Separability idempotents in C^*-algebras

Byung-Jay Kahng and Alfons Van Daele

Abstract. In this paper, we study the notion of a *separability idempotent* in the C^*-algebra framework. This is analogous to the notion in the purely algebraic setting, typically considered in the case of (finite-dimensional) algebras with identity, then later also considered in the multiplier algebra framework by the second-named author. The current work was motivated by the appearance of such objects in the authors' ongoing work on locally compact quantum groupoids.

1. Introduction

Consider a *groupoid* G over the set of units $G^{(0)}$, together with the maps $s\colon G \to G^{(0)}$ and $t\colon G \to G^{(0)}$ (the "source map" and the "target map"). This means that there is a set of "composable pairs" $G^{(2)} = \{(p, q) \in G \times G : s(p) = t(q)\}$, on which the product pq in G is defined. This product is assumed to be associative, in an appropriate sense. The set of units, $G^{(0)}$, may be naturally regarded as a subset of G. There exists also the inverse map $p \mapsto p^{-1}$ (so that $(p^{-1})^{-1} = p$), for which we have $s(p^{-1}) = t(p)$, $t(p^{-1}) = s(p)$, and satisfying some natural conditions. For a more detailed discussion on the definition and the basic theory of groupoids, refer to [5, 9]. The groupoid notion can be further extended to incorporate locally compact topology, which is the notion of a *locally compact groupoid*. For this, refer to [19, 20].

Suppose G is a groupoid and consider $A = K(G)$, the set of all complex-valued functions on G having finite support. For the time being, let us disregard any topology on G. Under the pointwise multiplication, A becomes a commutative algebra.

In particular, if G is a finite groupoid, the algebra A becomes unital ($1 \in A$). In that case, it is known that A can be given a structure of a *weak Hopf algebra*, together

with the map Δ from A into $A \otimes A$ (algebraic tensor product), defined by

$$(\Delta f)(p, q) := \begin{cases} f(pq), & \text{if } s(p) = t(q), \\ 0, & \text{otherwise.} \end{cases}$$

The Δ map is referred to as the "comultiplication" (or "coproduct"). A weak Hopf algebra is in general noncommutative, but includes the above example as a fundamental case. For more discussion on weak Hopf algebras, refer to [2, 3]. In a sense, a weak Hopf algebra is a finite quantum groupoid (see also [18, 23, 29]). Recently, at the purely algebraic level, this notion has been further generalized to include the case of non-finite groupoids (so the algebra A is non-unital), which is the notion of a *weak multiplier Hopf algebra* developed by one of us (Van Daele) and Wang. Refer to [26, 27].

If (A, Δ) is a weak multiplier Hopf algebra, there exists a certain canonical idempotent element E, playing the role of "$\Delta(1)$". Among the main properties of E is the fact that it is a *separability idempotent*. Namely, there exist algebras B and C such that E is an idempotent element contained in the multiplier algebra $M(B \otimes C)$, and it satisfies some number of conditions. The case that is relevant to us is when A is a *-algebra and is "regular" (see Section 4 of [27]). As a consequence, it turns out that there exist bijective anti-automorphisms $S_B: B \to C$ and $S_C: C \to B$ such that

$$E(b \otimes 1) = E\big(1 \otimes S_B(b)\big), \qquad (1 \otimes c)E = \big(S_C(c) \otimes 1\big)E,$$

for $b \in B$, $c \in C$.

In the special (commutative) case of the weak multiplier Hopf algebra $A = K(G)$ for a groupoid G, let B be the subalgebra of $M(A)$ given by the pull-back of the algebra $K(G^{(0)})$ via the source map $s: G \to G^{(0)}$, and $C \subseteq M(A)$ the pull-back of the algebra $K(G^{(0)})$ via the target map $t: G \to G^{(0)}$. Then we will have $E \in M(B \otimes C)$. The maps S_B, S_C would come from the antipode map of the weak multiplier Hopf algebra.

One of the main reasons for studying separability idempotents lies in the fact that they play an important role in the theory of weak multiplier Hopf algebras. In a sense, any separability idempotent arises from a weak multiplier Hopf algebra, and having an appropriate separability idempotent element can guide us to construct an example of a weak multiplier Hopf algebra: See Proposition 3.2 of [28]. See [24] (refer to both versions v1 and v2, as they are substantially different), where a systematic discussion is given on separability idempotents in the setting of multiplier algebras, together with some examples. Separability idempotents also arise naturally from the theory of discrete quantum groups (See Section 3 of [24]).

Suppose the groupoid G is now equipped with a compatible locally compact topology. A natural question to ask is whether we can formulate a C^*-algebraic theory similar to that of a weak multiplier Hopf algebra, together with a suitable separability idempotent element, keeping all the topological aspects.

The aim of the current paper is to define, develop, and clarify the notion of a separability idempotent in the setting of C^*-algebras. While the setting may be more general, this makes things a little restrictive as well, because we need the element to be compatible with the C^*-structures on B and C, unlike in the purely algebraic case. On the other hand, not every property from the purely algebraic case will carry over (for instance, the maps S_B and S_C will only have to be densely-defined). However, it is worth noting that we are still able to formulate below a reasonable notion.

Such separability idempotents turn out to be quite useful in developing a C^*-algebraic counterpart to the weak multiplier Hopf algebra theory, which would provide us with a (sub)class of *locally compact quantum groupoids*. This is discussed in the authors' upcoming work [10–12].

In the purely algebraic theory of separability idempotents (see [24]), the defining conditions imply the existence of two *distinguished linear functionals*, denoted φ_C and φ_B, on the algebras C and B respectively. In our formulation below (in the C^*-algebra setting), we assume instead the existence of two faithful KMS weights μ (on C) and ν (on B), from which the other properties follow, including the densely defined anti-homomorphisms. This approach is somewhat similar in philosophy to the general theory of locally compact quantum groups as given in [15, 16, 25].

We will see that the separability idempotent $E \in M(B \otimes C)$ is more or less determined by the C^*-algebra B and the weight ν, where $\otimes$ is now the C^*-algebra (spatial) tensor product. Meanwhile, the existence of E means that the pair (B, ν) cannot be arbitrarily chosen. It is an interesting question to explore the nature of the C^*-algebra B, and we will do this towards the end of the paper.

The paper is organized as follows. In Section 2, we gather some basic results concerning C^*-algebra weights. The purpose here is to set the terminologies and notations to be used in later sections. Our main definition and the properties are given in Sections 3 and 4. We introduce the notion of a separability triple and a separability idempotent in Section 3. As a consequence, we obtain the (densely-defined) anti-homomorphisms γ_B and γ_C (These are the maps corresponding to S_B and S_C in the purely algebraic case.). More results and properties are collected in Section 4, including the fact that a separability idempotent is "full".

In Section 5, some examples are considered. In addition to the natural example coming from a groupoid or a weak multiplier Hopf algebra, we included an example arising from a certain action groupoid and another example when the base C^*-algebra is the algebra of compact operators.

In Section 6, the separability idempotent notion is considered in the von Neumann algebra setting. We show that the C^*-algebra approach and the von Neumann algebra approach are equivalent, in the sense that one can start from the C^*-algebra framework, then construct a separability idempotent in the von Neumann algebra framework, then from this, one can recover the idempotent and the C^*-algebra we began with.

Finally, in Section 7, we explore the nature of the base C^*-algebra B, which would allow the existence of a separability idempotent. We claim that B has to be a postliminal C^*-algebra.

2. Preliminaries on C^*-algebra weights

In this section, we review some basic notations and results concerning the weights on C^*-algebras, which will be useful later. For standard terminologies and for a more complete treatment on C^*-algebra weights, refer to [6, 21, 22]. See also a nice survey given in [14].

Let A be a C^*-algebra. A function $\psi: A^+ \to [0, \infty]$ is called a weight on A, if $\psi(x + y) = \psi(x) + \psi(y)$ for all $x, y \in A^+$ and $\psi(\lambda x) = \lambda \psi(x)$ for all $x \in A^+$ and $\lambda \in \mathbb{R}^+$, with the convention that $0 \cdot \infty = 0$.

Given a weight ψ on A, we can consider the following subspaces:

$$\mathfrak{M}_\psi^+ := \{a \in A^+ : \psi(a) < \infty\}$$
$$\mathfrak{N}_\psi := \{a \in A : \psi(a^*a) < \infty\}$$
$$\mathfrak{M}_\psi := \mathfrak{N}_\psi^* \mathfrak{N}_\psi = \mathrm{span}\{y^*x : x, y \in \mathfrak{N}_\psi\}.$$

It is easy to see that $\mathfrak{N}_\psi$ is a left ideal in $M(A)$, the multiplier algebra of A. So $\mathfrak{M}_\psi \subseteq \mathfrak{N}_\psi$. Moreover, $\mathfrak{M}_\psi$ is a *-subalgebra of A spanned by $\mathfrak{M}_\psi^+$, which turns out to be its positive part. It is possible to naturally extend ψ on $\mathfrak{M}_\psi^+$ to a map from $\mathfrak{M}_\psi$ into $\mathbb{C}$, which we will still denote by ψ.

A weight ψ is said to be "faithful", if $\psi(a) = 0, a \in A^+$, implies $a = 0$. We say a weight ψ is densely-defined (or "semi-finite"), if $\mathfrak{N}_\psi$ is dense in A. Throughout this paper, a weight (on a C^*-algebra A) will always be assumed to be faithful, semi-finite, and also lower semi-continuous.

It is useful to consider the following sets, first considered by Combes [6]:

$$\mathcal{F}_\psi := \{\omega \in A_+^* : \omega(x) \leq \psi(x), \forall x \in A^+\},$$
$$\mathcal{G}_\psi := \{\alpha\omega : \omega \in \mathcal{F}_\psi, \text{ for } \alpha \in (0,1)\}.$$

Here A^* denotes the norm dual of A. Note that on $\mathcal{F}_\psi$, one can give a natural order inherited from A_+^*. Meanwhile, $\mathcal{G}_\psi$ is a directed subset of $\mathcal{F}_\psi$. That is, for every $\omega_1, \omega_2 \in \mathcal{G}_\psi$, there exists an element $\omega \in \mathcal{G}_\psi$ such that $\omega_1 \leq \omega, \omega_2 \leq \omega$. Because of this, $\mathcal{G}_\psi$ is often used as an index set (of a net). Since ψ is lower semi-continuous, we would have: $\psi(x) = \lim_{\omega \in \mathcal{G}_\psi} (\omega(x))$, for $x \in A^+$.

One can associate to ψ a GNS-construction $(\mathcal{H}_\psi, \pi_\psi, \Lambda_\psi)$. Here, $\mathcal{H}_\psi$ is a Hilbert space, $\Lambda_\psi \colon \mathfrak{N}_\psi \to \mathcal{H}_\psi$ is a linear map such that $\Lambda_\psi(\mathfrak{N}_\psi)$ is dense in $\mathcal{H}_\psi$, and $\langle \Lambda_\psi(a), \Lambda_\psi(b) \rangle = \psi(b^*a)$ for $a, b \in \mathfrak{N}_\psi$. Since ψ is assumed to be lower semi-continuous, we further have that Λ_ψ is closed. And, π_ψ is the GNS representation of A on $\mathcal{H}_\psi$, given by $\pi_\psi(a)\Lambda_\psi(b) = \Lambda_\psi(ab)$ for $a \in A$, $b \in \mathfrak{N}_\psi$. The GNS representation is non-degenerate, and the GNS construction is unique up to a unitary transformation.

Every $\omega \in A^*$ has a unique extension to the level of the multiplier algebra $M(A)$, which we may still denote by ω. From this fact, it follows easily that any proper weight ψ on A has a natural extension to the weight $\overline{\psi}$ on $M(A)$. For convenience, we will use the notations $\overline{\mathfrak{N}}_\psi = \mathfrak{N}_{\overline{\psi}}$ and $\overline{\mathfrak{M}}_\psi = \mathfrak{M}_{\overline{\psi}}$. Also, the GNS construction for a proper weight ψ on A has a natural extension to the GNS construction for $\overline{\psi}$ on $M(A)$, with $\mathcal{H}_{\overline{\psi}} = \mathcal{H}_\psi$.

To give somewhat of a control over the non-commutativity of A, one introduces the notion of KMS weights (see [6] and Chapter VIII of [22]). The notion as defined below (due to Kustermans [13]) is slightly different from the original one given by Combes, but equivalent.

Definition 2.1. Let ψ be a faithful, semi-finite, lower semi-continuous weight. It is called a *KMS weight*, if there exists a norm-continuous one-parameter group of automorphisms $(\sigma_t)_{t \in \mathbb{R}}$ of A such that $\psi \circ \sigma_t = \psi$ for all $t \in \mathbb{R}$, and

$$\psi(a^*a) = \psi\big(\sigma_{i/2}(a)\sigma_{i/2}(a)^*\big) \text{ for all } a \in \mathcal{D}(\sigma_{i/2}).$$

Here, $\sigma_{i/2}$ is the analytic generator of the one-parameter group (σ_t) at $z = i/2$, and $\mathcal{D}(\sigma_{i/2})$ is its domain. In general, with the domain properly defined, it is known that σ_z, $z \in \mathbb{C}$, is a closed map.

The one-parameter group (σ_t) is called the "modular automorphism group" for ψ. It is uniquely determined, as ψ is faithful. In the special case when ψ is a trace, that is, $\psi(a^*a) = \psi(aa^*)$ for $a \in \mathfrak{N}_\psi$, it is clear that ψ is KMS, with the modular automorphism group being trivial ($\sigma \equiv \mathrm{Id}$).

Basic properties of KMS weights can be found in [13, 14, 22]. In particular, there exists a (unique) anti-unitary operator J (the "modular conjugation") on $\mathcal{H}_\psi$ such that $J\Lambda_\psi(x) = \Lambda_\psi(\sigma_{i/2}(x)^*)$, $x \in \mathfrak{N}_\psi \cap \mathcal{D}(\sigma_{i/2})$. There exists also a strictly positive operator ∇ (the "modular operator") on $\mathcal{H}_\psi$ such that $\nabla^{it}\Lambda_\psi(a) = \Lambda_\psi(\sigma_t(a))$, for $a \in \mathfrak{N}_\psi$ and $t \in \mathbb{R}$. Following is a standard result, which can be easily extended to elements in the multiplier algebra:

Lemma 2.2. *Let ψ be a KMS weight on a C^*-algebra A, with GNS representation $(\mathcal{H}_\psi, \pi_\psi, \Lambda_\psi)$. Then we have:*

(1) *Let $a \in \mathcal{D}(\sigma_{i/2})$ and $x \in \mathfrak{N}_\psi$. Then*

$$xa \in \mathfrak{N}_\psi \quad \text{and} \quad \Lambda_\psi(xa) = J\pi_\psi\big(\sigma_{i/2}(a)\big)^* J\Lambda_\psi(x).$$

(2) *Let $a \in \mathcal{D}(\sigma_{-i})$ and $x \in \mathfrak{M}_\psi$. Then*

$$ax, x\sigma_{-i}(a) \in \mathfrak{M}_\psi \quad and \quad \psi(ax) = \psi\big(x\sigma_{-i}(a)\big).$$

(3) *Let $x \in \mathfrak{N}_\psi \cap \mathfrak{N}_\psi^*$ and $a \in \mathfrak{N}_\psi^* \cap \mathcal{D}(\sigma_{-i})$ be such that $\sigma_{-i}(a) \in \mathfrak{N}_\psi$. Then*

$$\psi(ax) = \psi\big(x\sigma_{-i}(a)\big).$$

It is known (see [14, 22]) that when we lift a KMS weight to the level of von Neumann algebra $\pi_\psi(A)''$ in an evident way, we obtain a normal, semi-finite, faithful ("n.s.f.") weight $\widetilde{\psi}$. In addition, the modular automorphism group $(\sigma_t^{\widetilde{\psi}})$ for the n.s.f. weight $\widetilde{\psi}$ leaves the C^*-algebra B invariant, and the restriction of $(\sigma_t^{\widetilde{\psi}})$ to B coincides with our (σ_t). The operators J, ∇ above are none other than the restrictions of the corresponding operators arising in the standard Tomita–Takesaki theory of von Neumann algebra weights.

In Tomita–Takesaki theory, a useful role is played by the Tomita *-algebra. In our case, consider the n.s.f. weight $\widetilde{\psi}$ on the von Neumann algebra $\pi_\psi(A)''$. Then the Tomita *-algebra for $\widetilde{\psi}$, denoted by $\mathcal{T}_{\widetilde{\psi}}$, is as follows:

$$\mathcal{T}_{\widetilde{\psi}} := \big\{ x \in \mathfrak{N}_{\widetilde{\psi}} \cap \mathfrak{N}_{\widetilde{\psi}}^* : x \text{ is analytic w.r.t. } \sigma^{\widetilde{\psi}}, \text{ and } \sigma_z^{\widetilde{\psi}}(x) \in \mathfrak{N}_{\widetilde{\psi}} \cap \mathfrak{N}_{\widetilde{\psi}}^*, \forall z \in \mathbb{C} \big\}.$$

It is a strongly *-dense subalgebra in $\pi_\psi(A)''$, and it satisfies many useful properties. Refer to the standard textbooks on the modular theory, for instance [21, 22]. Furthermore, it is known that $\mathcal{T}_\psi := \mathcal{T}_{\widetilde{\psi}} \cap A$ is (norm)-dense in the C^*-algebra A, so turns out to be useful as we work with the weight ψ in the C^*-algebra setting. In particular, it is easy to see that for every $z \in \mathbb{C}$, the domains $\mathcal{D}(\sigma_z)$, which contain $\mathcal{T}_\psi$, are dense in A.

3. Definition of a separability idempotent

In this section, we wish to give the proper definition of a separability idempotent element in the C^*-algebra framework. Let us begin by considering a triple (E, B, ν), where:

- B is a C^*-algebra,

- ν is a (faithful) KMS weight on B, together with its associated norm-continuous automorphism group (σ_t^ν),

- $E \in M(B \otimes C)$ is a self-adjoint idempotent element (so a projection, satisfying $E^2 = E = E^*$), for some C^*-algebra C such that there exists a *-anti-isomorphism $R: B \to C$.

Remark. In the above, we can see immediately that $C \cong B^{\mathrm{op}}$ as C^*-algebras. Even so, there is no reason to assume that C and B^{op} to be exactly same. Going forward, our triple (E, B, ν) will be understood in such a way that the C^*-algebra C and the $*$-anti-isomorphism $R\colon B \to C$ are implicitly fixed, with C possibly different from B^{op}.

The weight theory says $\mathfrak{N}_\nu$ is only a left ideal in B, but we have below a useful result involving $B_0 = \mathcal{D}(\sigma^\nu_{i/2})$, which is dense in B.

