CONSTRUCTION OF A C^*-ALGEBRAIC QUANTUM GROUPOID FROM A WEAK MULTIPLIER HOPF ALGEBRA

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ABSTRACT. Van Daele and Wang developed a purely algebraic notion of *weak multiplier Hopf algebras*, which extends the notions of Hopf algebras, multiplier Hopf algebras, and weak Hopf algebras. With an additional requirement of an existence of left or right integrals, this framework provides a self-dual class of algebraic quantum groupoids. The aim of this paper is to show that from this purely algebraic data, with only a minimal additional requirement ("quasi-invariance"), one can construct a C^**-algebraic quantum groupoid of separable type*, recently defined by the author, with Van Daele. The C^*-algebraic quantum groupoid is represented as an operator algebra on the Hilbert space constructed from the left integral, and the comultiplication is determined by means of a certain multiplicative partial isometry W, which is no longer unitary. In the last section (Appendix), we obtain some results in the purely algebraic setting, which have not appeared elsewhere.

0. Introduction

In a series of papers, Van Daele and Wang introduced a purely algebraic notion of *weak multiplier Hopf algebras* [30], [31], [32], [33]. In short, a weak multiplier Hopf algebra is a pair (A, Δ), where A is a non-degenerate idempotent algebra and Δ is a comultiplication, satisfying some number of conditions. They are natural generalizations of Hopf algebras (when A is unital and Δ is non-degenerate), multiplier Hopf algebras (when A is non-unital and Δ is non-degenerate), and weak Hopf algebras (when A is unital, but $\Delta(1) \neq 1 \otimes 1$). For a weak multiplier Hopf algebra, the algebra is not assumed to be unital and the comultiplication is no longer assumed to be non-degenerate.

In [33], Van Daele and Wang considered the situation in which a weak multiplier Hopf algebra possesses (a faithful family of) left/right invariant functionals (or "integrals"). While this is a purely algebraic framework, this category includes all compact and discrete quantum groups, all weak Hopf algebras and finite quantum groupoids. This framework does not contain all quantum groups/groupoids, and some classical groups are left out. Nevertheless, they showed that it is possible to construct a dual object within this category, thereby giving rise to a nice self-dual class of *algebraic quantum groupoids*.

Furthermore, this framework provided a strong motivational basis for a C^*-algebraic framework of *locally compact quantum groupoids of separable type*, by Van Daele and the author [10], [11]. There is a strong resemblance between the purely algebraic framework (= weak multiplier Hopf algebras) and the C^*-framework (= locally compact quantum groupoids of separable

Key words and phrases. Weak Hopf algebra, Weak multiplier Hopf algebra, Separability idempotent, Locally compact quantum groupoid, Multiplier Hopf algebra, Locally compact quantum group.

type). Having said this, it has never been made clear whether there is indeed a direct pathway from the purely algebraic setting to the C^*-setting without too many additional requirements. Naively speaking, this is about constructing a C^*-completion of the algebra. But there are some subtle issues to consider.

The main purpose of this paper is to clarify that we can indeed carry out the construction of a C^*-algebraic locally compact quantum groupoid (in the sense of [10], [11]) out of a weak multiplier Hopf *-algebra equipped with a faithful left (or right) integral. Just as the work of Kustermans and Van Daele of similar nature ([16]), constructing a C^*-algebraic quantum group from a purely algebraic object of a multiplier Hopf algebra, made fundamental contributions to the development of the theory of locally compact quantum groups, we hope that the current work can help us understand better the theory of quantum groupoids.

We describe how the paper is organized: In Section 1, we give an overview of the purely algebraic framework of weak multiplier Hopf algebras. This is needed, not only for motivational purposes but for clarifying the ingredients necessary for the construction of the C^*-algebraic quantum groupoid in what follows. The definition and the properties of weak multiplier Hopf algebras are given in this section, though most of the detailed proofs are skipped (Instead, the relevant theorems elsewhere are referred to.) We do not plan to describe the original definition (as in [31]), however. Instead, we take a more recent but equivalent approach, as in [8]: In that paper, it is shown that a (regular) weak multiplier bialgebra (in the sense of [2]) becomes a weak multiplier Hopf algebra, if it has sufficient number of left and right integrals. Being more recent is one thing, but actually, for our purposes of working with the integrals and studying the duality, this characterization turns out to be more convenient.

