Research Description and Plans

(Byung-Jay Kahng, Dept. Math/Stats)

My general area of research is in functional analysis, with special attention given to the C^*-algebraic approach to *non-commutative [or quantum] geometry*. This field naturally combines techniques from various disciplines of mathematics (analysis, algebra, geometry), while having many interesting connections with quantum physics and other areas. In the below, I provide further information about my research topics and plans for the next a few years.

I. Research on Quantum Groupoids and Mathematical Physics

In ordinary geometry/topology, the main object of study is a manifold M, or $C(M)$, the space of continuous functions on M. Whereas in noncommutative geometry [3], the $C(M)$ is replaced by a C^*-algebra A, which is more general than $C(M)$ and often considered as a continuous function space on a "noncommutative manifold".

Transition from ordinary geometry to noncommutative geometry is broadly called the *quantization*. In nature, this is essentially the same as the process in physics of jumping from classical mechanical observables to operators on a quantum mechanical system. In our case, we would further insist on working with continuous functions and the C^*-algebra framework. In this way, we keep the full aspects of topology as we go through the quantization process. In addition to the process itself being of interest, having a "quantum" perspective on a certain classical setting may often enable a deeper understanding of the situation. Naturally, it is worthwhile to have a good understanding of the both classical and quantum theories, while making efforts to further develop the inter-relationship between the two branches.

The main themes of my research have been *quantization* and *noncommutative geometry*, as described above. In particular, I have been studying for some time the **C^*-algebraic, locally compact quantum groups** [13], [14], [17], [21], trying to explore their roles as noncommutative geometric objects of study while viewing them as objects quantized from their classical counterparts, called Poisson–Lie groups. Recently, taking advantage of my sabbatical leave break at KU Leuven (Belgium) during 2012–2013, and supported by the research abroad grant from the U.S. Fulbright Foundation, I began working on the notion of **locally compact quantum groupoids**. The project is on-going, and I made some meaningful progress on this latest project. I am planning to continue further research in this direction.

(1). General theory of locally compact quantum groupoids.

Building on my years of research experience working on the topic of locally compact quantum groups, I recently began to study the notion of *quantum groupoids*. In a sense, this is an even more general framework than that of quantum groups. However, precisely because of this fact, there is enough flexibility to allow wider class of objects. For example, it is already known that graphs (things like networks) can be described in terms of certain class of groupoid C^*-algebras [12], [18]. The theory of quantum groupoids would include these examples, and likely include their quantum versions as well.

Ideally, one wishes to develop a functional analytic theory of quantum groupoids, based on the C^*-algebra framework. However, the known results so far on the subject are mostly primitive ones or the ones based on uncomfortably difficult techniques [16], [4], and the scope has been limited. So

I first began working on certain intermediate notions of "weak multiplier Hopf algebras" [22], [23] and "multiplier Hopf algebroids" [20], which have been developed by my colleague at KU Leuven, Professor Alfons Van Daele, and a few others. Along the way, Van Daele and I wrote a paper on the subject [11], gaining some useful insights.

The theories of weak multiplier Hopf algebras or multiplier Hopf algebroids are nice, but they are only intermediate theories, based on the purely algebraic level. One needs a theory in the framework of C^*-algebras or von Neumann algebras. Only then the resulting theory could be properly called the theory of *locally compact quantum groupoids*. This is the big project I am working on together with Van Daele, which began during my stay in Belgium.

Since it has to extend the existing quantum group theory, which is known to be rather difficult, there are numerous technical issues and subtleties involved. Nevertheless, I am happy to report that the project on developing the C^*-algebraic quantum groupoid theory is progressing well. We (Van Daele and I) have obtained some partial results, and developed some specific techniques that will turn out to be useful when working on the topic. One paper was produced on one of the partial results (on "separability idempotents" [10]), while the main work continues. We now have a working version of the theory, which will give us precise framework for a class of locally compact quantum groupoids. Eventually, this will lead us to a general C^*-algebraic theory that describes even wider class of quantum groupoids. Since last year, I had several occasions to give talks on this project, to report on the results that have been obtained and to report on upcoming results.