Proposition 3.1. *For any $x \in \mathfrak{N}_\nu$ and $b \in B_0$, we have: $xb \in \mathfrak{N}_\nu$. As a consequence, we have:*

$$\mathfrak{N}_\nu B_0 \subseteq \mathfrak{N}_\nu, \quad \mathfrak{M}_\nu B_0 \subseteq \mathfrak{M}_\nu, \quad B_0^* \mathfrak{M}_\nu \subseteq \mathfrak{M}_\nu.$$

Proof. See (1) of Lemma 2.2, from which it follows that

$$\nu\big((xb)^*(xb)\big) = \big\langle \Lambda_\nu(xb), \Lambda_\nu(xb) \big\rangle \leq \big\|\sigma^\nu_{i/2}(b)\big\|^2 \nu(x^*x).$$

The result $\mathfrak{N}_\nu B_0 \subseteq \mathfrak{N}_\nu$ is an immediate consequence. As for the next two results, use the fact that $\mathfrak{M}_\nu = \mathfrak{N}_\nu^* \mathfrak{N}_\nu$. $\qquad\square$

Consider the triple (E, B, ν) as above, with C and R understood. We now give the definition of the separability triple:

Definition 3.2. We will say that (E, B, ν) is a *separability triple*, if the following conditions hold:

(1) $(\nu \otimes \mathrm{id})(E) = 1$

(2) For $b \in B_0$, we have: $(\nu \otimes \mathrm{id})(E(b \otimes 1)) = R(\sigma^\nu_{i/2}(b))$.

If (E, B, ν) forms a separability triple, then we say E is a *separability idempotent*.

Remark. In Definition 3.2 above, (1) means that for any $\omega \in C_+^*$, we require $(\mathrm{id} \otimes \omega)(E) \in \overline{\mathfrak{M}}_\nu \ (\subseteq M(B))$ and that $\nu((\mathrm{id} \otimes \omega)(E)) = \omega(1)$. From this, it will follow that $(\mathrm{id} \otimes \omega)(E) \in \overline{\mathfrak{M}}_\nu$ for any $\omega \in C^*$, and that $\nu((\mathrm{id} \otimes \omega)(E)) = \omega(1)$, $\forall \omega \in C^*$. We are using here a natural extension of ν to the multiplier algebra $M(B)$, still denoted by ν. However, we will soon see that $(\mathrm{id} \otimes \omega)(E) \in B$, actually. See the corollary following Proposition 4.3.

By Proposition 3.1, we know $(\mathrm{id} \otimes \omega)(E)\, b \in \mathfrak{M}_\nu$, $\forall \omega \in C^*$ and $\forall b \in B_0$. What condition (2) of the definition is saying is that we further require $\nu((\mathrm{id} \otimes \omega)(E)\, b) = \omega((R \circ \sigma^\nu_{i/2})(b))$.

The following result provides sort of a uniqueness result for the separability idempotent E.

Proposition 3.3. *Let E be a separability idempotent. Then E is uniquely determined by the data (B, ν, R).*

Proof. From Definition 3.2 and Remark following it, we know that

$$\big(\nu(\cdot\, b) \otimes \omega\big)(E) = \omega\big((R \circ \sigma_{i/2}^{\nu})(b)\big), \quad \forall b \in B_0, \ \forall \omega \in C^*. \qquad (3.1)$$

Since B_0 is dense in B, we know $\{\nu(\cdot\, b) : b \in B_0\}$ is dense in B^*. This shows that E is completely determined by the data (B, ν, R). $\qquad \square$

As we regard a separability triple (E, B, ν) to implicitly fix the C^*-algebra C and the anti-isomorphism R (see an earlier remark), we can see from the above result that the E is more or less characterized by the pair (B, ν). However, the element $E \in M(B \otimes C)$ that is determined by (B, ν) via equation (3.1) does not necessarily have to be a projection. In other words, the existence of the separability idempotent E is really a condition on the pair (B, ν). In Section 7 below, we will discuss a little about the nature of the C^*-algebra B that could allow this to happen.

Let us now gather some basic properties of the separability triple (E, B, ν) and the separability idempotent $E \in M(B \otimes C)$. For convenience, let us write $\gamma_B(b) := (R \circ \sigma_{i/2}^{\nu})(b)$, for $b \in B_0$, which determines a map $\gamma_B \colon B_0 \to C$. Since $\sigma_{i/2}^{\nu}$ is a closed map (see comment given in Definition 2.1), we see that γ_B is also a closed map.

Proposition 3.4. *Considered as a map from B to C, the aforementioned map γ_B is a closed, densely-defined, injective map, whose range is dense in C. Moreover, for $b \in B_0$, we have:* $(\nu \otimes \mathrm{id})\big(E(b \otimes 1)\big) = \gamma_B(b)$.

Proof. Since $\mathcal{D}(\gamma_B) = B_0$ is dense in B, it is clear that γ_B is densely-defined. We have already seen that it is closed. The injectivity of γ_B is easy to see, because R is an anti-isomorphism and σ_t^{ν} is an automorphism for all t.

By the property of the Tomita algebra, we know that $\mathcal{T}_\nu \subseteq B_0$ and $\sigma_{i/2}^{\tilde{\nu}}(\mathcal{T}_{\tilde{\nu}}) = \mathcal{T}_{\tilde{\nu}}$. So we have: $\sigma_{i/2}^{\nu}(\mathcal{T}_\nu) = \sigma_{i/2}^{\nu}(\mathcal{T}_{\tilde{\nu}} \cap B) = \mathcal{T}_{\tilde{\nu}} \cap B = \mathcal{T}_\nu$, which is dense in B. In addition, since R is an anti-isomorphism between B and C, we see that $\mathrm{Ran}(\gamma_B) = R(\sigma_{i/2}^{\nu}(B_0)) \supseteq R(\mathcal{T}_\nu)$, which is dense in C.

The last result is just re-writing (2) of Definition 3.2. $\qquad \square$

For convenience, write $C_0 = \mathrm{Ran}(\gamma_B)$, which is dense in C. By the above proposition, we can consider $\gamma_B^{-1} \colon C_0 \to B$. Viewed as a map from C to B, it is clear that the map γ_B^{-1} is also a closed, densely-defined injective map, having a dense range. In fact, we will have: $\gamma_B^{-1} = \sigma_{-i/2}^{\nu} \circ R^{-1}$, and $\mathcal{D}(\gamma_B^{-1}) = \mathrm{Ran}(\gamma_B) = C_0$, while $\mathrm{Ran}(\gamma_B^{-1}) = \mathcal{D}(\gamma_B) = B_0$.

Proposition 3.5. (1) $B_0 = \mathcal{D}(\gamma_B)$ *is closed under multiplication, and γ_B is an anti-homomorphism.*

(2) $C_0 = \mathcal{D}(\gamma_B^{-1})$ *is closed under multiplication, and γ_B^{-1} is an anti-homomorphism.*

Proof. Suppose $b, b' \in B_0 = \mathcal{D}(\sigma_{i/2}^\nu)$. Being analytic elements, we have $bb' \in B_0$. Meanwhile, since $\gamma_B = R \circ \sigma_{i/2}^\nu$, with $\sigma_{i/2}^\nu$ being an automorphism and R an anti-isomorphism, we have $\gamma_B(bb') = \gamma_B(b')\gamma_B(b)$. Similarly, we have: $\gamma_B^{-1}(cc') = \gamma_B^{-1}(c')\gamma_B^{-1}(c)$ for $c, c' \in C_0$. $\qquad\qquad\square$

Here is a property that relates the C^*-algebras B and C, at the level of their dense subspaces:

Proposition 3.6. (1) *For $b \in B_0$, we have:* $E(b \otimes 1) = E(1 \otimes \gamma_B(b))$.

(2) *For $c \in C_0$, we have:* $E(1 \otimes c) = E(\gamma_B^{-1}(c) \otimes 1)$.

Proof. (1) For $b, b' \in B_0$,

$$
\begin{aligned}
\big(\nu(\cdot\, b') \otimes \mathrm{id}\big)\big(E(b \otimes 1)\big) &= (\nu \otimes \mathrm{id})\big(E(bb' \otimes 1)\big) = \gamma_B(bb') = \gamma_B(b')\gamma_B(b) \\
&= (\nu \otimes \mathrm{id})\big(E(b' \otimes 1)\big)\gamma_B(b) = (\nu \otimes \mathrm{id})\big(E(b' \otimes \gamma_B(b))\big) \\
&= (\nu \otimes \mathrm{id})\big(E(1 \otimes \gamma_B(b))(b' \otimes 1)\big) \\
&= \big(\nu(\cdot\, b') \otimes \mathrm{id}\big)\big(E(1 \otimes \gamma_B(b))\big).
\end{aligned}
$$

In the second equation, we are using (2) of Definition 3.2. The third equation holds because γ_B is an anti-homomorphism. Since ν is faithful, and since the result is true for all $b' \in B_0$, which is dense in B, we conclude that $E(b \otimes 1) = E\big(1 \otimes \gamma_B(b)\big)$.

(2) This is an immediate consequence of (1), because $C_0 = \mathrm{Ran}(\gamma_B)$ and $B_0 = \mathrm{Ran}(\gamma_B^{-1})$. $\qquad\qquad\square$

From $(\nu \otimes \mathrm{id})(E(b \otimes 1)) = R(\sigma_{i/2}^\nu(b))$, $b \in B_0$, take the adjoint. Knowing that R is a *-anti-isomorphism, we have:

$$
(\nu \otimes \mathrm{id})\big((b^* \otimes 1)E\big) = \big[R(\sigma_{i/2}^\nu(b))\big]^* = R\big([\sigma_{i/2}^\nu(b)]^*\big) = R\big(\sigma_{-i/2}^\nu(b^*)\big).
$$

This observation means that for $b \in B_0^* = \mathcal{D}(\sigma_{-i/2}^\nu)$, which is also dense in B, the expression $(\nu \otimes \mathrm{id})((b \otimes 1)E)$ is valid, and $(\nu \otimes \mathrm{id})((b \otimes 1)E) = (R \circ \sigma_{-i/2}^\nu)(b)$. Or, put another way, we have:

$$
\nu\big(b\,(\mathrm{id} \otimes \omega)(E)\big) = \omega\big((R \circ \sigma_{-i/2}^\nu)(b)\big), \quad \text{for } \omega \in C^*, b \in \mathcal{D}(\sigma_{-i/2}^\nu).
$$

So, by the same argument as in the case of the map γ_B, it is clear that the map $b \mapsto (R \circ \sigma_{-i/2}^\nu)(b)$ is closed and densely-defined on B, injective, and has a dense range in C. Also, its inverse map $c \to (\sigma_{i/2}^\nu \circ R)(c)$ is closed and densely-defined on C, injective, and has a dense range in B. So let us define the maps $\gamma_C : C \to B$ and $\gamma_C^{-1} : B \to C$ as follows:

Proposition 3.7. *Write $\gamma_C := \sigma_{i/2}^\nu \circ R^{-1}$ and $\gamma_C^{-1} := R \circ \sigma_{-i/2}^\nu$. Then:*

(1) *The map γ_C is closed and densely-defined on C, injective, and has a dense range in B.*

(2) *The map γ_C^{-1} is closed and densely-defined on B, injective, and has a dense range in C.*

(3) *Both maps γ_C and γ_C^{-1} are anti-homomorphisms: That is,*

$$\gamma_C(cc') = \gamma_C(c')\gamma_C(c), \ c, c' \in \mathcal{D}(\gamma_C),$$

and $\qquad \gamma_C^{-1}(bb') = \gamma_C^{-1}(b')\gamma_C^{-1}(b), \ b, b' \in \mathcal{D}(\gamma_C^{-1}).$

(4) *For $c \in \mathcal{D}(\gamma_C)$, we have:*

$$(1 \otimes c)E = \bigl(\gamma_C(c) \otimes 1\bigr)E.$$

(5) *For $b \in \mathcal{D}(\gamma_C^{-1})$, we have:*

$$(b \otimes 1)E = \bigl(1 \otimes \gamma_C^{-1}(b)\bigr)E.$$

Proof. (1) and (2) were already observed in the previous paragraph. Proof of (3) is done exactly as in Proposition 3.5. The anti-homomorphism properties are needed in the proof of the results (4) and (5), which is done using a similar argument as in Proposition 3.6. $\qquad \square$

Remark. The results of Proposition 3.6 and Proposition 3.7 suggest us that E behaves very much like a separability idempotent in the purely algebraic setting [24], justifying our definition given in Definition 3.2. Our maps γ_B and γ_C correspond to the maps S_B and S_C in [24]. Indeed, our Proposition 3.6 (1) and Proposition 3.7 (4) are exactly the defining axioms in the algebraic case (see Definition 1.4 of [24]), for S_B and S_C. The only difference is that in the algebraic case, the maps S_B and S_C are bijections, while our γ_B, γ_C maps are densely-defined with dense ranges.

Compared to the purely algebraic setting, we have here the $*$-structure. The next result is about the relationship between the $*$-structure and the maps γ_B, γ_C. Observe that in general, the maps γ_B, γ_C are not necessarily $*$-maps.

Proposition 3.8. (1) *For $b \in \mathcal{D}(\gamma_B)$, we have:*

$$\gamma_B(b)^* \in \mathcal{D}(\gamma_C) \quad and \quad \gamma_C\bigl(\gamma_B(b)^*\bigr)^* = b.$$

(2) *Similarly for $c \in \mathcal{D}(\gamma_C)$, we have*

$$\gamma_B\bigl(\gamma_C(c)^*\bigr)^* = c.$$

Proof. (1) If $b \in \mathcal{D}(\gamma) = \mathcal{D}(\sigma_{i/2}^{\nu})$, then we know from Proposition 3.6 that $E(b \otimes 1) = E(1 \otimes \gamma_B(b))$. Taking adjoints, we have: $(b^* \otimes 1)E = (1 \otimes \gamma_B(b)^*)E$.

Note that $b^* \in \mathcal{D}(\sigma_{-i/2}^{\nu}) = D(\gamma_C^{-1})$. Comparing with Proposition 3.7, it follows that $\gamma_B(b)^* \in \mathcal{D}(\gamma_C)$ and $\gamma_C(\gamma_B(b)^*) = b^*$. This is equivalent to saying that $\gamma_C(\gamma_B(b)^*)^* = b$.

(2) Similar argument will show that: $\gamma_B(\gamma_C(c)^*)^* = c, c \in \mathcal{D}(\gamma_C)$. $\qquad \square$

Notice the properties of the maps γ_B, γ_C obtained so far, namely, being closed and densely-defined anti-homomorphisms satisfying the result like Proposition 3.8 for the *-structure. One may observe that the behavior of the maps γ_B, γ_C resemble that of the antipode map for a locally compact quantum group. Indeed, in a certain context, it turns out that the maps γ_B, γ_C are indeed the antipode map of a locally compact quantum groupoid (A, Δ), restricted to the subalgebras B and C, which are essentially the "source algebra" and the "target algebra". And, our anti-isomorphism R would be the restriction to B of the "unitary antipode" $R_A \colon A \to A$. More systematic treatment in this direction will be given in our upcoming works [11, 12].

4. Properties of the separability idempotent

Suppose we have a separability triple (E, B, ν) in the sense of the previous section, with C and R understood. We have already observed that E is uniquely determined by the pair (B, ν). Now, using the *-anti-isomorphism $R \colon B \to C$, we can also define a faithful weight μ on C, by

$$\mu := \nu \circ R^{-1} = \nu \circ \gamma_C.$$

The second characterization is true because $\gamma_C = \sigma^\nu_{i/2} \circ R^{-1}$, while ν is σ^ν-invariant. It is not difficult to show that μ is also a KMS weight on the C^*-algebra C, together with its modular automorphism group $(\sigma^\mu_t)_{t \in \mathbb{R}}$, given by $\sigma^\mu_t := R \circ \sigma^\nu_{-t} \circ R^{-1}$. It turns out that the pair (C, μ) behaves a lot like (B, ν).

Proposition 4.1. *Let μ be as above. Then we have:*

$$(\mathrm{id} \otimes \mu)(E) = 1.$$

Proof. As in the remark following Definition 3.2, the above equation means that $(\theta \otimes \mathrm{id})(E) \in \overline{\mathfrak{M}}_\mu$ for all $\theta \in B^*$ (naturally extended to the multiplier algebra level), and that $\mu((\theta \otimes \mathrm{id})(E)) = \theta(1)$.

We can verify this for $\theta = \nu(\cdot b) \in B^*$, where $b \in \mathcal{D}(\gamma_B)$. Such functionals are dense in B^*, so that will prove the proposition. To see this, consider an arbitrary $b \in \mathcal{D}(\gamma_B)$. Then by Definition 3.2 (2), we have $(\nu \otimes \mathrm{id})(E(b \otimes 1)) = (R \circ \sigma^\nu_{i/2})(b) = \gamma_B(b)$. Apply here μ. Then we have:

$$\mu\big((\nu \otimes \mathrm{id})(E(b \otimes 1))\big) = \mu\big((R \circ \sigma^\nu_{i/2})(b)\big) = \nu(b),$$

where, we used the fact that $\mu = \nu \circ R^{-1}$ and that ν is σ^ν-invariant. Observe that this equation can be re-written as

$$\mu\big((\theta \otimes \mathrm{id})(E)\big) = \theta(1),$$

proving the claim. $\qquad\square$

Corollary. *As a consequence of the previous proposition, we have the following:*

$$(\mathrm{id} \otimes \mu)\big((1 \otimes c)E\big) = \gamma_C(c),$$

for $c \in \mathcal{D}(\gamma_C)$.

Proof. Use the fact that $(1 \otimes c)E = (\gamma_C(c) \otimes 1)E$, for $c \in D(\gamma_C)$ and that $(\mathrm{id} \otimes \mu)(E) = 1$. Note also that $\mathcal{D}(\gamma_C) = \mathcal{D}((\sigma^\nu_{i/2} \circ R^{-1})) = \mathcal{D}(\sigma^\mu_{-i/2})$. □

We also have the following alternative descriptions for the maps γ_B, γ_B^{-1}, γ_C, γ_C^{-1}, which will be useful later:

Proposition 4.2. *We have:*

$$\gamma_B = R \circ \sigma^\nu_{i/2} = \sigma^\mu_{-i/2} \circ R,$$
$$\gamma_B^{-1} = \sigma^\nu_{-i/2} \circ R^{-1} = R^{-1} \circ \sigma^\mu_{i/2},$$
$$\gamma_C = \sigma^\nu_{i/2} \circ R^{-1} = R^{-1} \circ \sigma^\mu_{-i/2},$$
$$\gamma_C^{-1} = R \circ \sigma^\nu_{-i/2} = \sigma^\mu_{i/2} \circ R.$$

Proof. The definitions involving the weight ν have been already observed in the previous section. As for the descriptions involving the weight μ, we just use the fact that $\sigma^\mu_{i/2} = R \circ \sigma^\nu_{-i/2} \circ R^{-1}$ and $\sigma^\mu_{-i/2} = R \circ \sigma^\nu_{i/2} \circ R^{-1}$. □

In the below is the result that is usually assumed as part of the definition of a separability idempotent in the purely algebraic setting:

Proposition 4.3. *For all $b \in B$ and all $c \in C$, we have:*

$$E(1 \otimes c) \in B \otimes C, \quad (1 \otimes c)E \in B \otimes C, \quad (b \otimes 1)E \in B \otimes C, \quad E(b \otimes 1) \in B \otimes C.$$

Here, $\otimes$ is the (spatial) C^*-tensor product.

