von Neumann algebraic framework (*measured quantum groupoids*). The C^*-algebraic framework is currently not developed yet.

Let us begin with the definition of a *multiplier Hopf algebra* by Van Daele, which we referred to earlier. In the below $M(A)$ denotes the multiplier algebra of an algebra A, which is unital and contains A as an ideal. Even though A is not necessarily a C^*-algebra, the definition of $M(A)$ is more or less the same as in the C^*-algebra setting.

[Definition]: Let A be an algebra with a non-degenerate product (i. e. for every $a \in A$, $a \neq 0$, we have $ab \neq 0$ and $ba \neq 0$ for all $b \in A$), while the linear span of AA is equal to A.
A *multiplier bialgebra* is a non-degenerate algebra A equipped with a non-degenerate homomorphism $\Delta : A \to M(A \otimes A)$ such that

- the following subsets of $M(A \otimes A)$ are contained in $A \otimes A \subseteq M(A \otimes A)$:

$$\Delta(A)(1 \otimes A), \quad \Delta(A)(A \otimes 1), \quad (A \otimes 1)\Delta(A), \quad (1 \otimes A)\Delta(A).$$

- Δ is coassociative: $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$.

In addition, if the linear maps T_1 and T_2, defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are bijective maps from $A \otimes A$ to $A \otimes A$, then (A, Δ) is called a *multiplier Hopf algebra*. A multiplier Hopf algebra (A, Δ) is "regular", if $(A^{\mathrm{op}}, \Delta)$ and $(A, \Delta^{\mathrm{cop}})$ are also multiplier Hopf algebras.

[Remark]: Every Hopf algebra is a multiplier Hopf algebra, and multiplier Hopf algebras are similar to Hopf algebras in many respects. For instance, they also possess a counit and an antipode. By incorporating an involution, we can also consider the notion of a "multiplier Hopf *-algebra". As a natural analogue of the Haar weight of a locally compact group, one can introduce the concept of a left and right "integral" on a multiplier Hopf algebra. They are fundamental to the duality theory of multiplier Hopf algebras.

A multiplier Hopf *-algebra with a positive left integral and a positive right integral is called an **algebraic quantum group**. Intuitively speaking, it may be considered as a "quantized $C_c(G)$" of a locally compact group G. Even though this remains a purely algebraic theory, it is remarkable to note that the framework of multiplier Hopf algebras and algebraic quantum groups essentially contain compact quantum groups, discrete quantum groups, and much more. Nowadays, this theory would be contained in the theory of locally compact quantum groups. However, it was this theory that led to the development of the Kustermans–Vaes framework of locally compact quantum groups, and due to the fact that this is technically simpler, it still remains an important and active area of research.

Next we turn to the definition of a *weak multiplier Hopf algebra* (Van Daele), modeled after the notion of a groupoid. In this setting, the comultiplication map Δ is not "unital", and the canonical maps T_1, T_2 are no longer assumed to be bijective. To avoid being too technical, we will skip some details.

[Definition]: A pair (A, Δ) is a **weak multiplier Hopf algebra**, if

- A is an idempotent algebra (i. e. $AA = A$) with a non-degenerate product
- $\Delta : A \to M(A \otimes A)$ is "full" coproduct with a (unique) counit ϵ
- There is an idempotent element $E \in M(A \otimes A)$ such that

$$\Delta(A)(1 \otimes A) = E(A \otimes A) \qquad \text{and} \qquad (A \otimes 1)\Delta(A) = (A \otimes A)E$$

  and also $(\Delta \otimes \mathrm{id})E = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1)$.

- The kernels of the canonical maps are also determined by E in a certain way.

[Remark]: In the definition, the element E is actually the smallest idempotent so that $E\Delta(a) = \Delta(a)E = \Delta(a)$ for all $a \in A$. And, there is a unique extension of Δ satisfying

$$\Delta(1) = E, \qquad \text{and similarly,} \qquad (\Delta \otimes \mathrm{id})(1) = E \otimes 1, \quad (\mathrm{id} \otimes \Delta)(1) = 1 \otimes E.$$

As a consequence of the definition, there exists a unique *antipode* S giving "generalized inverses" of the canonical maps. It is a linear map from A to $M(A)$ and it is both an anti-algebra and an anti-coalgebra map.