So far, my co-author and I have some good understanding about the definition and some fundamental properties of the locally compact quantum groupoids. As I polish up this work, I am also planning to work on possible applications. Among others, I describe two: Construction of quantum groupoids as quantizations of certain Poisson structures; and Generalized duality theory.

(2). CONSTRUCTION OF QUANTUM GROUPS AND QUANTUM GROUPOIDS

A Poisson manifold is a manifold M, equipped with a certain Poisson bracket structure defined on $C^\infty(M)$, the space of smooth functions. It is a generalization of the phase space in classical mechanics, and this is the object ready to be quantized. By *deformation quantization*, we mean a process of quantizing the pointwise product on $C^\infty(M)$ to a noncommutative one, in such a way that its first-order approximation is given by the Poisson bracket [1], [24]. If this can be carried out in the C^*-algebra framework, it is called the "strict deformation quantization" [19].

Some time ago, I have developed a method of deforming/quantizing a certain class of non-linear Poisson brackets that may be regarded as a "cocycle perturbation" of the linear Poisson bracket on the dual vector space of a Lie algebra [5]. The method is fairly general, using the framework of "twisted crossed product C^*-algebras". In fact, using this method, I was further able to construct a class of non-compact locally compact quantum groups [6], [7], [9].

Meanwhile, among the more fundamental cases of a Poisson structure is given by the so-called "cotangent bundle" T^*M of a manifold M. In this case, we do not have a Lie algebra but instead a Lie algebroid. Using this observation, Landsman and Ramazan have shown that its (semi-strict) deformation quantization can be carried out using groupoids [15]. Thus a natural problem arises, whether my own result mentioned above can be further extended. It is likely that we will need to introduce a type of a non-linear Poisson structure that can be viewed as a "cocycle perturbation" of a linear Poisson bracket on the dual of a Lie algebroid. And, the quantum object likely will be given by a quantum groupoid. If so, this would provide us with a class of useful examples for our newly-developed theory of locally compact quantum groupoids.

In addition, it may be possible to view some of the known quantum algebras as examples of quantum groupoids. For instance, I am investigating to see if certain algebras arising quantum principal bundles or some non-commutative geometric objects like "quantum teardrops" or "quantum lens space", ... can be considered in this way. If successful, such added perspectives will benefit our attempts at further understanding these interesting objects.

(3). Generalized Fourier transform and duality theory.

Fourier transform is among the more important tools in classical analysis, which has wide applications in sciences and engineering. A few years ago, I studied and produced a paper on the notion of the (generalized) Fourier transform and the inverse Fourier transform, in the setting of locally compact quantum groups [8]. So far, it is only at the primitive stages and is not quite deep technically, but analogues of some classical results were obtained.

In classical and abstract harmonic analysis, the Fourier transform plays a fundamental role in both the theory and the applications, including group representation theory, special function theory, or even signal processing. I hope that this generalized Fourier transform can be developed further so that it can play a similar role in the quantum setting, enriching the general theory of locally compact quantum groups, as well as their applications.

Meanwhile, it turns out that Fourier transform is naturally tied with the Pontryagin-type duality between a quantum group and its dual object, which is again a quantum group. It is technically difficult to analyze duality, so having the generalized Fourier transform is helpful. In addition, the duality picture can be further generalized to the category of quantum groupoids. It is rather too early to tell, but I am interested in exploring to see whether some version of Fourier transform can be considered in this larger class.

II. Future Plans

During the year 2012–2013, I was granted a one-year sabbatical leave from the college, and this opportunity allowed me to begin my research project with Professor Alfons Van Daele, leading to some successful results.