Proof. Since $E \in M(B \otimes C)$, we know that $E(b \otimes c) \in B \otimes C$, for any $b \in B$ and any $c \in C$. If, in particular, $b_0 \in \mathcal{D}(\gamma_B)$, we have:

$$E\big(1 \otimes \gamma_B(b_0)c\big) = E\big(1 \otimes \gamma_B(b_0)\big)(1 \otimes c) = E(b_0 \otimes 1)(1 \otimes c) = E(b_0 \otimes c) \in B \otimes C.$$

But the elements $\gamma_B(b_0)c$, for $b_0 \in \mathcal{D}(\gamma_B)$, $c \in C$, are dense in C, because $\mathrm{Ran}(\gamma_B)$ is dense in C and $C^2 = C$. It follows that $E(1 \otimes c) \in B \otimes C$ for all $c \in C$. Similar argument holds for each of the other three cases. □

Corollary. *As a consequence of the proposition, we have:*

$$(\mathrm{id} \otimes \omega)(E) \in B, \ \forall \omega \in C^*, \quad and \quad (\theta \otimes \mathrm{id})(E) \in C, \ \forall \theta \in B^*.$$

Proof. Let $b \in \mathcal{D}(\gamma_B) \subseteq B$. Then, we know from the above proposition that $E(b \otimes 1) \in B \otimes C$. So

$$(\nu(\cdot b) \otimes \mathrm{id})(E) = (\nu \otimes \mathrm{id})(E(b \otimes 1)) \in C.$$

Since the $\nu(\cdot b)$ form a dense subspace of B^*, this means that $(\theta \otimes \mathrm{id})(E) \in C$, for all $\theta \in B^*$. Similarly, we have $(\mathrm{id} \otimes \omega)(E) \in B$, for all $\omega \in C^*$. $\qquad\square$

We already knew that $(\mathrm{id} \otimes \omega)(E) \in M(B)$, but we now see that it is actually contained in B. Moreover, we see below that such elements form a dense subspace in B. Similar for C.

Proposition 4.4. *The separability idempotent E is "full", in the sense that:*

$$\{(\theta \otimes \mathrm{id})(E(b \otimes 1)) : b \in B, \theta \in B^*\} \text{ is dense in } C,$$
$$\{(\theta \otimes \mathrm{id})((b \otimes 1)E) : b \in B, \theta \in B^*\} \text{ is dense in } C,$$
$$\{(\mathrm{id} \otimes \omega)((1 \otimes c)E) : c \in C, \omega \in C^*\} \text{ is dense in } B,$$
$$\{(\mathrm{id} \otimes \omega)(E(1 \otimes c)) : c \in C, \omega \in C^*\} \text{ is dense in } B.$$

Proof. By the corollary following Proposition 4.3, we are assured that containment statement for each set is true. So we just need to show that they are dense subsets.

Consider arbitrary $b_1, b_2 \in \mathcal{D}(\gamma_B)$, which is dense in B. We know from Proposition 3.5 that $\mathcal{D}(\gamma_B)$ is closed under multiplication, so $b_1 b_2 \in \mathcal{D}(\gamma_B)$. For $\theta = \nu(\cdot b_2) \in B^*$, we have:

$$(\theta \otimes \mathrm{id})(E(b_1 \otimes 1)) = (\nu \otimes \mathrm{id})(E(b_1 b_2 \otimes 1)) = \gamma_B(b_1 b_2).$$

This shows that

$$\{(\theta \otimes \mathrm{id})(E(b \otimes 1)) : b \in B, \theta \in B^*\}$$

contains $\{\gamma_B(b_1 b_2) : b_1, b_2 \in \mathcal{D}(\gamma_B)\}$, which is dense in C because γ_B was shown to have a dense range in C. This proves the first statement. Proofs for the other statements are very much similar, knowing that the maps $\gamma_B, \gamma_B^{-1}, \gamma_C, \gamma_C^{-1}$ all have dense ranges. $\qquad\square$

Here is a result that is related to the fact that E is full. While it is possible to give the proof using the above proposition, we instead chose to give a direct proof, which seems simpler.

Proposition 4.5. (1) *If $(1 \otimes c)E = 0$, $c \in C$, then necessarily $c = 0$.*

(2) *If $E(1 \otimes c) = 0$, $c \in C$, then necessarily $c = 0$.*

(3) *If $E(b \otimes 1) = 0$, $b \in B$, then necessarily $b = 0$.*

(4) *If $(b \otimes 1)E = 0$, $b \in B$, then necessarily $b = 0$.*

Proof. (1) Let $c \in C$ be such that $(1 \otimes c)E = 0$. While, for any $b \in \mathcal{D}(\gamma_B)$, we know $\gamma_B(b) = (\nu \otimes \mathrm{id})\big(E(b \otimes 1)\big)$. So we have:

$$c\gamma_B(b) = c(\nu \otimes \mathrm{id})\big(E(b \otimes 1)\big) = (\nu \otimes \mathrm{id})\big((1 \otimes c)E(b \otimes 1)\big) = 0.$$

Since $\mathrm{Ran}(\gamma_B)$ is dense in C, this means that $c = 0$.

The results (2), (3), (4) can be proved similarly. $\qquad\square$

Remark. The "fullness" of $E \in M(B \otimes C)$, as given in Proposition 4.4, means that the left leg of E is B and the right leg of E is C. In the purely algebraic setting, the fullness of E was part of the definition of E being a separability idempotent [24]. There, from the defining axioms, one obtains the existence of certain "distinguished linear functionals", φ_B and φ_C. In our setting, however, we begin first with the weights μ and ν, then obtain the results on E, including its fullness, the maps γ_B, γ_C, and the like.

In the next proposition, we see that the analytic generators σ_{-i}^{μ} and σ_{-i}^{ν} can be characterized in terms of the maps γ_B and γ_C.

Proposition 4.6. *We have:*

(1) $\sigma_{-i}^{\mu}(c) = (\gamma_B \circ \gamma_C)(c)$, *for* $c \in \mathcal{D}(\gamma_B \circ \gamma_C)$.

(2) $\sigma_{-i}^{\nu}(b) = (\gamma_B^{-1} \circ \gamma_C^{-1})(b)$, *for* $b \in \mathcal{D}(\gamma_B^{-1} \circ \gamma_C^{-1})$.

Proof. (1) Use $\gamma_B = \sigma_{-i/2}^{\mu} \circ R$ and $\gamma_C = R^{-1} \circ \sigma_{-i/2}^{\mu}$, observed in Proposition 4.2.

(2) Use $\gamma_B^{-1} = \sigma_{-i/2}^{\nu} \circ R^{-1}$ and $\gamma_C^{-1} = R \circ \sigma_{-i/2}^{\nu}$, again from Proposition 4.2. $\quad\square$

Remark. The significance of Proposition 4.6 is that the maps $\gamma_B \circ \gamma_C$ and $\gamma_B^{-1} \circ \gamma_C^{-1}$ provide "modular automorphisms" for the weights μ and ν, respectively. To be more precise, recall Lemma 2.2. Our result says that for $c \in \mathcal{D}(\gamma_B \circ \gamma_C) = \mathcal{D}(\sigma_{-i}^{\mu})$ and $x \in \mathfrak{M}_{\mu}$, we have: $cx, x\sigma_{-i}^{\mu}(c) \in \mathfrak{M}_{\mu}$, and $\mu(cx) = \mu\big(x\sigma_{-i}^{\mu}(c)\big) = \mu\big(x(\gamma_B \circ \gamma_C)(c)\big)$. Similar also for the weight ν.

We will conclude this section by showing a few different characterizations of the idempotent E.

Proposition 4.7. *For any* $t \in \mathbb{R}$, *we have:* $(\sigma_t^{\nu} \otimes \sigma_{-t}^{\mu})(E) = E$.

Proof. Suppose $b \in \mathcal{D}(\sigma_{i/2}^{\nu}) = \mathcal{D}(\gamma_B)$ be arbitrary, and compute:

$$
\begin{aligned}
&(\nu \otimes \mathrm{id})\big((\sigma_t^{\nu} \otimes \sigma_{-t}^{\mu})(E)(b \otimes 1)\big) \\
&\quad = (\nu \otimes \mathrm{id})\big((\sigma_t^{\nu} \otimes \sigma_{-t}^{\mu})[E(\sigma_{-t}^{\nu}(b) \otimes 1)]\big) \\
&\quad = (\nu \otimes \mathrm{id})\big((\mathrm{id} \otimes \sigma_{-t}^{\mu})[E(\sigma_{-t}^{\nu}(b) \otimes 1)]\big) = \sigma_{-t}^{\mu}\big((\nu \otimes \mathrm{id})[E(\sigma_{-t}^{\nu}(b) \otimes 1)]\big) \\
&\quad = \sigma_{-t}^{\mu}\big(\gamma_B(\sigma_{-t}^{\nu}(b))\big) = \sigma_{-t}^{\mu}\big((R \circ \sigma_{i/2}^{\nu} \circ \sigma_{-t}^{\nu})(b)\big) = \sigma_{-t}^{\mu}\big((R \circ \sigma_{-t}^{\nu} \circ \sigma_{i/2}^{\nu})(b)\big) \\
&\quad = (R \circ \sigma_t^{\nu} \circ R^{-1})\big((R \circ \sigma_{-t}^{\nu} \circ \sigma_{i/2}^{\nu})(b)\big) = (R \circ \sigma_{i/2}^{\nu})(b) = \gamma_B(b) \\
&\quad = (\nu \otimes \mathrm{id})\big(E(b \otimes 1)\big).
\end{aligned}
$$

In the first equality, we are using the fact that σ_t^ν is an automorphism. In the second equality, we used $\nu \circ \sigma_t^\nu = \nu$.

This is true for any $b \in \mathcal{D}(\sigma_{i/2}^\nu)$. By the uniqueness property observed in Proposition 3.3, we conclude that $(\sigma_t^\nu \otimes \sigma_{-t}^\mu)(E) = E$, for any $t \in \mathbb{R}$. $\qquad\square$

Write ς to denote the flip map on $M(B \otimes C)$. So we will have $\varsigma E \in M(C \otimes B)$. In the below, we wish to show that $(\gamma_C \otimes \gamma_B)(\varsigma E) = E$. However, as of now, we do not know if the expression $(\gamma_C \otimes \gamma_B)(\varsigma E)$ even makes sense as a bounded element. To make sense of all this, and anticipating other future applications, we prove first the following lemma:

Lemma 4.8. *Suppose $b \in \mathcal{T}_\nu$ and $c \in \mathcal{T}_\mu$. We have:*

$$(\gamma_C \otimes \gamma_B)\big((\gamma_C^{-1}(b) \otimes \gamma_B^{-1}(c))(\varsigma E)\big) = E(b \otimes c),$$
$$(\gamma_C \otimes \gamma_B)\big((\varsigma E)(\gamma_C^{-1}(b) \otimes \gamma_B^{-1}(c))\big) = (b \otimes c)E,$$
$$(\gamma_C \otimes \gamma_B)\big((1 \otimes \gamma_B^{-1}(c))(\varsigma E)(\gamma_C^{-1}(b) \otimes 1)\big) = (b \otimes 1)E(1 \otimes c),$$
$$(\gamma_C \otimes \gamma_B)\big((\gamma_C^{-1}(b) \otimes 1)(\varsigma E)(1 \otimes \gamma_B^{-1}(c))\big) = (1 \otimes c)E(b \otimes 1).$$

Proof. To verify the last equation, note that $(1 \otimes c)E(b \otimes 1) \in \mathfrak{M}_{\nu \otimes \mu}$. Applying $\nu \otimes \mu$, we have:

$$
\begin{aligned}
(\nu \otimes \mu)\big((1 \otimes c)E(b \otimes 1)\big) &= (\nu \otimes \mu)\big((1 \otimes c)EE(b \otimes 1)\big) \\
&= (\nu \otimes \mu)\big([\gamma_C(c) \otimes 1]E[1 \otimes \gamma_B(b)]\big) \\
&= (\mu \otimes \nu)\big([1 \otimes \gamma_C(c)](\varsigma E)[\gamma_B(b) \otimes 1]\big) \\
&= (\mu \otimes \nu)\big([\sigma_i^\mu(\gamma_B(b)) \otimes 1](\varsigma E)[1 \otimes \sigma_{-i}^\nu(\gamma_C(c))]\big).
\end{aligned}
$$

By Proposition 4.6, we know that $\sigma_i^\mu(\gamma_B(b)) = (\gamma_C^{-1} \circ \gamma_B^{-1})(\gamma_B(b)) = \gamma_C^{-1}(b)$, and $\sigma_{-i}^\nu(\gamma_C(c)) = (\gamma_B^{-1} \circ \gamma_C^{-1})(\gamma_C(c)) = \gamma_B^{-1}(c)$. So we have:

$$(\nu \otimes \mu)\big((1 \otimes c)E(b \otimes 1)\big) = (\mu \otimes \nu)\big([\gamma_C^{-1}(b) \otimes 1](\varsigma E)[1 \otimes \gamma_B^{-1}(c)]\big).$$

Since $\mu = \nu \circ \gamma_C$ and $\nu = \mu \circ \gamma_B$, we thus have:

$$(\nu \otimes \mu)\big((1 \otimes c)E(b \otimes 1)\big) = (\nu \circ \gamma_C \otimes \mu \circ \gamma_B)\big([\gamma_C^{-1}(b) \otimes 1](\varsigma E)[1 \otimes \gamma_B^{-1}(c)]\big).$$

This result is true for any $b \in \mathcal{T}_\nu$, $c \in \mathcal{T}_\mu$, which are dense in B and C, respectively. Moreover, we know that the weights ν, μ are faithful. It follows that

$$(1 \otimes c)E(b \otimes 1) = (\gamma_C \otimes \gamma_B)\big([\gamma_C^{-1}(b) \otimes 1](\varsigma E)[1 \otimes \gamma_B^{-1}(c)]\big),$$

proving the claim. Other cases can be proved similarly. $\qquad\square$

By Lemma 4.8, we can now prove the following result:

Proposition 4.9. *We have:* $(\gamma_C \otimes \gamma_B)(\varsigma E) = E$ *and* $(\gamma_B \otimes \gamma_C)(E) = \varsigma E.$

Proof. As written, we do not know whether $(\gamma_C \otimes \gamma_B)(\varsigma E)$ is bounded. However, for $b \in \mathcal{T}_\nu$ (dense in B) and $c \in \mathcal{T}_\mu$ (dense in C), we know from Lemma 4.8 that it can be made sense as follows:

$$\big[(\gamma_C \otimes \gamma_B)(\varsigma E)\big](b \otimes c) := (\gamma_C \otimes \gamma_B)\big((\gamma_C^{-1}(b) \otimes \gamma_B^{-1}(c))(\varsigma E)\big) = E(b \otimes c),$$

$$(b \otimes c)\big[(\gamma_C \otimes \gamma_B)(\varsigma E)\big] := (\gamma_C \otimes \gamma_B)\big((\varsigma E)(\gamma_C^{-1}(b) \otimes \gamma_B^{-1}(c))\big) = (b \otimes c)E.$$

This means that $(\gamma_C \otimes \gamma_B)(\varsigma E)$ coincides with $E \in M(B \otimes C)$ as a left and right multiplier map, on a dense subset of $B \otimes C$. Since E is bounded, this implies that $(\gamma_C \otimes \gamma_B)(\varsigma E)$ can be canonically extended to a left and right multiplier map on all of $B \otimes C$, which would mean that $(\gamma_C \otimes \gamma_B)(\varsigma E) \in M(B \otimes C)$, and that $(\gamma_C \otimes \gamma_B)(\varsigma E) = E$. By taking the flip map, we also have: $(\gamma_B \otimes \gamma_C)(E) = \varsigma E.$ $\square$

Corollary. *We have:* $(R^{-1} \otimes R)(\varsigma E) = E$ *and* $(R \otimes R^{-1})(E) = \varsigma E.$

Proof. From the proposition, we know: $\varsigma E = (\gamma_B \otimes \gamma_C)(E) \in M(C \otimes B)$. Apply $R^{-1} \otimes R$ to both sides. By Proposition 4.2, we know $\gamma_B = R \circ \sigma_{i/2}^\nu$ and $\gamma_C = R^{-1} \circ \sigma_{-i/2}^\mu$. Thus we have:

$$(R^{-1} \otimes R)(\varsigma E) = (R^{-1} \otimes R)(\gamma_B \otimes \gamma_C)(E) = \big(\sigma_{i/2}^\nu \otimes \sigma_{-i/2}^\mu\big)(E) = E.$$

For the last equality, we used the result of Proposition 4.7.

By taking the flip map, we also have: $(R \otimes R^{-1})(E) = \varsigma E.$ $\square$

5. Examples and special cases

5.1. Groupoids. As indicated in the Introduction, the theory of weak multiplier Hopf algebras is motivated by attempts to generalize the notion of a groupoid. Our theory is essentially a C^*-algebraic counterpart to the weak multiplier Hopf algebra theory. While it is true that this is not fully general due to the fact that the existence condition for the separability idempotent element is rather strong to be compatible with some topological aspects, it remains the case that a certain subclass of groupoids provide us with typical examples. See below.

Example 5A. Let G be a discrete groupoid (equipped with the discrete topology). Set the notations for the unit space $G^{(0)}$, the source map $s_G: G \to G^{(0)}$, and the target map $t_G: G \to G^{(0)}$, as in the Introduction. Consider the commutative C^*-algebras $A = C_0(G)$ and $M(A) = C_b(G)$. Let $s, t: C_0(G^{(0)}) \to M(A)$ be the pull-back maps corresponding to s_G and t_G, respectively. That is, for $f \in C_0(G^{(0)})$, we have $s(f) \in M(A) = C_b(G)$ such that $s(f)(p) = f\big(s_G(p)\big)$, and similarly for the map t. Let B and C be the images under the maps s and t, which are C^*-subalgebras of $M(A)$. We have $M(B)$ and $M(C)$ contained in $M(A)$ as subalgebras.

Let $E \in M(A \otimes A) = C_b(G \times G)$ be such that

$$E(p, q) := \begin{cases} 1, & \text{if } s_G(p) = t_G(q), \\ 0, & \text{otherwise.} \end{cases}$$

Since $E(p, q) = E\big(s_G(p), t_G(q)\big)$, we observe easily that $E \in M(B \otimes C)$.

Consider next the counting measure on $G^{(0)}$, which naturally determines a weight on $C_0(G^{(0)})$. Using the pull-back maps, we can define weights ν on B and μ on C. They are faithful, tracial weights (so KMS).

To describe the R map, consider $s(f) = f \circ s_G \in B$, where $f \in C_0(G^{(0)})$. Then we have $R(f \circ s_G) \in C$, given by $R(f \circ s_G)(p) = (f \circ t_G)(p^{-1})$. It is a well-defined map, because whenever $s_G(p) = s_G(q) \in G^{(0)}$, we have $t_G(p^{-1}) = t_G(q^{-1})$. It is clear that it is an (anti)-isomorphism.