[Example]: Let $\mathcal{G}$ be a groupoid and let A be the algebra $K(\mathcal{G})$ of complex functions on $\mathcal{G}$ with finite support. For $f \in K(\mathcal{G})$, define:

$$\Delta(f)(p, q) := f(p, q), \text{ if } (p, q) \in \mathcal{G}^{(2)}, \qquad \Delta(f)(p, q) = 0, \text{ otherwise.}$$

Then (A, Δ) is a regular weak multiplier Hopf algebra. The idempotent E is given by the function that is 1 on $(p, q) \in \mathcal{G}^{(2)}$ and 0 on other pairs. The antipode is given by $S(f)(p) = f(p^{-1})$.
[Example]: Other non-trivial examples exist. In particular, any finite-dimensional weak Hopf algebra (in the sense of Böhm, Nill, Szlachányi) is a regular weak multiplier Hopf algebra.

Suppose (A, Δ) is a regular multiplier Hopf algebra. Then one can define the *source* and *range* algebras by

$$A_s = \big\{y \in M(A) : \Delta(y) = E(1 \otimes y)\big\}, \qquad A_r = \big\{x \in M(A) : \Delta(x) = (x \otimes 1)E\big\}.$$

From this data and using the antipode map, one can define the source and range maps.

One can also introduce the notion of "left integrals" and "right integrals", although there is no longer uniqueness result. With some extra requirements, one can consider the notion of a **algebraic quantum groupoid**, at which level a nice duality theorem holds. There is a hope that this framework would provide us a way to formulate a C^*-algebraic notion of a quantum groupoid.

[Measured quantum groupoids]: Motivated by the theory of (von Neumann algebraic) locally compact quantum groups by Kustermans–Vaes, and by modeling $L^\infty(\mathcal{G}, \mu)$, Enock, Vallin, Lesieur, ... have developed the theory of **measured quantum groupoids**, in the von Neumann algebra setting. The theory is quite technical, so we will not attempt to explain it here. However, here are a few aspects of the theory:

- A very important technical tool is the notion of the Connes–Sauvageot *relative tensor product*: One starts from a von Neumann algebra N and Hilbert spaces H and K with an antirepresentation and a representation of N, respectively, and produces a new Hilbert space $H \otimes_N K$ by factoring out the actions of N on H and K. The precise construction is rather complicated, and needs to work with a f.n.s. weight on N.

- We also need the notion of *fiber product*: Let M_1 and M_2 be von Neumann algebras on Hilbert spaces H_1 and H_2, respectively, such that M_1 is a right N-module and M_2 is a left N-module for a von Neumann algebra N. The Hilbert spaces H_1 and H_2 have natural N-module structures so that one can form the relative tensor product $H_1 \otimes_N H_2$. The fiber product of M_1 and M_2, denoted by $M_1 \,{}_N^*\, M_2$, is a certain von Neumann algebra contained in $\mathcal{B}(H_1 \otimes_N H_2)$.

- A *Hopf-von Neumann bimodule* consists of a von Neumann algebra M, a von Neumann algebra N (the *base*), with a representation and an antirepresentation $r, s : N \to M$ (the *range* map and the *source* map), and the *comultiplication* $\Delta : M \to M_s \,{}_N^*\, {}_r M$, being mapped into the fiber product. The maps satisfy:

  1. r, s, Δ are normal, injective, unital
  2. $\Delta\big(s(x)\big) = 1 \underset{N}{\otimes} s(x)$ and $\Delta\big(r(x)\big) = r(x) \underset{N}{\otimes} 1$ for all $x \in N$
  3. Δ is coassociative, in the sense that $(\Delta \,{}_N^*\, \mathrm{id}_M) \circ \Delta = (\mathrm{id}_M \,{}_N^*\, \Delta) \circ \Delta$ as maps $M \to M_s \,{}_N^*\, {}_r M_s \,{}_N^*\, {}_r M$.

- A Hopf-von Neumann bimodule (M, N, r, s, Δ) is a **measured quantum groupoid** if there exist a left invariant and a right invariant operator valued weights T and T' from M on $s(N)$ and $r(N)$ respectively, together with a certain weight ν on N, which is relatively invariant with respect to T and T'.

- A nice duality picture exists in this setting.

- Examples include locally compact groupoids, and inclusions of factors. And, under some assumptions, one can consider matched pair of groupoids.

- The concept of *pseudo-multiplicative unitary operators* can be considered, and as in the quantum group theory, they play a similar fundamental role in the theory of measured quantum groupoids.

- The relationship between the framework of weak multiplier Hopf algebras (or algebraic quantum groupoids) and the framework of measured quantum groupoid is not yet completely clarified, but it is expected that one can pass from one to the other.

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