The project is a rather large one, so it is still very much on-going. My three research topics described above are all very closely related. At present, I am giving most effort on topic (1), with some preliminary work on topic (2). I expect to achieve some fruitful results that can be announced in mathematics conferences and published in professional journals. If successful, I believe that the result will be quite helpful in enriching the fields of locally compact quantum groups and quantum groupoids, as well as their applications, including some in mathematical physics.

In addition to being beneficial for my personal advancement, I expect that the research will benefit my department and our students. Some time ago, I had a successful experience guiding student research as a P.I. for the NSF-sponsored, Research Experiences for Undergraduates (R.E.U.) program my department hosted. The experience to be gained through my on-going research could help me as I do similar work in the future for our mathematics major students.

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization I, II*, Ann. Phys. **110** (1978), 61–110, 111–151.

[2] B. Blackadar, *Operator Algebras: Theory of C^*-algebras and von Neumann Algebras, Operator Algebras and Non-commutative Geometry III*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer, 2006.

[3] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[4] M. Enock, *Measured Quantum Groupoid in Action*, Mémoires de la SMF, no. 114, Soc. Math. Fr., 2008.

[5] B. J. Kahng, *Deformation quantization of certain non-linear Poisson structures*, Internat. J. Math. **9** (1998), 599–621.

[6] ———, *Non-compact quantum groups arising from Heisenberg type Lie bialgebras*, J. Operator Theory **44** (2000), 303–334.

[7] ———, *Quantizations of some Poisson–Lie groups: the bicrossed product construction*, J. Geom. Phys. **56** (2006), 485–511.

[8] ———, *Fourier transform on locally compact quantum groups*, J. Operator Theory **64** (2010), 69–87.

[9] ———, *A $(2n+1)$-dimensional quantum group constructed from a skew-symmetric matrix*, J. Geom. Phys. **61** (2011), 2081–2097.

[10] B.J. Kahng and A. Van Daele, *Separability idempotents in C^*-algebras*, preprint.

[11] ———, *The Larson–Sweedler theorem for weak multiplier Hopf algebras*, 2014, preprint (available as arXiv:1406.0299 at http://lanl.arXiv.org).

[12] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids, and Cuntz–Krieger algberas*, J. Funct. Anal. **144** (1997), 505–341.

[13] J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Scient. Éc. Norm. Sup., 4^e série **t. 33** (2000), 837–934.

[14] ———, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand. **92** (2003), no. 1, 68–92.

[15] N.P. Landsman and B. Ramazan, *Quantization of Poisson algberas associated to Lie algebroids*, Contemp. Math. **282** (2001), 159–192.

[16] F. Lesieur, *Measured Quantum Groupoids*, Mémoires de la SMF, no. 109, Soc. Math. Fr., 2007.

[17] T. Masuda, Y. Nakagami, and S. Woronowicz, *A C^*-algebraic framework for quantum groups*, Internat. J. Math. **14** (2003), no. 9, 903–1001.

[18] I. Raeburn, *Graph Algebras*, Amer. Math. Soc., 2005.

[19] M. A. Rieffel, *Deformation quantization of Heisenberg manifolds*, Comm. Math. Phys. **122** (1989), 531–562.

[20] T. Timmermann and A. Van Daele, *Regular multiplier Hopf algebroids. Basic theory and examples*, 2013, preprint (available as arXiv:1307.0769 at http://lanl.arXiv.org).

[21] A. Van Daele, *Locally compact quantum groups. A von Neumann algebra approach*, SIGMA **10** (2014), no. 082.

[22] A. Van Daele and S. Wang, *Weak multiplier Hopf algebras. Preliminaries, motivation and basic examples*, Operator algebras and quantum groups, vol. 98, Banach Center Publications, 2012, pp. 367–415.

[23] ______, *Weak multiplier Hopf algebras. The main theory*, 2012, preprint (available as arXiv:1210.4395 at http://lanl.arXiv.org), to appear in J. Reine Angew. Math. (Crelles Journal).

[24] J. Vey, *Déformation du crochet de Poisson sur une variété symplectique*, Comm. Math. Helv. **50** (1975), 421–454 (French).