It is not difficult to see that (E, B, ν) forms a separability triple in the sense of Definition 3.2, and therefore E is a separability idempotent.

Example 5B. Consider again a discrete groupoid G. Denote by $K(G)$ the space of all complex functions on G having finite support. In particular, for $p \in G$, define $\lambda_p \in K(G)$ by

$$\lambda_p(x) := \delta_{p,x} = \begin{cases} 1, & \text{if } x = p, \\ 0, & \text{otherwise.} \end{cases}$$

Note that any function $f \in K(G)$ can be expressed in the form $f = \sum_{p \in G} f(p)\lambda_p$. We can give $K(G)$ a *-algebra structure by letting $\lambda_p \lambda_q = \lambda_{pq}$, valid only when pq is defined and 0 otherwise; while at the same time letting $\lambda_p^* = \lambda_{p^{-1}}$. This is none other than the convolution *-algebra structure.

Similar to the group case, one can consider the (left) regular representation of $K(G)$ on $l^2(G)$, given by the counting measure on G. The regular representation extends $K(G)$ to a C^*-algebra $A = C_r^*(G)$. Unless $G^{(0)}$ is a finite set, the algebra A is in general non-unital. However, at the level of the multiplier algebra, the unit element is $1 = \sum \lambda_e$, where the sum is taken over all $e \in G^{(0)}$. We will skip the details (left Haar system, and the like) and refer to the standard textbooks ([19, 20]), as the issue at hand is more about the separability idempotents than about groupoid algebras. More discussion on the (quantum) groupoid aspects will be given in the authors' upcoming papers [11, 12].

Note that if $e \in G^{(0)}$, we have $s_G(e) = t_G(e) = e$. Because of this, the base algebras B and C, which are generated by $\{\lambda_{s_G(p)} : p \in G\}$ and $\{\lambda_{t_G(p)} : p \in G\}$ respectively, will turn out to be isomorphic to $C_0(G^{(0)})$. The counting measure on $G^{(0)}$ would provide the weights ν and μ. Being faithful and tracial, they are KMS weights. Let $R: B \to C$ be given by $\lambda_{s_G(p)} \mapsto \lambda_{t_G(p^{-1})}$. It is a well-defined map, which is clearly an (anti)-isomorphism.

Now consider $E := \sum \lambda_e \otimes \lambda_e$, where the sum is taken over all $e \in G^{(0)}$. It is not difficult to show that it becomes a separability idempotent.

Remark. (1) Since G is discrete, the topology does not play much significant role here. At the *-algebra level, the two examples above can be described in the framework of weak multiplier Hopf algebras [26, 27]. For a general weak multiplier Hopf algebra, the discussion about E being a separability idempotent in terms of its "distinguished linear functionals" (μ and ν in our case) can be found in Section 4 of [24].

(2) There also exists the notion of a *measured quantum groupoid* by Lesieur and Enock, in the von Neumann algebra framework [8, 17]. It generalizes the notion of a groupoid, so in some special cases (for instance, when the base algebra is finite-dimensional), they provide a separability idempotent. But we will postpone the discussion on all these cases related to "quantum groupoids" to our future papers [11, 12].

5.2. An example coming from an action groupoid.

Consider the following specific example: The group $(\mathbb{Z}, +)$ acts on $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\}$ by translation and leaving ∞ fixed, and let $\overline{\mathbb{Z}} \times \mathbb{Z}$ be the corresponding action groupoid (or, transformation group groupoid). It can be considered as a locally compact groupoid, together with the discrete topology on $\mathbb{Z}$, and $\overline{\mathbb{Z}}$ being the compactification to ∞. Then consider $G := (\overline{\mathbb{Z}} \times \mathbb{Z})|_{\overline{\mathbb{N}}}$, the restriction of the action groupoid to $\overline{\mathbb{N}} \subset \overline{\mathbb{Z}}$.

G is also a groupoid (the "Cuntz groupoid", see [4, 19, 20]), where $G^{(0)} = \overline{\mathbb{N}}$ and equipped with the target map $t: (m, p) \mapsto m$ and the source map $s: (m, p) \mapsto m + p$. We have: $(m, p) \cdot (n, q) = \delta_{m+p,n}(m, p + q)$. It is a locally compact groupoid, whose topology is inherited from that of the action groupoid $\overline{\mathbb{Z}} \times \mathbb{Z}$. The (left) Haar system is given by the discrete measure on each t-fiber.

We can certainly consider, as in Example 5B above, a natural separability idempotent $E_0 := \sum_{m \in \overline{\mathbb{N}}} \lambda_{m,0} \otimes \lambda_{m,0}$, which is characterized by $B_0 \cong C_0(G^{(0)})$ and ν_0, the counting measure on B_0, namely, $\nu_0(\lambda_{m,0}) = 1$, $\forall m \in \overline{\mathbb{N}}$. However, for this next example, we instead wish to consider all of $\mathbb{C}G$ as the base algebra for our (soon to be constructed) separability idempotent.

Consider $B := C^*(G) = \mathbb{C}G$. As in Example 5B, it can be realized as a completion of the convolution algebra $C_c(G)$ given by the convolution product $\lambda_{m,p}\lambda_{n,q} = \delta_{m+p,n}\lambda_{m,p+q}$ and the involution $\lambda_{m,p}^* = \lambda_{m+p,-p}$. As before, we are regarding the generators $\lambda_{m,p}$ as functions on G, given by $\lambda_{m,p}(n, q) = \delta_{m,n}\delta_{p,q}$. It is known that B is isomorphic to the C^*-algebra generated by the unilateral shift operator on $l^2(\mathbb{N})$, but we do not need this fact for our purposes (see [20]).

Fix a positive constant $\hbar > 0$. Define:

$$E := \sum_{m,p \in \mathbb{N}} e^{-(p+m)\hbar/2}\big(1 - e^{-\hbar}\big)\lambda_{m,p-m} \otimes \lambda_{m,p-m}. \tag{5.1}$$

Actually, the sum is taken over all $(m, p - m) \in G$, so $t(m, p - m) = m \in \overline{\mathbb{N}}$ and $s(m, p - m) = p \in \overline{\mathbb{N}}$, but with the convention that $e^{-\infty} = 0$. In what follows,

we will show that there exists a suitable weight $\nu_\hbar$ on B so that $(E, B, \nu_\hbar)$ forms a separability triple.

Proposition 5.1. *E is self-adjoint and idempotent.*

Proof. (1) By definition, $\lambda^*_{m,p-m} = \lambda_{m+(p-m),-(p-m)} = \lambda_{p,m-p}$. So we have

$$E^* = \sum_{m,p \in \mathbb{N}} e^{-(p+m)\hbar/2}\left(1 - e^{-\hbar}\right)\lambda_{p,m-p} \otimes \lambda_{p,m-p} = E.$$

(2) Since $\lambda_{m,p-m}\lambda_{n,q-n} = \delta_{m+(p-m),n}\lambda_{m,(p-m)+(q-n)} = \delta_{p,n}\lambda_{m,q-m}$, we have:

$$
\begin{aligned}
E^2 &= \sum_{m,p,n,q \in \mathbb{N}} e^{-(p+m)\hbar/2}e^{-(q+n)\hbar/2}\left(1 - e^{-\hbar}\right)^2 \lambda_{m,p-m}\lambda_{n,q-n} \otimes \lambda_{m,p-m}\lambda_{n,q-n} \\
&= \sum_{m,p,q \in \mathbb{N}} e^{-p\hbar}e^{-(q+m)\hbar/2}\left(1 - e^{-\hbar}\right)^2 \lambda_{m,q-m} \otimes \lambda_{m,q-m} \\
&= \sum_{m,q \in \mathbb{N}} e^{-(q+m)\hbar/2}\left(1 - e^{-\hbar}\right)\lambda_{m,q-m} \otimes \lambda_{m,q-m} = E.
\end{aligned}
$$

The second equality is using the fact that $n = p$, and the third equality is because $\sum_{p \in \mathbb{N}} e^{-p\hbar}(1 - e^{-\hbar}) = 1$. $\qquad\square$

Let $C = C^*(G)$, which will be considered as an opposite C^*-algebra of B, via the map $R: B \to C$, defined on the generators by $R(\lambda_{n,q}) = \lambda_{n+q,-q}$. Coming from the involution, it is clear that R is indeed a $*$-anti-isomorphism.

Next, define the weight $\nu_\hbar$ on B, whose values on the generators are

$$
\begin{aligned}
\nu_\hbar(\lambda_{m,0}) &= e^{m\hbar}(1 - e^{-\hbar})^{-1}, \quad \text{for } m = 0, 1, 2, \dots; \\
\nu_\hbar(\lambda_{m,p}) &= 0, \text{ when } p \neq 0.
\end{aligned}
\tag{5.2}
$$

So for $f \in C_c(G)$, we have: $\nu_\hbar(f) = \sum_{m=0}^\infty e^{m\hbar}(1 - e^{-\hbar})^{-1}f(m,0)$. With the values known for the generators, it is clear that $\nu_\hbar$ is a proper weight on B. It is in fact a KMS weight, together with the one-parameter group $(\sigma_t^\hbar)_{t \in \mathbb{R}}$, defined by $\sigma_t^\hbar(\lambda_{n,q}) := e^{-itq\hbar}\lambda_{n,q}$. See proposition below.

Proposition 5.2. *Let E be as in Equation (5.1), and consider the weight $\nu_\hbar$ on B as in Equation (5.2). Then:*

(1) $\nu_\hbar$, *together with the one-parameter group $(\sigma_t^\hbar)_{t \in \mathbb{R}}$, is a KMS weight on B.*

(2) *For each $\lambda_{n,q}$, we have: $E(\lambda_{n,q} \otimes 1) = E(1 \otimes e^{q\hbar/2}\lambda_{n+q,-q})$.*

(3) $(\nu_\hbar \otimes \mathrm{id})(E) = 1$.

(4) $(\nu_\hbar \otimes \mathrm{id})(E(\lambda_{n,q} \otimes 1)) = R(\sigma_{i/2}^\hbar(\lambda_{n,q}))$.

Proof. (1) Let $\nu_\hbar(f^*f) = 0$, where $f = \sum_{m,p} f(m, p)\lambda_{m,p}$. Since $\lambda_{n,q}^*\lambda_{m,p} = \lambda_{n+q,-q}\lambda_{m,p} = \delta_{n,m}\lambda_{n+q,p-q}$, we have:

$$f^*f = \sum_{m,p,n,q} \overline{f(n,q)}f(m,p)\lambda_{n,q}^*\lambda_{m,p} = \sum_{m,p,q} \overline{f(m,q)}f(m,p)\lambda_{m+q,p-q}.$$

Note also that $\nu_\hbar(\lambda_{m+q,p-q}) = 0$ unless $p = q$. It follows that

$$\nu_\hbar(f^*f) = \sum_{m,p} \overline{f(m,p)}f(m,p)\nu_\hbar(\lambda_{m+p,0}).$$

But we always have $\nu_\hbar(\lambda_{m+p,0}) > 0$, while $\overline{f(m,p)}f(m,p) \geq 0$. This means that $\nu_\hbar(f^*f) = 0$ only if $f(m,p) \equiv 0$, showing that $\nu_\hbar$ is faithful.

Meanwhile, with the $(\sigma_t^\hbar)$ given above, we have:

$$\nu_\hbar\big(\sigma_t^\hbar(\lambda_{n,q})\big) = \nu_\hbar\big(e^{-itq\hbar}\lambda_{n,q}\big) = \begin{cases} 0, & \text{if } q \neq 0, \\ \nu_\hbar(\lambda_{n,0}), & \text{if } q = 0. \end{cases}$$

So for any $t \in \mathbb{R}$, we see that $\nu_\hbar\big(\sigma_t^\hbar(\lambda_{n,q})\big) = \nu_\hbar(\lambda_{n,q})$, $\forall n, q$, showing the invariance of $\nu_\hbar$ under the $(\sigma_t^\hbar)$. In addition,

$$\begin{aligned} \nu_\hbar\big(\sigma_{i/2}^\hbar(\lambda_{n,q})\sigma_{i/2}^\hbar(\lambda_{n,q})^*\big) &= \nu_\hbar\big((e^{q\hbar/2}\lambda_{n,q})(e^{q\hbar/2}\lambda_{n+q,-q})\big) = e^{q\hbar}\nu_\hbar(\lambda_{n,0}) \\ &= e^{(q+n)\hbar}(1-e^{-\hbar})^{-1}, \end{aligned}$$

which coincides with $\nu_\hbar(\lambda_{n,q}^*\lambda_{n,q}) = \nu_\hbar(\lambda_{n+q,-q}\lambda_{n,q}) = \nu_\hbar(\lambda_{n+q,0})$. Therefore, by Definition 2.1, we see that $\nu_\hbar$ is indeed a KMS weight.

(2) Let E be as in equation (5.1). Then:

$$\begin{aligned} E(\lambda_{n,q} \otimes 1) &= \sum_{m,p} e^{-(p+m)\hbar/2}\big(1-e^{-\hbar}\big)\lambda_{m,p-m}\lambda_{n,q} \otimes \lambda_{m,p-m} \\ &= \sum_m e^{-(n+m)\hbar/2}\big(1-e^{-\hbar}\big)\lambda_{m,n-m+q} \otimes \lambda_{m,n-m} \end{aligned}$$

while

$$\begin{aligned} E(1 \otimes e^{q\hbar/2}\lambda_{n+q,-q}) &= \sum_{m,p} e^{-(p+m)\hbar/2}\big(1-e^{-\hbar}\big)e^{q\hbar/2}\lambda_{m,p-m} \otimes \lambda_{m,p-m}\lambda_{n+q,-q} \\ &= \sum_m e^{-(n+q+m)\hbar/2}\big(1-e^{-\hbar}\big)e^{q\hbar/2}\lambda_{m,n+q-m} \otimes \lambda_{m,n-m}. \end{aligned}$$

Comparing, we see that $E(\lambda_{n,q} \otimes 1) = E(1 \otimes e^{q\hbar/2}\lambda_{n+q,-q})$.

(3) With the definition of $\nu_\hbar$ given in Equation (5.2), we have:

$$
\begin{aligned}
&(\nu_\hbar \otimes \mathrm{id})\big[E(\lambda_{n,q} \otimes 1)\big] \\
&= (\nu_\hbar \otimes \mathrm{id})\Big[\sum_m e^{-(n+m)\hbar/2}\big(1 - e^{-\hbar}\big)\lambda_{m,n-m+q} \otimes \lambda_{m,n-m}\Big] \\
&= \sum_m e^{-(n+m)\hbar/2}\big(1 - e^{-\hbar}\big)e^{m\hbar}\big(1 - e^{-\hbar}\big)^{-1}\delta_{n-m+q,0}\lambda_{m,n-m} = e^{q\hbar/2}\lambda_{n+q,-q}.
\end{aligned}
$$

$$(5.3)$$

By (2), this can be realized as $(\nu_\hbar \otimes \mathrm{id})[E(1 \otimes e^{q\hbar/2}\lambda_{n+q,-q})] = e^{q\hbar/2}\lambda_{n+q,-q}$, which is equivalent to $(\nu_\hbar \otimes \mathrm{id})[E(1 \otimes \lambda_{m,p})] = \lambda_{m,p}$, true for any generator $\lambda_{m,p}$. This shows that $(\nu_\hbar \otimes \mathrm{id})(E) = 1$.

(4) By definition, we know $\sigma^{\hbar}_{i/2}(\lambda_{n,q}) = e^{q\hbar/2}\lambda_{n,q}$. So $R(\sigma^{\hbar}_{i/2}(\lambda_{n,q})) = e^{q\hbar/2}\lambda_{n+q,-q}$. Then the result obtained in equation (5.3) is none other than saying

$$
(\nu_\hbar \otimes \mathrm{id})\big(E(\lambda_{n,q} \otimes 1)\big) = e^{q\hbar/2}\lambda_{n+q,-q} = R\big(\sigma^{\hbar}_{i/2}(\lambda_{n,q})\big). \qquad \square
$$

By Definition 3.2, we conclude from Proposition 5.2 that $(E, B, \nu_\hbar)$ forms a separability triple, and therefore, E is a separability idempotent. We may have an occasion in the future to explore this example further.

5.3. A case when $B = \mathcal{B}_0(\mathcal{H})$. We wish to study an example that generalizes the example given in Section 4 of [24]. Consider a Hilbert space $\mathcal{H}$, possibly infinite-dimensional. Fix an orthonormal basis $(\xi_j)_{j \in J}$ for $\mathcal{H}$, where J is an index set. For any $\mathbf{v} \in \mathcal{H}$, we may write $\mathbf{v} = \sum_{j \in J} v_j \xi_j$, where $v_j = \langle \mathbf{v}, \xi_j \rangle$. Consider also $\mathrm{Tr}(\cdot)$, the canonical trace on $\mathcal{B}(\mathcal{H})$.

Define $E_0 := \sum_{i,j \in J} e_{ij} \otimes e_{ij}$, where the $e_{ij} \in \mathcal{B}(\mathcal{H})$ are the matrix units, given by

$$
e_{ij}(\mathbf{v}) := \langle \mathbf{v}, \xi_j \rangle \xi_i = v_j \xi_i, \quad \text{for } \mathbf{v} \in \mathcal{H}.
$$

Lemma 5.3. *With the notations above, we have the following results:*

(1) *For $a \in \mathcal{B}(\mathcal{H})$ and $\mathbf{v} \in \mathcal{H}$, we have, for $i \in J$, $(a\mathbf{v})_i = \sum_{j \in J} a_{ij} v_j$, where $a_{ij} = \langle a\xi_j, \xi_i \rangle$.*

(2) *For $a \in \mathcal{B}(\mathcal{H})$, we can consider $a^T \in \mathcal{B}(\mathcal{H})$, the "transpose" of a, defined by $a^T_{ij} = a_{ji} = \langle a\xi_i, \xi_j \rangle$, for $i, j \in J$. We will have:*

$$
\mathrm{Tr}(a) = \sum_{j \in J} \langle a\xi_j, \xi_j \rangle = \sum_{j \in J} a_{jj} = \mathrm{Tr}(a^T).
$$

(3) *For the (unbounded) operator E_0 defined above, we have:*

$$
E_0(\mathbf{v} \otimes \mathbf{w}) = \sum_{i,j \in J} v_j w_j \xi_i \otimes \xi_i, \quad \text{for } \mathbf{v}, \mathbf{w} \in \mathcal{H}.
$$

Proof. Straightforward: Basic linear algebra. $\qquad \square$

Proposition 5.4. *Let $a \in \mathcal{B}(\mathcal{H})$. Then we have:*

$$E_0(a \otimes 1) = E_0(1 \otimes a^T).$$

Proof. For $\mathbf{v}, \mathbf{w} \in \mathcal{H}$, by using Lemma 5.3, we have:

$$
\begin{aligned}
E_0(a \otimes 1)(\mathbf{v} \otimes \mathbf{w}) &= E_0(a\mathbf{v} \otimes \mathbf{w}) \\
&= \sum_{i,j \in J} (a\mathbf{v})_j w_j \xi_i \otimes \xi_i = \sum_{i,j,k \in J} (a_{jk} v_k) w_j \xi_i \otimes \xi_i \\
&= \sum_{i,j,k \in J} v_k (a_{kj}^T w_j) \xi_i \otimes \xi_i = \sum_{i,k \in J} v_k (a^T \mathbf{w})_k \xi_i \otimes \xi_i \\
&= E_0(\mathbf{v} \otimes a^T \mathbf{w}).
\end{aligned}
$$

This is true for all $\mathbf{v}, \mathbf{w} \in \mathcal{H}$. So $E_0(a \otimes 1) = E_0(1 \otimes a^T)$. $\qquad\square$

Consider $r \in \mathcal{B}(\mathcal{H})$, which is invertible and $\mathrm{Tr}(r^*r) = 1$. So r is a compact operator of Hilbert–Schmidt type. This means that r^{-1} is unbounded in general, unless $\mathcal{H}$ is finite-dimensional. Next, define a new operator E, by

$$E := (r \otimes 1) E_0(r^* \otimes 1). \tag{5.4}$$

To learn more about the property of E, we will first consider a lemma.

Lemma 5.5. *Let $e_{ij} \in \mathcal{B}(\mathcal{H})$ be as above. Then we have:*

(1) $e_{ij} e_{kl} = \delta_{jk} e_{il}$.

(2) $e_{ij}^* = e_{ji}$.

(3) *For $a \in \mathcal{B}(\mathcal{H})$, we have:* $\sum_{j \in J} e_{ij} a e_{jl} = \mathrm{Tr}(a) e_{il}$.

Proof. (1) Let $\mathbf{v} \in \mathcal{H}$ be arbitrary. We have:

$$e_{ij} e_{kl}(\mathbf{v}) = e_{ij}\big(e_{kl}(\mathbf{v})\big) = e_{ij}(v_l \xi_k) = v_l \delta_{jk} \xi_i = \delta_{jk} e_{il}(\mathbf{v}).$$

(2) Let $\mathbf{v}, \mathbf{w} \in \mathcal{H}$ be arbitrary. We have:

$$\big\langle e_{ij}(\mathbf{v}), \mathbf{w} \big\rangle = \langle v_j \xi_i, \mathbf{w} \rangle = v_j w_i = \langle \mathbf{v}, w_i \xi_j \rangle = \big\langle \mathbf{v}, e_{ji}(\mathbf{w}) \big\rangle.$$

(3) Let $\mathbf{v} \in \mathcal{H}$ be arbitrary. We have:

$$
\begin{aligned}
\sum_{j \in J} e_{ij} a e_{jl}(\mathbf{v}) &= \sum_{j \in J} e_{ij} a(v_l \xi_j) = \sum_{j \in J} v_l e_{ij}\left(\sum_{k \in J} a_{kj} \xi_k\right) \\
&= \sum_{j,k \in J} v_l a_{kj} e_{ij}(\xi_k) = \sum_{j,k \in J} v_l a_{kj} \delta_{jk} \xi_i \\
&= \sum_{j \in J} a_{jj} v_l \xi_i = \mathrm{Tr}(a) e_{il}(\mathbf{v}). \qquad\qquad\square
\end{aligned}
$$

By using the result of Lemma 5.5, we obtain the following proposition:

Proposition 5.6. *Let E be defined as in (5.4) above. Then $E \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, and we have: $E^* = E = E^2$.*

Proof. By Lemma 5.5 (2), we observe that

$$E_0^* = \Big(\sum_{i,j \in J} e_{ij} \otimes e_{ij} \Big)^* = \sum_{i,j \in J} e_{ji} \otimes e_{ji} = E_0.$$

It follows that $E^* = ((r \otimes 1)E_0(r^* \otimes 1))^* = E$.

By Lemma 5.5 (1) and (3), we see that

$$
\begin{aligned}
E_0(r^*r \otimes 1)E_0 &= \sum_{i,j,k,l \in J} (e_{ij} \otimes e_{ij})(r^*r \otimes 1)(e_{kl} \otimes e_{kl}) \\
&= \sum_{i,j,k,l \in J} e_{ij} r^* r e_{kl} \otimes \delta_{jk} e_{il} = \sum_{i,j,l \in J} e_{ij} r^* r e_{jl} \otimes e_{il} \\
&= \sum_{i,l \in J} \mathrm{Tr}(r^*r) e_{il} \otimes e_{il} = \mathrm{Tr}(r^*r) E_0 = E_0.
\end{aligned}
$$

In the last line, we used the assumption that $\mathrm{Tr}(r^*r) = 1$. It follows that

$$E^2 = (r \otimes 1)E_0(r^*r \otimes 1)E_0(r^* \otimes 1) = (r \otimes 1)E_0(r^* \otimes 1) = E.$$

We see that E is a self-adjoint idempotent, so a bounded projection in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.
$\square$

Next, define $B := \mathcal{B}_0(\mathcal{H})$, the C^*-algebra of compact operators on $\mathcal{H}$. Let us consider a suitable weight on B:

Proposition 5.7. *Consider the map $\nu := \mathrm{Tr}(r^{-1} \cdot r^{*-1}) = \mathrm{Tr}(p \cdot)$, where*

$$p = r^{*-1} r^{-1} = (rr^*)^{-1}.$$

It is a KMS weight on the C^*-algebra B. The associated norm-continuous automorphism group, the "modular automorphism group" for ν, is $(\sigma_t^\nu)_{t \in \mathbb{R}}$, defined by $\sigma_t^\nu(b) = p^{it} b p^{-it}$.

Proof. From the properties of $\mathrm{Tr}(\cdot)$, which is itself a proper weight, it is easy to see that ν is a lower semi-continuous weight and is densely-defined. It is also faithful, because if $\nu(b^*b) = \mathrm{Tr}(r^{-1}b^*br^{*-1}) = 0$, it necessarily means $br^{*-1} = 0$, so $b = 0$.

With the definition of the (σ_t^ν) given above and for $b \in \mathfrak{M}_\nu$, observe that

$$\nu\big(\sigma_t^\nu(b)\big) = \mathrm{Tr}(pp^{it}bp^{-it}) = \mathrm{Tr}(p^{-it}pp^{it}b) = \mathrm{Tr}(pb) = \nu(b), \quad t \in \mathbb{R}.$$

This shows the invariance of ν under the (σ_t^ν). While, for $b \in \mathcal{D}(\sigma_{i/2}^\nu)$, we have:

$$\nu\big(\sigma_{i/2}^\nu(b)\sigma_{i/2}^\nu(b)^*\big) = \mathrm{Tr}\big(p(p^{-\frac{1}{2}}bp^{\frac{1}{2}})(p^{-\frac{1}{2}}bp^{\frac{1}{2}})^*\big) = \mathrm{Tr}(p^{\frac{1}{2}}bpb^*p^{-\frac{1}{2}})$$
$$= \mathrm{Tr}(pb^*p^{-\frac{1}{2}}p^{\frac{1}{2}}b) = \mathrm{Tr}(pb^*b) = \nu(b^*b).$$

By the conditions given in Definition 2.1, we see that ν is a KMS weight. $\qquad\square$

Let $C = \mathcal{B}_0(\mathcal{H})$. Note that $M(B) = M(C) = \mathcal{B}(\mathcal{H})$. Meanwhile, by polar decomposition, write $r^* = u|r^*|$, where $|r^*| = (rr^*)^{\frac{1}{2}} = p^{-\frac{1}{2}}$. Now let $R: B \to C$ be defined by $R(b) := (ubu^*)^T$. It is a *-anti-isomorphism, with $R^{-1}(c) = u^*c^Tu$. It will soon be made clear why we choose R in this way.

With the definition given in Equation (5.4), we showed in Proposition 5.6 that E is self-adjoint and idempotent such that $E \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) = M(B \otimes C)$. In the next proposition, we see that (E, B, ν) indeed forms a separability triple.

Proposition 5.8. *Consider the triple (E, B, ν) as given above. Then*

(1) $(\nu \otimes \mathrm{id})(E) = 1_{\mathcal{B}(\mathcal{H})}$, *where $1_{\mathcal{B}(\mathcal{H})}$ is the identity in $M(B) = \mathcal{B}(\mathcal{H})$.*

(2) *For $b \in \mathcal{D}(\sigma_{i/2}^\nu)$, we have: $(\nu \otimes \mathrm{id})(E(b \otimes 1)) = R(\sigma_{i/2}^\nu(b))$.*

By Definition 3.2, we see that (E, B, ν) forms a separability triple, and therefore, E is a separability idempotent.

Proof. (1) To be rigorous, as noted in the Remark following Definition 3.2, we should understand (1) as saying that $(\mathrm{id} \otimes \omega)(E) \in \mathfrak{M}_\nu$ for $\omega \in C^*$ and that $\nu((\mathrm{id} \otimes \omega)(E)) = \omega(1)$. As $C = \mathcal{B}_0(\mathcal{H})$, any $\omega \in C^*$ may be expressed in the form $\omega = \mathrm{Tr}(\cdot\, a)$, where a is a trace-class operator. Then, since $(e_{ij}a)_{mn} = \delta_{m,i}a_{jn}$, we have:

$$(\mathrm{id} \otimes \omega)(E_0) = (\mathrm{id} \otimes \mathrm{Tr})\big(E_0(1 \otimes a)\big) = \sum_{i,j,m \in J} e_{ij}(\delta_{m,i}a_{jm}) = \sum_{i,j \in J} e_{ij}a_{ji} = a^T.$$
$$(5.5)$$

With $p = r^{*-1}r^{-1}$, it is obvious from Equation (5.5) that

$$\mathrm{Tr}(a^T) = \mathrm{Tr}(r^*pra^T) = \mathrm{Tr}(pra^Tr^*) = \mathrm{Tr}\big(pr(\mathrm{id} \otimes \omega)(E_0)r^*\big).$$

Since we can write $r(\mathrm{id} \otimes \omega)(E_0)r^* = (\mathrm{id} \otimes \omega)\big((r \otimes 1)E_0(r^* \otimes 1)\big) = (\mathrm{id} \otimes \omega)(E)$, and since $\nu = \mathrm{Tr}(p \,\cdot\,)$, we can now see that $(\mathrm{id} \otimes \omega)(E) \in \mathfrak{M}_\nu$. We also have:

$$\nu\big((\mathrm{id} \otimes \omega)(E)\big) = \mathrm{Tr}\big(pr(\mathrm{id} \otimes \omega)(E_0)r^*\big) = \mathrm{Tr}(a^T) = \mathrm{Tr}(a) = \omega(1),$$

proving the claim.

(2) Recall the polar decomposition $r^* = u|r^*| = up^{-\frac{1}{2}}$. For $b \in B$, note that $r^*br^{*-1} = u|r^*|b|r^*|^{-1}u^* = up^{-\frac{1}{2}}bp^{\frac{1}{2}}u^*$. We see that r^*br^{*-1} is bounded if

and only if $p^{-\frac{1}{2}}bp^{\frac{1}{2}} = \sigma^\nu_{i/2}(b)$ is bounded. In particular, if $b \in \mathcal{D}(\sigma^\nu_{i/2})$, we know that r^*br^{*-1} is bounded.

Consider an arbitrary trace-class operator a, and consider $\omega = \mathrm{Tr}(\,\cdot\,a)$. For the time being, write $s = (r^*br^{*-1})^T$, which is also bounded. As the algebra of trace-class operators forms an ideal in $\mathcal{B}(\mathcal{H})$, it follows that sa is also trace-class. Then as in Equation (5.5), we will have:

$$(\mathrm{id} \otimes \omega)\big(E_0(1 \otimes s)\big) = (\mathrm{id} \otimes \mathrm{Tr})\big(E_0(1 \otimes sa)\big) = (sa)^T.$$

By Proposition 5.4, this can be re-written as: $(\mathrm{id} \otimes \omega)(E_0(s^T \otimes 1)) = (sa)^T$. With $p = r^{*-1}r^{-1}$ and $s = (r^*br^{*-1})^T$, it follows that

$$\begin{aligned}
\mathrm{Tr}((sa)^T) &= \mathrm{Tr}(r^* p r (sa)^T) = \mathrm{Tr}(p r (sa)^T r^*) \\
&= \mathrm{Tr}\big(p r (\mathrm{id} \otimes \omega)(E_0(s^T \otimes 1)) r^*\big) \\
&= \mathrm{Tr}\big(p r (\mathrm{id} \otimes \omega)(E_0(r^*br^{*-1} \otimes 1)) r^*\big) \\
&= \mathrm{Tr}\big(p (\mathrm{id} \otimes \omega)((r \otimes 1) E_0(r^*b \otimes 1))\big) \\
&= \mathrm{Tr}\big(p (\mathrm{id} \otimes \omega)(E(b \otimes 1))\big).
\end{aligned}$$

Therefore, we can conclude that $(\mathrm{id} \otimes \omega)\big(E(b \otimes 1)\big) \in \mathfrak{M}_\nu$, and that

$$\nu\big((\mathrm{id} \otimes \omega)(E(b \otimes 1))\big) = \mathrm{Tr}((sa)^T) = \mathrm{Tr}(sa) = \omega(s) = \omega\big((r^*br^{*-1})^T\big).$$

This is true for any $\omega \in C^*$. In other words, we have:

$$(\nu \otimes \mathrm{id})\big(E(b \otimes 1)\big) = (r^*br^{*-1})^T = \big(u\sigma^\nu_{i/2}(b)u^*\big)^T = R\big(\sigma^\nu_{i/2}(b)\big). \qquad \square$$

Since $E \in M(B \otimes C)$ is a separability triple, all the results of Section 3 and Section 4 will hold. Here we gather some of them, which may be useful in the future.

Consider $\mu := \nu \circ R^{-1}$. By general theory (see Section 4), we know that it is a KMS weight on $C = \mathcal{B}_0(\mathcal{H})$, such that $(\mathrm{id} \otimes \mu)(E) = 1_{\mathcal{B}(\mathcal{H})}$. The following proposition gathers its properties.

Proposition 5.9. *With the definition above, we have:* $\mu = \mathrm{Tr}(q \,\cdot\,)$, *where* $q = ((r^*r)^{-1})^T$. *The modular automorphism group for μ is* $(\sigma^\mu_t)_{t \in \mathbb{R}}$, *given by*

$$\sigma^\mu_t(c) = (R \circ \sigma^\nu_{-t} \circ R^{-1})(c) = q^{it} c q^{-it}, \quad c \in C.$$

Proof. (1) If $c \in \mathfrak{M}_\mu$, we will have:

$$\mu(c) = \nu\big(R^{-1}(c)\big) = \nu(u^* c^T u) = \mathrm{Tr}(p u^* c^T u).$$

From the polar decomposition $r^* = u|r^*| = up^{-\frac{1}{2}}$, we have: $r = p^{-\frac{1}{2}}u^*$, and $r^{-1} = up^{\frac{1}{2}}$. So we can write:

$$\mu(c) = \mathrm{Tr}(p^{\frac{1}{2}} p p^{-\frac{1}{2}} u^* c^T u) = \mathrm{Tr}(u p^{\frac{1}{2}} p p^{-\frac{1}{2}} u^* c^T) = \mathrm{Tr}(r^{-1} p r c^T).$$

Since $p = r^{*-1}r^{-1}$, we thus have:

$$\mu(c) = \mathrm{Tr}(r^{-1}r^{*-1}c^T) = \mathrm{Tr}\big((r^{-1}r^{*-1}c^T)^T\big) = \mathrm{Tr}\big(c(r^{-1}r^{*-1})^T\big) = \mathrm{Tr}(qc),$$

where $q = (r^{-1}r^{*-1})^T = \big((r^*r)^{-1}\big)^T$.

(2) We can check the result directly, by using the characterization given in (1) and by imitating the proof of Proposition 5.7. Or, we can compute the following:

$$\sigma_t^\mu(c) = (R \circ \sigma_{-t}^\nu \circ R^{-1})(c) = (up^{-it}u^* c^T up^{it}u^*)^T = (up^{it}u^*)^T c(up^{-it}u^*)^T.$$

But,

$$up^{it}u^* = (upu^*)^{it} = (up^{\frac{1}{2}}p^{\frac{1}{2}}u^*)^{it} = (r^{-1}r^{*-1})^{it} = (q^T)^{it}.$$

It follows that $(up^{it}u^*)^T = q^{it}$. Also, $(up^{-it}u^*)^T = q^{-it}$. In this way, we can see that $\sigma_t^\mu(c) = q^{it}cq^{-it}$. $\qquad\square$

The next proposition describes the (densely-defined) anti-homomorphism maps γ_B, γ_B^{-1}, γ_C, γ_C^{-1} that we saw in Sections 3 and 4.

Proposition 5.10. (1) *For $b \in \mathcal{D}(\sigma_{i/2}^\nu)$, write $\gamma_B(b) := (R \circ \sigma_{i/2}^\nu)(b) = (r^*br^{*-1})^T$.*
We have:

$$E(b \otimes 1) = E\big(1 \otimes \gamma_B(b)\big) = E\big(1 \otimes (r^*br^{*-1})^T\big).$$

(2) *For $c \in \mathrm{Ran}(\gamma_B) = \mathcal{D}(\sigma_{i/2}^\mu)$, write $\gamma_B^{-1}(c) = (R^{-1} \circ \sigma_{i/2}^\mu)(c) = r^{*-1}c^Tr^*$.*
We have:

$$E(1 \otimes c) = E\big(\gamma_B^{-1}(c) \otimes 1\big) = E\big(r^{*-1}c^Tr^* \otimes 1\big).$$

(3) *For $c \in \mathcal{D}(\sigma_{-i/2}^\mu)$, write $\gamma_C(c) := (R^{-1} \circ \sigma_{-i/2}^\mu)(c) = (\sigma_{i/2}^\nu \circ R^{-1})(c) = rc^Tr^{-1}$.*
We have:

$$(1 \otimes c)E = \big(\gamma_C(c) \otimes 1\big)E = \big(rc^Tr^{-1} \otimes 1\big)E.$$

(4) *For $b \in \mathrm{Ran}(\gamma_C) = \mathcal{D}(\sigma_{-i/2}^\nu)$, write $\gamma_C^{-1}(b) = (R \circ \sigma_{-i/2}^\nu)(b) = (r^{-1}br)^T$.*
We have:

$$(b \otimes 1)E = \big(1 \otimes \gamma_C^{-1}(b)\big)E = \big(1 \otimes (r^{-1}br)^T\big)E.$$

(5) *The maps γ_B, γ_B^{-1}, γ_C, γ_C^{-1} are anti-homomorphisms such that*

$$\gamma_C\big(\gamma_B(b)^*\big)^* = b, \quad \text{for } b \in \mathcal{D}(\sigma_{i/2}^\nu);$$

and
$$\gamma_B\big(\gamma_C(c)^*\big)^* = c, \quad \text{for } c \in \mathcal{D}(\sigma_{-i/2}^\mu).$$

Proof. See Propositions 3.5, 3.6, 3.7, 3.8, 4.2. $\qquad\square$

6. Separability idempotents in von Neumann algebras

6.1. Separability idempotent in the von Neumann algebra setting. It is possible to consider the notion of a separability idempotent in the setting of von Neumann algebras. We will do this here. Later in the section, we will also explore how the C^*-algebra approach and the von Neumann algebra approach are related.

We begin by considering a triple $(E, N, \widetilde{\nu})$, where

- N is a von Neumann algebra;
- $\widetilde{\nu}$ is an n.s.f. weight on N, together with its associated modular automorphism group $(\sigma_t^{\widetilde{\nu}})$;
- E is a self-adjoint idempotent (so a projection) element contained in the von Neumann algebra tensor product $N \otimes L$, where L is a von Neumann algebra such that there exists a *-anti-isomorphism $R\colon N \to L$.

Consider the triple $(E, N, \widetilde{\nu})$ as above. As in Section 3, we consider L and $R\colon N \to L$ to be implicitly understood. We now give the definition of the separability triple:

Definition 6.1. We will say that $(E, N, \widetilde{\nu})$ is a *separability triple*, if the following conditions hold:

(1) $(\widetilde{\nu} \otimes \mathrm{id})(E) = 1$.

(2) For $b \in \mathcal{D}(\sigma_{i/2}^{\widetilde{\nu}})$, we have: $(\widetilde{\nu} \otimes \mathrm{id})(E(b \otimes 1)) = R(\sigma_{i/2}^{\widetilde{\nu}}(b))$.

If $(E, N, \widetilde{\nu})$ forms a separability triple, then we say E is a *separability idempotent*.

Let $(E, N, \widetilde{\nu})$ be a separability triple. Then, using the *-anti-isomorphism $R\colon N \to L$, we can define the n.s.f. weight $\widetilde{\mu}$ on L, given by $\widetilde{\mu} := \widetilde{\nu} \circ R^{-1}$. The arguments given in Sections 3 and 4 will all go through, including the existence of the densely-defined anti-homomorphisms γ_N, γ_L and γ_N^{-1}, γ_L^{-1}:

Proposition 6.2. *We have the maps*

$$\gamma_N = R \circ \sigma_{i/2}^{\widetilde{\nu}} = \sigma_{-i/2}^{\widetilde{\mu}} \circ R \quad and \quad \gamma_L = \sigma_{i/2}^{\widetilde{\nu}} \circ R^{-1} = R^{-1} \circ \sigma_{-i/2}^{\widetilde{\mu}}.$$

They are closed and densely-defined, injective anti-homomorphisms, having dense ranges. We can also consider γ_N^{-1} and γ_L^{-1}, having similar properties. In addition, we have:

(1) *For $b \in \mathcal{D}(\sigma_{i/2}^{\widetilde{\nu}})$, we have:* $E(b \otimes 1) = E(1 \otimes \gamma_N(b))$.

(2) *For $c \in \mathcal{D}(\sigma_{i/2}^{\widetilde{\mu}})$, we have:* $E(1 \otimes c) = E(\gamma_N^{-1}(c) \otimes 1)$.

(3) *For $c \in \mathcal{D}(\sigma_{-i/2}^{\widetilde{\mu}})$, we have:* $(1 \otimes c)E = (\gamma_L(c) \otimes 1)E$.

(4) *For $b \in \mathcal{D}(\sigma_{-i/2}^{\widetilde{\nu}})$, we have:* $(b \otimes 1)E = (1 \otimes \gamma_L^{-1}(b))E$.

We can also show that E is full. The proof is done in essentially the same way as in Proposition 4.4. The only difference is that while the sets were norm dense in the C^*-algebra setting, now the sets are strongly*-dense in the current setting:

Proposition 6.3. *The separability idempotent* $E \in N \otimes L$ *is "full", in the sense that:*

$$\{(\theta \otimes \mathrm{id})(E(b \otimes 1)) : b \in N, \theta \in N_*\} \text{ is dense in } L,$$
$$\{(\theta \otimes \mathrm{id})((b \otimes 1)E) : b \in N, \theta \in N_*\} \text{ is dense in } L,$$
$$\{(\mathrm{id} \otimes \omega)((1 \otimes c)E) : c \in L, \omega \in L_*\} \text{ is dense in } N,$$
$$\{(\mathrm{id} \otimes \omega)(E(1 \otimes c)) : c \in L, \omega \in L_*\} \text{ is dense in } N.$$

Other results in Sections 3 and 4 also have corresponding results, including the following:

Proposition 6.4. *Let the notation be as above, and as before, denote by ς the flip map on $N \otimes L$. Then:*

(1) $(\sigma_t^{\tilde{\nu}} \otimes \sigma_{-t}^{\tilde{\mu}})(E) = E$, *for all* $t \in \mathbb{R}$.

(2) $(\gamma_L \otimes \gamma_N)(\varsigma E) = E$ *and* $(\gamma_N \otimes \gamma_L)(E) = \varsigma E$.

(3) $(R^{-1} \otimes R)(\varsigma E) = E$ *and* $(R \otimes R^{-1})(E) = \varsigma E$.

Proof. See Propositions 4.7 and 4.9. $\qquad\square$

6.2. Comparison with the C^*-algebraic notion. The results above seem to indicate that the C^*-algebra approach and the von Neumann algebra approach to the separability idempotent are equivalent. To prove this, let us first begin with the C^*-algebra framework.

Suppose (E, B, ν) is a separability triple in the C^*-algebra setting, with the C^*-algebra C and the *-ani-isomorphism $R \colon B \to C$ understood, and $\mu = \nu \circ R^{-1}$. Let π_ν and π_μ be the GNS representations of B and C, on $\mathcal{H}_\nu$ and $\mathcal{H}_\mu$, respectively. For convenience, we may regard $B = \pi_\nu(B) \subseteq \mathcal{B}(\mathcal{H}_\nu)$, and also $C = \pi_\mu(C) \subseteq \mathcal{B}(\mathcal{H}_\mu)$. Then we have $E = (\pi_\nu \otimes \pi_\mu)(E) \subseteq \mathcal{B}(\mathcal{H}_\nu \otimes \mathcal{H}_\mu)$.

Consider $N := \pi_\nu(B)''$. Denote by $\tilde{\nu}$ the weight on N lifted from ν. Since ν is a KMS weight, we know that $\tilde{\nu}$ should be an n.s.f. weight on N. Similarly, define $L := \pi_\mu(C)''$. We can perform the lifting of μ and obtain $\tilde{\mu}$, an n.s.f. weight on L. We expect that our separability idempotent will be described at the level of these von Neumann algebras. First, let us show that the *-anti-isomorphism $R \colon B \to C$ extends to $\tilde{R} \colon N \to L$, also a *-anti-isomorphism.

Lemma 6.5. *We have:*

$$C = \overline{\{(\nu \otimes \mathrm{id})((b_1^* \otimes 1)E(b_2 \otimes 1)) : b_1, b_2 \in \mathfrak{N}_\nu\}}^{\|\ \|},$$
$$B = \overline{\{(\mathrm{id} \otimes \mu)((1 \otimes c_1^*)E(1 \otimes c_2)) : c_1, c_2 \in \mathfrak{N}_\mu\}}^{\|\ \|}.$$

Proof. First statement is a consequence of E being "full" (Proposition 4.4), together with the fact that $\mathfrak{N}_\nu$ is dense in B. Note also that the functionals of the form $\nu(b_1^* \cdot b_2)$, $b_1, b_2 \in \mathfrak{N}_\nu$, are dense in B^*. Second statement is also a consequence of E being full. □

Lemma 6.6. (1) $b \in \mathfrak{N}_\nu$ *if and only if* $R(b)^* \in \mathfrak{N}_\mu$.

(2) $c \in \mathfrak{N}_\mu$ *if and only if* $R^{-1}(c)^* \in \mathfrak{N}_\nu$.

Proof. Use $\nu = \mu \circ R$. We have: $\nu(b^*b) = \mu(R(b^*b)) = \mu(R(b)R(b)^*)$, because R is a *-anti-isomorphism. So $b \in \mathfrak{N}_\nu$ if and only if $R(b)^* \in \mathfrak{N}_\mu$. Similar proof for the other statement. □

Proposition 6.7. *The (C^*-algebraic) *-anti-isomorphism $R \colon B \to C$ extends to $\widetilde{R} \colon N \to L$, which is also a (von Neumann algebraic) *-anti-isomorphism.*

Proof. From Lemma 6.5, we know the elements $(\mathrm{id} \otimes \mu)((1 \otimes c_1^*)E(1 \otimes c_2))$, $c_1, c_2 \in \mathfrak{N}_\mu$, generate B. Under R, since $\mu = \nu \circ R^{-1}$, we have:

$$
\begin{aligned}
R\big((\mathrm{id} \otimes \mu)((1 \otimes c_1^*)E(1 \otimes c_2))\big) \\
&= (\mathrm{id} \otimes \nu)\big((R \otimes R^{-1})((1 \otimes c_1^*)E(1 \otimes c_2))\big) \\
&= (\mathrm{id} \otimes \nu)\big((1 \otimes R^{-1}(c_2))[(R \otimes R^{-1})(E)](1 \otimes R^{-1}(c_1)^*)\big) \\
&= (\nu \otimes \mathrm{id})\big(R^{-1}(c_2) \otimes 1)E(R^{-1}(c_1)^* \otimes 1)\big),
\end{aligned}
$$

using the fact that $(R \otimes R^{-1})(E) = \varsigma E$ (Corollary of Proposition 4.9). Similarly, we also have:

$$
R^{-1} \colon (\nu \otimes \mathrm{id})\big((b_1^* \otimes 1)E(b_2 \otimes 1)\big) \mapsto (\mathrm{id} \otimes \mu)\big((1 \otimes R(b_2))E(1 \otimes R(b_1)^*)\big).
$$

Recall that

$$
N = \pi_\nu(B)'' = \big\{(\mathrm{id} \otimes \mu)((1 \otimes c_1^*)E(1 \otimes c_2)) : c_1, c_2 \in \mathfrak{N}_\mu\big\}''
$$

and $\qquad L = \pi_\mu(C)'' = \big\{(\nu \otimes \mathrm{id})((b_1^* \otimes 1)E(b_2 \otimes 1)) : b_1, b_2 \in \mathfrak{N}_\nu\big\}''.$

In view of Lemma 6.6, looking at the generators of N and L, it is now clear that the R map extends to an injective map $\widetilde{R}$ from N onto L. The preservation of the *-structure and the anti-multiplicativity of $\widetilde{R}$ are all straightforward because of the corresponding properties of R. □

Remark. For convenience, we may still write R instead of $\widetilde{R}$, with its inverse $R^{-1} \colon L \to N$ also a *-anti-isomorphism. It is evident that we will have $\widetilde{\mu} = \widetilde{\nu} \circ R^{-1}$.

Finally, observe that $E \in M(B \otimes C) \subseteq N \otimes L$. Since B, C are dense in N, L, and since $\widetilde{\nu}$, $\widetilde{\mu}$ naturally extend ν, μ, it follows that $(E, N, \widetilde{\nu})$ forms a separability triple in the sense of Definition 6.1.

Let us next consider the problem of going back. So we begin with a separability triple $(E, N, \widetilde{\nu})$ in the von Neumann algebra setting, again with the von Neumann algebra L and the anti-isomorphism $R: N \to L$ understood. Also recall that $\widetilde{\mu} = \widetilde{\nu} \circ R^{-1}$.

Let $\mathcal{T}_{\widetilde{\mu}}$ be the Tomita *-algebra, which is dense in $\mathfrak{N}_{\widetilde{\mu}}$. Let $c_1, c_2 \in \mathcal{T}_{\widetilde{\mu}}$. Observe:

$$(\mathrm{id} \otimes \widetilde{\mu})\big((1 \otimes c_2^*) E (1 \otimes c_1)\big) = \big(\mathrm{id} \otimes \langle \cdot \, \Lambda_{\widetilde{\mu}}(c_1), \Lambda_{\widetilde{\mu}}(c_2) \rangle \big)(E) = (\mathrm{id} \otimes \omega_{\xi, \eta})(E),$$

where $\xi = \Lambda_{\widetilde{\mu}}(c_1)$, $\eta = \Lambda_{\widetilde{\mu}}(c_2)$. Considering the result of Proposition 6.3, we see that the $(\mathrm{id} \otimes \omega_{\xi, \eta})(E)$ are strongly*-dense in the von Neumann algebra N. This suggests us to consider the norm closure of the space spanned by the elements $(\mathrm{id} \otimes \omega_{\xi, \eta})(E)$, for $\xi, \eta \in \mathcal{H}_{\widetilde{\mu}}$. Namely,

$$B := \overline{\big\{ (\mathrm{id} \otimes \omega_{\xi, \eta})(E) : \xi, \eta \in \mathcal{H}_{\widetilde{\mu}} \big\}}^{\| \, \|} = \overline{\big\{ (\mathrm{id} \otimes \omega)(E) : \omega \in \mathcal{B}(\mathcal{H}_{\widetilde{\mu}})_* \big\}}^{\| \, \|}. \quad (6.1)$$

Similarly, define:

$$C := \overline{\big\{ (\theta \otimes \mathrm{id})(E) : \theta \in \mathcal{B}(\mathcal{H}_{\widetilde{\nu}})_* \big\}}^{\| \, \|}. \quad (6.2)$$

We gather some results on these subspaces $B \ (\subseteq N)$ and $C \ (\subseteq L)$.

Proposition 6.8. *Let $(E, N, \widetilde{\nu})$ be a separability triple in the von Neumann algebraic sense of Definition 6.1, so that $E \in N \otimes L$ is a separability idempotent. Consider the subspace B of N, given in Equation (6.1), and the subspace C of L, given in Equation (6.2). Then we have:*

(1) $\mathcal{D}(\gamma_N) \cap B$ *is dense in B, and γ_N restricted to this space has a dense range in L.*

(2) B *is a *-subalgebra of N.*

(3) $\mathcal{D}(\gamma_L) \cap C$ *is dense in C, and γ_L restricted to this space has a dense range in N.*

(4) C *is a *-subalgebra of L.*

Proof. (1) Let $c_1, c_2 \in \mathcal{T}_{\widetilde{\mu}}$, and consider $(\mathrm{id} \otimes \omega_{\Lambda_{\widetilde{\mu}}(c_1), \Lambda_{\widetilde{\mu}}(c_2)})(E) \in B$. But,

$$\begin{aligned}
(\mathrm{id} \otimes \omega_{\Lambda_{\widetilde{\mu}}(c_1), \Lambda_{\widetilde{\mu}}(c_2)})(E) &= (\mathrm{id} \otimes \widetilde{\mu})\big((1 \otimes c_2^*) E (1 \otimes c_1)\big) \\
&= (\mathrm{id} \otimes \widetilde{\mu})\big(E[1 \otimes c_1 \sigma_{-i}^{\widetilde{\mu}}(c_2^*)]\big) = \gamma_N^{-1}\big(c_1 \sigma_{-i}^{\widetilde{\mu}}(c_2^*)\big).
\end{aligned}$$
$$(6.3)$$

So we see that $(\mathrm{id} \otimes \omega_{\Lambda_{\widetilde{\mu}}(c_1), \Lambda_{\widetilde{\mu}}(c_2)})(E) \in \mathcal{D}(\gamma_N) \cap B$. We know from Equation (6.1) that such elements span a dense subspace of B. In addition, the elements of the form $c_1 \sigma_{-i}^{\widetilde{\mu}}(c_2^*)$, $c_1, c_2 \in \mathcal{T}_{\widetilde{\mu}}$, are known to be dense in L. As a consequence, we see that under γ_N, the space $\mathcal{D}(\gamma_N) \cap B$ is sent to a dense subspace in L.

(2) Let $b \in \mathcal{D}(\gamma_N) \cap B$, and consider $\widetilde{b} = (\mathrm{id} \otimes \omega)(E)$, for $\omega \in \mathcal{B}(\mathcal{H}_{\widetilde{\mu}})_*$. Such elements are dense in B. Observe that

$$\widetilde{b}b = (\mathrm{id} \otimes \omega)(E)\, b = (\mathrm{id} \otimes \omega)\big(E(b \otimes 1)\big) = (\mathrm{id} \otimes \omega)\big(E(1 \otimes \gamma_N(b))\big) = (\mathrm{id} \otimes \theta)(E),$$

where $\theta = \omega(\cdot \, \gamma_N(b))$. This shows that B is closed under multiplication. Meanwhile, observe also that $[(\mathrm{id} \otimes \omega)(E)]^* = (\mathrm{id} \otimes \bar{\omega})(E)$, by the self-adjointness of E. So we see that B is a *-subalgebra of N.

(3), (4) Proof is similar to that of (1), (2). $\qquad\qquad\qquad\qquad\qquad\square$

Corollary. (1) B *is a C^*-subalgebra of N, and N is the von Neumann algebra closure of B.*

(2) C *is a C^*-subalgebra of L, and L is the von Neumann algebra closure of C.*

Proof. For B, being a norm-closed *-subalgebra of N, that is strongly *-dense in N, the result follows immediately. Similar for C in L. $\qquad\qquad\square$

In the next proposition, we show that $E \in M(B \otimes C)$ and that E is a separability idempotent:

Proposition 6.9. *Let $E \in N \otimes L$ be a separability idempotent in the von Neumann algebraic sense, and consider the C^*-subalgebras B of N and C of L, as defined above. Denote by ν the weight on B, restricted from $\widetilde{\nu}$ on N, and similarly, consider μ on C, restricted from $\widetilde{\mu}$ on L. Then:*

(1) *The $\sigma_t^{\widetilde{\nu}}$, $t \in \mathbb{R}$, leaves B invariant. So we can consider $\sigma_t^{\nu} := \sigma_t^{\widetilde{\nu}}|_B$, for $t \in \mathbb{R}$. Similar for (σ_t^{μ}), on C.*

(2) ν *is a KMS weight on B, equipped with the (norm-continuous) automorphism group (σ_t^{ν}). Similarly, μ is a KMS weight on C, equipped with the automorphism group (σ_t^{μ}).*

(3) $E \in M(B \otimes C)$.

(4) (E, B, ν) *forms a separability triple in the C^*-algebraic sense of Definition* 3.2.

Proof. (1) Consider $(\mathrm{id} \otimes \omega)(E) \in B$, where $\omega \in \mathcal{B}(\mathcal{H}_{\widetilde{\mu}})_*$. For any $t \in \mathbb{R}$, we know from Proposition 6.4 that $(\sigma_{-t}^{\widetilde{\nu}} \otimes \sigma_t^{\widetilde{\mu}})(E) = E$. So we have:

$$\sigma_t^{\widetilde{\nu}}\big((\mathrm{id} \otimes \omega)(E)\big) = \sigma_t^{\widetilde{\nu}}\big((\mathrm{id} \otimes \omega)[(\sigma_{-t}^{\widetilde{\nu}} \otimes \sigma_t^{\widetilde{\mu}})(E)]\big) = \big(\mathrm{id} \otimes (\omega \circ \sigma_t^{\widetilde{\mu}})\big)(E). \quad (6.4)$$

Since $B = \overline{\{(\mathrm{id} \otimes \omega)(E) : \omega \in \mathcal{B}(\mathcal{H}_{\widetilde{\mu}})_*\}}^{\|\ \|}$, this means that $\sigma_t^{\widetilde{\nu}}(B) = B$, for all $t \in \mathbb{R}$. By a similar argument, we can show that $\sigma_t^{\widetilde{\mu}}(C) = C$, for all $t \in \mathbb{R}$.

(2) Let $\nu = \widetilde{\nu}|_B$. It is a faithful weight because $\widetilde{\nu}$ is. To see that ν is semi-finite, consider $(\mathrm{id} \otimes \omega_{\Lambda_{\widetilde{\mu}}(c_1), \Lambda_{\widetilde{\mu}}(c_2)})(E) \in B$, for $c_1, c_2 \in \mathcal{T}_{\widetilde{\mu}}$. We know that such elements span a dense subspace of B. By Equation (6.3), we have:

$$\nu\big((\mathrm{id} \otimes \omega_{\Lambda_{\widetilde{\mu}}(c_1), \Lambda_{\widetilde{\mu}}(c_2)})(E)\big) = \widetilde{\nu}\big((\mathrm{id} \otimes \omega_{\Lambda_{\widetilde{\mu}}(c_1), \Lambda_{\widetilde{\mu}}(c_2)})(E)\big) = \widetilde{\nu}\big(\gamma_N^{-1}(c_1 \sigma_{-i}^{\widetilde{\mu}}(c_2^*))\big)$$
$$= \widetilde{\mu}\big(c_1 \sigma_{-i}^{\widetilde{\mu}}(c_2^*)\big) < \infty.$$

In the last line, we used the fact that $\gamma_N^{-1} = R^{-1} \circ \sigma_{-i/2}^{\widetilde{\mu}}$ (analogous to Proposition 4.2), so $\widetilde{\nu} \circ \gamma_N^{-1} = \widetilde{\mu} \circ \sigma_{-i/2}^{\widetilde{\mu}} = \widetilde{\mu}$. In this way, we have shown that ν is valid on the dense subspace $\{(\mathrm{id} \otimes \omega_{\Lambda_{\widetilde{\mu}}(c_1), \Lambda_{\widetilde{\mu}}(c_2)})(E) : c_1, c_2 \in \mathcal{T}_{\widetilde{\mu}}\}$, which forms a core for ν.

Meanwhile, since $\sigma_t^{\widetilde{\nu}}(B) = B$, we may write $\sigma_t^{\nu} = \sigma_t^{\widetilde{\nu}}|_B$, for $t \in \mathbb{R}$. With $t \mapsto \sigma_t^{\widetilde{\nu}}$ being strongly continuous, and by Equation (6.4), we have that for $b \in B$, the function $t \mapsto \sigma_t^{\nu}(b)$ is norm-continuous. From the properties of the modular automorphism group $(\sigma_t^{\widetilde{\nu}})_{t \in \mathbb{R}}$ for $\widetilde{\nu}$, we can show easily the analogous properties for $(\sigma_t^{\nu})_{t \in \mathbb{R}}$, and this means that ν is a KMS weight on the C^*-algebra B. By construction, it is clear that the W^*-lift of ν will be just $\widetilde{\nu}$ on N.

Similarly, if we let $\mu = \widetilde{\mu}|_C$, then it is a KMS weight on the C^*-algebra C, together with the one-parameter group of automorphisms $(\sigma_t^{\mu})_{t \in \mathbb{R}}$, where $\sigma_t^{\mu} = \sigma_t^{\widetilde{\mu}}|_C$. The W^*-lift of μ is $\widetilde{\mu}$ on L.

(3) We already know that E is a self-adjoint idempotent contained in $N \otimes L$. Therefore, by Proposition 6.3, we have for any $\varepsilon > 0$ and for any $\xi \in \mathcal{H}_{\widetilde{\nu}}$, $\eta \in \mathcal{H}_{\widetilde{\mu}}$, we can find a finite number of elements $p_1, p_2, \ldots, p_n \in B$ of the form $(\mathrm{id} \otimes \omega)(E)$, $\omega \in \mathcal{B}(\mathcal{H}_{\widetilde{\mu}})_*$ and $q_1, q_2, \ldots, q_n \in C$ of the form $(\theta \otimes \mathrm{id})(E)$, $\theta \in \mathcal{B}(\mathcal{H}_{\widetilde{\nu}})_*$, such that

$$\left\|\Big(E - \sum_{k=1}^{n} p_k \otimes q_k\Big)(\xi \otimes \eta)\right\| < \varepsilon.$$

Since N (also B) acts on $\mathcal{H}_{\widetilde{\nu}}$ in a non-degenerate way, and similarly for L (also C) on $\mathcal{H}_{\widetilde{\mu}}$, without loss of generality we may let $\xi = \Lambda_{\nu}(b_0) = \Lambda_{\widetilde{\nu}}(b_0)$ and $\eta = \Lambda_{\mu}(c_0) = \Lambda_{\widetilde{\mu}}(c_0)$, where $b_0 \in \mathfrak{N}_{\nu}$, $c_0 \in \mathfrak{N}_{\mu}$. Then:

$$\left\|(\Lambda_{\widetilde{\nu}} \otimes \Lambda_{\widetilde{\mu}})\Big(E(b_0 \otimes c_0) - \sum_{k=1}^{n}(p_k \otimes q_k)(b_0 \otimes c_0)\Big)\right\| < \varepsilon.$$

Since b_0, c_0 are arbitrary, while $\mathfrak{N}_{\nu}$ is dense in B and $\mathfrak{N}_{\mu}$ is dense in C, we can see that E is contained in the closure of $B \otimes C$ under the strict topology. In other words, $E \in M(B \otimes C)$.

(4) From (3), we saw that $E \in M(B \otimes C)$. From (1), (2), we saw that ν is a KMS weight on B, together with its associated norm-continuous automorphism group (σ_t^{ν}).

Meanwhile, note that for $(\mathrm{id}\otimes\omega)(E) \in B\ (\subseteq N)$, by Proposition 6.4 (3), we have:

$$R\big((\mathrm{id}\otimes\omega)(E)\big) = R\big((\omega\otimes\mathrm{id})(\varsigma E)\big)$$
$$= R\big((\omega\otimes\mathrm{id})[(R\otimes R^{-1})(E)]\big) = \big((\omega\circ R)\otimes\mathrm{id}\big)(E) \in C.$$

This shows that the *-anti-isomorphism $R\colon N \to L$ restricts to $R\colon B \to C$. This will be also a *-anti-isomorphism. Since $R\colon B \to C$ is a restriction of $R\colon N \to L$ and since $\sigma_t^\nu = \sigma_t^{\widetilde{\nu}}|_B$, the conditions of Definition 6.1 for $(E, N, \widetilde{\nu})$ immediately give rise to the conditions of Definition 3.2. This means that (E, B, ν) is a separability triple in the C^*-algebraic sense. And, the associated γ_B, γ_C maps are none other than the restrictions of γ_N and γ_L to B and C, respectively, which have already appeared in $(1), (3)$ of Proposition 6.8. $\qquad\square$

7. The nature of the base C^*-algebra B

The requirements for (E, B, ν) being a separability triple means that some possible restrictions will be needed on the pair (B, ν). This question was considered in the algebraic setting (see [24]), and the conclusion in that framework was that B has to be a direct sum of matrix algebras. While the current situation is different, we do expect that some similar conditions on the base C^*-algebra B will exist. We wish to explore this question in this section.

To begin, assume that B is a C^*-algebra, ν a KMS weight on B, such that the pair (B, ν) satisfies the relevant conditions that give us a *separability idempotent* $E \in M(B\otimes C)$. Here C is the opposite C^*-algebra of B.

7.1. The subalgebra BbB for a fixed $b \in B$. Let us fix $b \in B$. By Proposition 4.3, we know that $(b\otimes 1)E$ is contained in the C^*-tensor product $B\otimes C$. So, for any $k \in \mathbb{N}$, we can find a finite number of elements $b_i^{(k)} \in B$, $c_i^{(k)} \in C$ $(i = 1, 2, \ldots, N_k)$, such that

$$\left\| (b\otimes 1)E - \sum_{i=1}^{N_k} b_i^{(k)} \otimes c_i^{(k)} \right\| < \frac{1}{k}.$$

Next, let $\widetilde{b} \in B$ be arbitrary and let $\varepsilon > 0$. By Proposition 4.4 (since E is "full"), we can find $\widetilde{c} \in C$, $\widetilde{\omega} \in C^*$, such that $\left\| \widetilde{b} - (\mathrm{id}\otimes\widetilde{\omega})(E(1\otimes\widetilde{c})) \right\| < \varepsilon/(2\|b\|)$. Then we have:

$$\left\| b\widetilde{b} - b(\mathrm{id}\otimes\widetilde{\omega})(E(1\otimes\widetilde{c})) \right\| < \frac{\varepsilon}{2}.$$

Consider $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon / (2\|\widetilde{\omega}\| \|\widetilde{c}\|)$, and as above, find corresponding $b_i^{(k)} \in B$, $c_i^{(k)} \in C$ $(i = 1, 2, \ldots, N_k)$. Then:

$$
\left\| b(\mathrm{id} \otimes \widetilde{\omega})(E(1 \otimes \widetilde{c})) - \sum_{i=1}^{N_k} b_i^{(k)} \widetilde{\omega}(c_i^{(k)} \widetilde{c}) \right\|
$$
$$
= \left\| (\mathrm{id} \otimes \widetilde{\omega})\left(\left[(b \otimes 1)E - \sum_{i=1}^{N_k} b_i^{(k)} \otimes c_i^{(k)} \right](1 \otimes \widetilde{c}) \right) \right\| < \frac{1}{k} \cdot \|\widetilde{\omega}\| \|\widetilde{c}\| < \frac{\varepsilon}{2}.
$$

Combine this with the earlier inequality to obtain (by triangle inequality):

$$
\left\| b\widetilde{b} - \sum_{i=1}^{N_k} b_i^{(k)} \widetilde{\omega}(c_i^{(k)} \widetilde{c}) \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

In the above, each $\widetilde{\omega}(c_i^{(k)} \widetilde{c}) \in \mathbb{C}$. So we have shown that any $b\widetilde{b}$ is within ε-distance from $\mathrm{span}(b_i^{(k)} : k \in \mathbb{N}, i = 1, 2, \ldots, N_k)$. And, this result is true for arbitrary $\widetilde{b} \in B$ and any $\varepsilon > 0$. We thus have:

$$
bB \subseteq \overline{\mathrm{span}\left(b_i^{(k)} : k \in \mathbb{N}, i = 1, 2, \ldots, N_k \right)}^{\|\ \|}.
$$

From the countable generating set $\{b_i^{(k)}\}$ for bB, consider a maximal linearly independent (basis) subset $\{b_\lambda\}_{\lambda \in \Lambda}$, indexed by a countable set Λ. We obtain the following result:

Proposition 7.1. (1) *For a fixed $b \in B$, we can find a countable collection of elements $\{b_\lambda\}_{\lambda \in \Lambda}$ contained in B, such that*

$$
bB \subseteq \overline{\mathrm{span}\left(b_\lambda : \lambda \in \Lambda \right)}^{\|\ \|}.
$$

(2) *For any $k \in \mathbb{N}$, we can find a finite set $F_k \subset \Lambda$ and a finite number of elements $\{c_\lambda\}_{\lambda \in F_k}$ in C, such that*

$$
\left\| (b \otimes 1)E - \sum_{\lambda \in F_k} b_\lambda \otimes c_\lambda \right\| < \frac{1}{k}.
$$

(3) *For any $z \in bB$ and any $\varepsilon > 0$, we can find a finite set $\widetilde{F} \subset \Lambda$ and a finite number of elements $\{\widetilde{c}_\lambda\}_{\lambda \in \widetilde{F}}$ in C, such that*

$$
\left\| (z \otimes 1)E - \sum_{\lambda \in \widetilde{F}} b_\lambda \otimes \widetilde{c}_\lambda \right\| < \varepsilon.
$$

Proof. (1) From the way $\{b_\lambda\}_{\lambda \in \Lambda}$ was constructed, it is clear that each generator $b_i^{(k)}$ for bB can be expressed as a linear combination of the $\{b_\lambda\}$. So the result follows.

(2) As a consequence of (1), for any finite sum of the form $\sum_{k,i} b_i^{(k)} \otimes c_i^{(k)}$, we can find a finite set $F \subset \Lambda$ so that

$$\sum_{k,i} b_i^{(k)} \otimes c_i^{(k)} = \sum_{\lambda \in F} b_\lambda \otimes c_\lambda,$$

where the c_λ are appropriate elements in C. In particular, for a fixed $k \in \mathbb{N}$, we can find a finite set $F_k \subset \Lambda$ and elements $\{c_\lambda\}_{\lambda \in F_k}$, such that

$$\sum_{i=1}^{N_k} b_i^{(k)} \otimes c_i^{(k)} = \sum_{\lambda \in F_k} b_\lambda \otimes c_\lambda.$$

Then we have:

$$\left\| (b \otimes 1)E - \sum_{\lambda \in F_k} b_\lambda \otimes c_\lambda \right\| < \frac{1}{k}.$$

(3) Find $\widetilde{b} \in B$ such that $\|z - b\widetilde{b}\| < \frac{\varepsilon}{2}$. Without loss of generality, we may assume that $\widetilde{b} \in \mathcal{D}(\gamma_C^{-1})$, such that $(\widetilde{b} \otimes 1)E = (1 \otimes \gamma_C^{-1}(\widetilde{b}))E$. Note that

$$(b\widetilde{b} \otimes 1)E = \big(b \otimes \gamma_C^{-1}(\widetilde{b})\big)E = \big(1 \otimes \gamma_C^{-1}(\widetilde{b})\big)(b \otimes 1)E.$$

Find $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon/(2\|\gamma_C^{-1}(\widetilde{b})\|)$, then by (2) we have a finite subset $\widetilde{F} = F_k \subset \Lambda$ such that:

$$\left\| (b\widetilde{b} \otimes 1)E - \sum_{\lambda \in \widetilde{F}} b_\lambda \otimes \gamma_C^{-1}(\widetilde{b})c_\lambda \right\| = \left\| \big(1 \otimes \gamma_C^{-1}(\widetilde{b})\big)\Big[(b \otimes 1)E - \sum_{\lambda \in \widetilde{F}} b_\lambda \otimes c_\lambda\Big] \right\|$$

$$< \frac{1}{k} \cdot \|\gamma_C^{-1}(\widetilde{b})\| < \frac{\varepsilon}{2}.$$

Write $\widetilde{c}_\lambda = \gamma_C^{-1}(\widetilde{b})c_\lambda$. Then,

$$\left\| (b\widetilde{b} \otimes 1)E - \sum_{\lambda \in \widetilde{F}} b_\lambda \otimes \widetilde{c}_\lambda \right\| < \frac{\varepsilon}{2}.$$

Meanwhile,

$$\left\| (z \otimes 1)E - (b\widetilde{b} \otimes 1)E \right| \le \|z - b\widetilde{b}\| \|E\| < \frac{\varepsilon}{2}.$$

By the triangle inequality, we have:

$$\left\| (z \otimes 1)E - \sum_{\lambda \in \widetilde{F}} b_\lambda \otimes \widetilde{c}_\lambda \right\| < \varepsilon. \qquad \square$$

Next, for a typical generator b_λ (for a fixed $\lambda \in \Lambda$), we wish to explore the space Bb_λ. Since $E(b_\lambda \otimes 1)$ is contained in the C^*-tensor product $B \otimes C$, for any $m \in \mathbb{N}$ we can find a finite number of elements $b_{\lambda,j}^{(m)} \in B$, $c_{\lambda,j}^{(m)} \in C$ ($j = 1, 2, \ldots, N_m$), such that

$$\left\| E(b_\lambda \otimes 1) - \sum_{j=1}^{N_m} b_{\lambda,j}^{(m)} \otimes c_{\lambda,j}^{(m)} \right\| < \frac{1}{m}.$$

From the set $\{b_{\lambda,j}^{(m)} : m \in \mathbb{N}, j = 1, 2, \ldots, N_m\}$, find a maximal linearly independent subset $\{b_{\lambda,q}\}_{q \in Q_\lambda}$, indexed by a countable set Q_λ. Then we will have similar results as before:

Proposition 7.2. (1) $Bb_\lambda \subseteq \overline{\mathrm{span}(b_{\lambda,q} : q \in Q_\lambda)}^{\|\ \|}$.

(2) *For any $y \in Bb_\lambda$ and any $\varepsilon > 0$, we can find a finite set $\widetilde{F} \subset Q_\lambda$ and a finite number of elements $\{\widetilde{c}_\lambda\}_{\lambda \in \widetilde{F}}$ in C, such that*

$$\left\| E(y \otimes 1) - \sum_{q \in \widetilde{F}} b_{\lambda,q} \otimes \widetilde{c}_\lambda \right\| < \varepsilon.$$

Proof. Proof is similar to that of Proposition 7.1.   □

Proposition 7.2 is valid for any b_λ, for $\lambda \in \Lambda$. Meanwhile, it is evident that $BbB \subseteq \overline{\cup_{\lambda \in \Lambda} Bb_\lambda}$. So

$$BbB \subseteq \overline{\mathrm{span}\left(b_{\lambda,q} : \lambda \in \Lambda, q \in Q_\lambda\right)}^{\|\ \|}.$$

Again, similarly as above, find $\{b_p\}_{p \in P}$, indexed by a countable set P and is a maximal linearly independent subset of the $\{b_{\lambda,q}\}$. Then we have the following result:

Theorem 7.3. (1) *For a fixed $b \in B$, we can find a countable set of elements $\{b_p\}_{p \in P}$ in B, such that*

$$BbB \subseteq \overline{\mathrm{span}(b_p : p \in P)}^{\|\ \|}.$$

(2) *For any $x \in BbB$ and any $\varepsilon > 0$, we can find a finite set $\widetilde{F} \subset P$ and a finite number of elements $\{\widetilde{c}_p\}_{p \in \widetilde{F}}$ in C, such that*

$$\left\| (x \otimes 1)E - \sum_{p \in \widetilde{F}} b_p \otimes \widetilde{c}_p \right\| < \varepsilon.$$

Proof. (1) This is a consequence of the previous paragraph.

(2) Let $x \in BbB$ and $\varepsilon > 0$ be arbitrary. We can find a finite number of elements $b_1', b_1''; b_2', b_2''; \ldots, b_n', b_n'' \in B$ such that

$$\left\| x - \left(b_1'bb_1'' + b_2'bb_2'' + \cdots + b_n'bb_n''\right) \right\| < \frac{\varepsilon}{3}.$$

Since γ_C^{-1} is densely-defined in B, we may as well take each $b_i'' \in \mathcal{D}(\gamma_C^{-1})$. Then

$$(b_i'bb_i'' \otimes 1)E = \left(b_i'b \otimes \gamma_C^{-1}(b_i'')\right)E = \left(b_i' \otimes \gamma_C^{-1}(b_i'')\right)(b \otimes 1)E.$$

We know from Proposition 7.1 (2) that the expression $(b \otimes 1)E$ can be approximated arbitrarily closely by a finite sum of the form $\sum_{\lambda \in F_k} b_\lambda \otimes c_\lambda$. This means that for each i, the expression $(b_i'bb_i'' \otimes 1)E$ can be approximated arbitrarily closely by a finite sum of the form $\sum_{\lambda \in F_k} b_i'b_\lambda \otimes \gamma_C^{-1}(b_i'')c_\lambda$. Meanwhile, by Proposition 7.2 and part (1) above that each $b_i'b_\lambda$ can be expressed as a finite linear combination in terms of the $\{b_p\}_{p \in P}$. So we can find a finite set $P_i \subset P$ such that

$$\left\| (b_i'bb_i'' \otimes 1)E - \sum_{p \in P_i} b_p \otimes c_p^{(i)} \right\| < \frac{\varepsilon}{3n},$$

for some appropriate $\{c_p^{(i)}\}_{p \in P_i} \in C$. This can be done for each $(b_i'bb_i'' \otimes 1)E$, $i = 1, 2, \ldots, n$.

We are able to find a finite set $\widetilde{F} \subset P_1 \cup P_2 \cup \cdots \cup P_n \subset P$ and suitable elements $\{\widetilde{c}_p\}_{p \in \widetilde{F}}$, such that

$$\sum_{p \in \widetilde{F}} b_p \otimes \widetilde{c}_p = \sum_{i=1}^{n} \sum_{p \in P_i} b_p \otimes c_p^{(i)}.$$

Therefore,

$$\left\| (x \otimes 1)E - \sum_{p \in \widetilde{F}} b_p \otimes \widetilde{c}_p \right\| \leq \left\| \left(\left[x - \left(b_1'bb_1'' + b_2'bb_2'' + \cdots + b_n'bb_n''\right)\right] \otimes 1\right)E \right\|$$
$$+ \sum_{i=1}^{n} \left\| (b_i'bb_i'' \otimes 1)E - \sum_{p \in P_i} b_p \otimes c_p^{(i)} \right\|$$
$$< \frac{\varepsilon}{3}\|E\| + n \cdot \frac{\varepsilon}{3n} < \varepsilon. \qquad \square$$

Remark. The choice of the generators $\{b_p\}_{p \in P}$ is determined by b, but of course not unique. Even so, to avoid confusion, we will from now on fix the choice of the $\{b_p\}$ as the generators for BbB. Without loss of generality, we may assume that each generator b_p is a self-adjoint element. In addition, since $\mathfrak{N}_\nu$ is dense in B, we may as well take each $b_p \in \mathfrak{N}_\nu$, or even $b_p \in \mathcal{T}_\nu$ (Tomita subalgebra). This would in turn means that each b_p is contained in the domains of the maps γ_B and γ_C^{-1}.

In the below is a lemma concerning the elements $\widetilde{c}_p \in C$:

Lemma 7.4. *In Theorem 7.3 (2) above, as we approximate $(x \otimes 1) E$ by the finite sums of the form $\sum b_p \otimes \widetilde{c}_p$, it is possible to choose each $\widetilde{c}_p$ to be contained in $\gamma_C^{-1}(BbB)$.*

Proof. Since $\mathcal{D}(\gamma_B)$ is dense in B, we may, without loss of generality, assume that $x \in \mathcal{D}(\gamma_B) \cap BbB$. Let $\varepsilon > 0$. As in (2) above, we can find a finite subset $\widetilde{F} \subset P$ and the elements $\{c'_p\}_{p \in \widetilde{F}}$ in C, such that

$$\left\| (x \otimes 1) E - \sum_{p \in \widetilde{F}} b_p \otimes c'_p \right\| < \frac{\varepsilon}{2}.$$

Since $\mathcal{D}(\gamma_C)$ is dense in C, we may assume here that each $c'_p \in \mathcal{D}(\gamma_C)$.

Apply $\gamma_B \otimes \gamma_C$. Since γ_B is anti-multiplicative, we then have:

$$\left\| (\gamma_B \otimes \gamma_C)(E)(\gamma_B(x) \otimes 1) - \sum_{p \in \widetilde{F}} \gamma_B(b_p) \otimes \gamma_C(c'_p) \right\| < \frac{\varepsilon}{2}.$$

Recall from Proposition 4.9 that $(\gamma_B \otimes \gamma_C)(E) = \varsigma E$. So applying the flip map, this becomes

$$\left\| E(1 \otimes \gamma_B(x)) - \sum_{p \in \widetilde{F}} \gamma_C(c'_p) \otimes \gamma_B(b_p) \right\| < \frac{\varepsilon}{2},$$

or equivalently (by Proposition 3.6), we have:

$$\left\| E(x \otimes 1) - \sum_{p \in \widetilde{F}} \gamma_C(c'_p) \otimes \gamma_B(b_p) \right\| < \frac{\varepsilon}{2}. \tag{7.1}$$

Let $M = |\widetilde{F}| \cdot \max(\|b_p\| : p \in \widetilde{F})$. For each $p \in \widetilde{F}$, consider $\omega \in C^*$, such that

$$\omega(\gamma_B(b_p)) = \frac{1}{M} \quad \text{and} \quad \omega(\gamma_B(b_q)) = 0 \quad \text{if } q \in \widetilde{F} \text{ with } q \neq p.$$

This is do-able because γ_B being an injective anti-homomorphism, the linear independence of the $\{b_p\}$ implies the linear independence of the $\{\gamma_B(b_p)\}$. Note that $\|\omega\| = \frac{1}{M}$. Applying $\mathrm{id} \otimes \omega$ to Equation (7.1), we obtain:

$$\left\| (\mathrm{id} \otimes \omega)[E(x \otimes 1)] - \gamma_C(c'_p) \right\| < \frac{\varepsilon}{2M}.$$

Note that $(\mathrm{id} \otimes \omega)[E(x \otimes 1)] \in BbB$. In other words, we must have each $\gamma_C(c'_p)$ to be within $\frac{\varepsilon}{2M}$-distance from BbB.

While this in itself does not automatically mean each $\gamma_C(c'_p)$ must be contained in BbB, this allows us to choose instead $\widetilde{c}_p \in \gamma_C^{-1}(BbB)$ such that

$$\|c'_p - \widetilde{c}_p\| = \|\gamma_C(c'_p) - \gamma_C(\widetilde{c}_p)\| < \frac{\varepsilon}{2M},$$

for each $p \in \widetilde{F}$. We would then have:

$$\left\|(x \otimes 1)E - \sum_{p \in \widetilde{F}} b_p \otimes \widetilde{c}_p\right\| \leq \left\|(x \otimes 1)E - \sum_{p \in \widetilde{F}} b_p \otimes c'_p\right\|$$
$$+ \sum_{p \in \widetilde{F}} \left\|b_p \otimes (c'_p - \widetilde{c}_p)\right\|$$
$$< \frac{\varepsilon}{2} + |\widetilde{F}| \cdot \max\big(\|b_p\| : p \in \widetilde{F}\big) \cdot \frac{\varepsilon}{2M} < \varepsilon. \qquad \square$$

7.2. BbB is a (separable) liminal C^*-algebra. Assume now that the fixed element $b \in B$ is self-adjoint. Then BbB is a (two-sided, closed) ideal in B, so itself a C^*-algebra. Also by the characterization given in Theorem 7.3, we see that BbB is a separable C^*-algebra. Let us explore what else we can learn about the nature of this C^*-algebra BbB.

Since ν is a KMS weight on B, we can perform the GNS construction, and obtain $(\pi, \mathcal{H}, \Lambda)$, where Λ is the (injective) GNS map. We may regard $B = \pi(B) \subseteq \mathcal{B}(\mathcal{H})$. Clearly, π is also a representation for the C^*-algebra BbB, and we have:

$$BbB = \pi(BbB) \subseteq \mathcal{B}(\mathcal{H}).$$

Consider now the subspace $\mathcal{H}_b := \overline{\mathrm{span}(\Lambda(b_p) : p \in P)}^{\|\ \|_{\mathcal{H}}} \subseteq \mathcal{H}$, where the $b_p \in \mathcal{T}_\nu$ are as in Theorem 7.3. Under the representation π, the C^*-algebra BbB leaves the subspace $\mathcal{H}_b$ invariant. In this way, we obtain a subrepresentation π_b of BbB, acting on $\mathcal{H}_b$. Meanwhile, by Gram–Schmidt process, without loss of generality, we may further assume that $\nu(b_q b_p) = \langle \Lambda(b_p), \Lambda(b_q)\rangle = 0$ if $p, q \in P$ is such that $p \neq q$, while $\nu(b_p b_p) = 1$, $\forall p \in P$. (Recall that we are assuming each b_p to be self-adjoint).

Definition 7.5. For any $p, q \in P$, denote by $\langle b_p, b_q\rangle_E$ the operator acting on $\mathcal{H}_b$ such that

$$\langle b_p, b_q\rangle_E \xi := \langle \xi, \Lambda(b_q)\rangle \Lambda(b_p).$$

Clearly, these are finite-rank (actually, rank one) operators contained in $\mathcal{B}(\mathcal{H}_b)$.

Proposition 7.6. *Any operator in $\pi_b(BbB)$ can be approximated by a linear combination of the operators $\langle b_p, b_q\rangle_E$, $p, q \in P$.*

Proof. Since π_b arises from the GNS representation, we know that for $x \in BbB$ and $\Lambda(\widetilde{b}) \in \mathcal{H}_b$, we have:

$$\pi_b(x)\Lambda(\widetilde{b}) = \Lambda(x\widetilde{b}).$$

Let $\widetilde{b} = b_m$, for $m \in P$. As $\widetilde{b} \in \mathcal{D}(\gamma_B)$, we can write $\widetilde{b} = (\mathrm{id} \otimes \mu)(E(\widetilde{b} \otimes 1))$. Then

$$x\widetilde{b} = (\mathrm{id} \otimes \mu)\big((x \otimes 1)E(\widetilde{b} \otimes 1)\big) = (\mathrm{id} \otimes \mu)\big((x \otimes 1)E(1 \otimes \gamma_B(\widetilde{b}))\big).$$

Let $\varepsilon > 0$ be arbitrary. By Theorem 7.3 (2), we can find a finite subset $F \subset P$ and $\{\widetilde{c}_p\}_{p \in F}$ in C, such that

$$\left\| (x \otimes 1)E - \sum_{p \in F} b_p \otimes \widetilde{c}_p \right\| < \frac{\varepsilon}{2\|\widetilde{b}\|}.$$

By Lemma 7.4, we may assume that each $\widetilde{c}_p \in \gamma_C^{-1}(BbB)$.

Observe that:

$$\left\| x\widetilde{b} - \sum_{p \in F} \mu(\widetilde{c}_p \gamma_B(\widetilde{b}))b_p \right\| = \left\| (\mathrm{id} \otimes \mu)\Big(\Big[(x \otimes 1)E - \sum_{p \in F} b_p \otimes \widetilde{c}_p\Big](1 \otimes \gamma_B(\widetilde{b}))\Big) \right\|$$

$$< \frac{\varepsilon}{2\|\widetilde{b}\|} \cdot \|\gamma_B(\widetilde{b})\| = \frac{\varepsilon}{2}.$$

We know $\mu = \nu \circ \gamma_C$. So $\mu(\widetilde{c}_p \gamma_B(\widetilde{b})) = \nu((\gamma_C \circ \gamma_B)(\widetilde{b})\gamma_C(\widetilde{c}_p))$. And, using the fact that $\gamma_C \circ \gamma_B = \sigma_i^\nu$ (see Proposition 4.6), we see that

$$\mu(\widetilde{c}_p \gamma_B(\widetilde{b})) = \nu(\gamma_C(\widetilde{c}_p)\widetilde{b}).$$

As $\gamma_C(\widetilde{c}_p) \in BbB$, we can approximate it arbitrarily closely by a finite linear combination of the b_q, for $q \in F^{(p)} \subset P$. This can be done for each $p \in F$. Therefore, it is evident that we can find a finite linear combination of the expressions $\nu(b_q\widetilde{b})b_p = \langle \Lambda(\widetilde{b}), \Lambda(b_q) \rangle \, b_p$, that is within $\frac{\varepsilon}{2}$-distance from $\sum_{p \in F} \mu(\widetilde{c}_p \gamma_B(\widetilde{b}))b_p$.

It follows that $x\widetilde{b}$ can be approximated within ε-distance by a finite linear combination of the $\langle \Lambda(\widetilde{b}), \Lambda(b_q) \rangle \, b_p$. Applying the GNS-map Λ, we see that $\pi_b(x)\Lambda(\widetilde{b})$ can be approximated arbitrarily closely by finite linear combinations of the expressions

$$\big\langle \Lambda(\widetilde{b}), \Lambda(b_q) \big\rangle \Lambda(b_p) = \langle b_p, b_q \rangle_E \, \Lambda(\widetilde{b}).$$

All this would hold for each of the basis element $\Lambda(b_m)$, $m \in P$. In this way, we prove that any $\pi_b(x)$, $x \in BbB$, can be approximated by a linear combination of the operators $\langle b_p, b_q \rangle_E$, $p, q \in P$. $\qquad\square$

Write $V = \mathrm{span}(\langle b_p, b_q \rangle_E : p, q \in \Gamma) \, (\subseteq \mathcal{B}(\mathcal{H}_b))$. Then:

Proposition 7.7. *For the operators* $\langle b_p, b_q \rangle_E, \langle b_{p'}, b_{q'} \rangle_E \in V$, *we have:*

(1) $\langle b_p, b_q \rangle_E \langle b_{p'}, b_{q'} \rangle_E = \delta_{p',q} \langle b_p, b_{q'} \rangle_E$

(2) $\langle b_p, b_q \rangle_E^* = \langle b_q, b_p \rangle_E$.

As a result, we conclude that $\overline{V}^{\|\ \|_{\mathrm{op}}} \cong \mathcal{B}_0(\mathcal{H}_b)$, *the algebra of compact operators on* $\mathcal{H}_b$.

Proof. These are rather standard and straightforward.

(1) Let $\xi \in \mathcal{H}_b$ be arbitrary. we have:

$$\langle b_p, b_q \rangle_E \langle b_{p'}, b_{q'} \rangle_E \xi = \langle \xi, \Lambda(b_{q'}) \rangle \langle b_p, b_q \rangle_E \Lambda(b_{p'})$$
$$= \langle \xi, \Lambda(b_{q'}) \rangle \langle \Lambda(b_{p'}), \Lambda(b_q) \rangle \Lambda(b_p).$$
$$= \nu(b_q^* b_{p'}) \langle b_p, b_{q'} \rangle_E \xi = \delta_{p',q} \langle b_p, b_{q'} \rangle_E \xi.$$

(2) Let $\xi, \zeta \in \mathcal{H}_b$. Then

$$\langle \langle b_p, b_q \rangle_E \xi, \zeta \rangle = \langle \langle \xi, \Lambda(b_q) \rangle \Lambda(b_p), \zeta \rangle = \langle \xi, \Lambda(b_q) \rangle \langle \Lambda(b_p), \zeta \rangle$$
$$= \langle \xi, \overline{\langle \Lambda(b_p), \zeta \rangle} \Lambda(b_q) \rangle = \langle \xi, \langle \zeta, \Lambda(b_p) \rangle \Lambda(b_q) \rangle$$
$$= \langle \xi, \langle b_q, b_p \rangle_E \zeta \rangle.$$

It follows that $\langle b_p, b_q \rangle_E^* = \langle b_q, b_p \rangle_E$.

(3) The results of (1), (2) show that V is a *-algebra contained in $\mathcal{B}(\mathcal{H}_b)$, and the generators behave exactly like the matrix units (see, for instance, Lemma 5.5). Therefore, the operator norm closure of V coincides with $\mathcal{B}_0(\mathcal{H}_b)$. $\qquad \square$

The following results are immediate:

Theorem 7.8. *Let $b \in B$ be a fixed self-adjoint element. Then we have:*

(1) $BbB = \pi_b(BbB) \subseteq \mathcal{B}_0(\mathcal{H}_b)$.

(2) *The C^*-algebra BbB is liminal.*

Proof. (1) The result of Proposition 7.6 is that $\pi_b(BbB)$ is contained in the closure of V. Therefore, by Proposition 7.7, the result follows.

(2) Since BbB is a C^*-algebra contained in the algebra of compact operators, it is *liminal*, in the sense of [7]. $\qquad \square$

7.3. B is a postliminal C^*-algebra. By definition, a C^*-algebra A is said to be *postliminal*, if every non-zero quotient C^*-algebra of A possesses a non-zero liminal closed two-sided ideal (see [1, 7]).

In our case, suppose J is any nontrivial proper (closed, two-sided) ideal of B, and let B/J be its corresponding quotient C^*-algebra. Since $J \neq B$, we can consider a self-adjoint element $b \in B$ with $b \notin J$. Consider the ideal BbB of B. Then $BbB + J$ is an ideal of B such that $(BbB + J)/J \cong BbB/(BbB \cap J)$. Since the latter is a quotient of a liminal C^*-algebra BbB, it is also liminal. In this way, we have shown that the quotient C^*-algebra B/J contains an ideal $(BbB + J)/J$, which is isomorphic to a liminal C^*-algebra. By definition above, we can see that B is *postliminal*:

Theorem 7.9. *Let (E, B, ν) be a separability triple. Then B is a postliminal C^*-algebra.*

Remark. For separable C^*-algebras, it is well known that being postliminal is equivalent to being *type I*. The result that B is postliminal (or type I) seems compatible with the result in the purely algebraic case [24], where it was shown that the base algebra B has to be a direct sum of matrix algebras.

Finally, here is the result when B is commutative:

Proposition 7.10. *If B is commutative, its spectrum is totally disconnected.*

Proof. If B is commutative, any of its ideals is also commutative. In particular, each building-block ideal BbB (for b self-adjoint), being a commutative subalgebra contained in the algebra of compact operators, would be isomorphic to $\mathbb{C}$, spanned by a single projection. Our B would be spanned by its projections. If we write $B \cong C_0(X)$, where X is the spectrum of B, this means that X is totally disconnected. $\qquad\square$

Remark. In fact, if B is commutative, $B \cong C_0(X)$, its spectrum X turns out to be discrete. See Remark following Proposition 2.2 in [11].

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B.-J. Kahng, Department of Mathematics and Statistics, Canisius College, Buffalo, NY 14208, USA

E-mail: kahngb@canisius.edu

A. Van Daele, Department of Mathematics, University of Leuven, Celestijnenlaan 200B, B-3001 Heverlee, Belgium

E-mail: alfons.vandaele@wis.kuleuven.be