One added benefit is that we get to consider the case of a weak multiplier Hopf *-algebra here. While the discussion about the case with an involution has appeared in the original literatures on weak multiplier Hopf algebras and separability idempotents, typically they have appeared in a scattered way.

From the purely algebraic data given above, we construct in Section 2 the "base" C^*-algebras B and C, as well as their multiplier algebras $M(B)$ and $M(C)$. They are essentially the source algebra and the target algebra. They are equipped with certain KMS-weights ν and μ.

The construction of the C^*-algebra A and the comultiplication Δ is carried out in Section 3, whose representation is given in terms of a certain partial isometry W. The construction of the canonical idempotent $E \in M(A \otimes A)$ is also given in this section.

In Section 4, we construct two KMS weights, φ and ψ, on A, satisfying the left invariance condition and the right right invariant condition, respectively. To make things work, we require a certain *quasi-invariance* condition at the *-algebra level. With the existence of a left-invariant weight and a right-invariant weight, we can show that what we have obtained so far fits well into the C^*-algebraic framework developed in [10], [11], as expected.

We chose not to pursue the construction of the dual object here, which should be more or less similar. The reason for not doing this is because the C^*-algebraic framework for the duality of the locally compact quantum groupoids of separable type is still in the works [12]. In that paper, we plan to give a clarification of the duality picture in the C^*-algebraic framework (for a related work, refer also to [7]). We will postpone to a future occasion to verify the expected result that the C^*-algebraic counterpart of the algebraic dual $(\widehat{\mathcal{A}}, \widehat{\Delta})$ is indeed isomorphic to the C^*-algebraic dual of (A, Δ) obtained here.

In Appendix (Section 5), we gathered some results in the purely algebraic framework regarding the modular element δ. Even though this is done in the purely algebraic setting, the author could not find a suitable reference, and some of the results here may be new. As such, all proofs are given for the results in the Appendix section.

Acknowledgments: The author wishes to thank Alfons Van Daele (Leuven), who inspired him to pursue research projects on weak multiplier Hopf algebras and locally compact quantum groupoids. The author is always indebted to his constant support and guidance. The initial seed for the current work arose from talking with Thomas Timmermann (Münster), while both of us were visiting Albert Sheu (Kansas) in 2017, who has always been supportive of the author's projects.

1. The algebraic framework: Weak multiplier Hopf algebra with integrals

1.1. **Preliminaries.** As indicated in Introduction, we will mostly follow the description in [8]. We will only consider (associative) algebras over $\mathbb{C}$. We require that the algebras are non-degenerate, which means that the product on an algebra is non-degenerate as a bilinear form. So $\mathcal{A}$ is a non-degenerate algebra, if $[ab = 0, \forall b \in \mathcal{A}] \Rightarrow [a = 0]$; also $[ba = 0, \forall b \in \mathcal{A}] \Rightarrow [a = 0]$. We also require that the algebras are idempotent, written $\mathcal{A}^2 = \mathcal{A}$, meaning that every element in the algebra can be written as a sum of products of two elements. It is evident that if an algebra $\mathcal{A}$ is unital or has local units, it is automatically non-degenerate and idempotent. Typically, however, we do not expect our algebras to be unital. We will denote by $\mathcal{A}^*$ the dual vector space of $\mathcal{A}$, consisting of the linear functionals on $\mathcal{A}$.

In most cases, we will consider *-algebras, equipped with an involution. For a non-degenerate *-algebra $\mathcal{A}$, we can define its multiplier algebra $M(\mathcal{A})$, which is a unital *-algebra containing $\mathcal{A}$ as an essential self-adjoint ideal. It is the largest such, and is unique. It can be characterized in terms of double centralizers. If $\mathcal{A}$ is unital, then we have $M(\mathcal{A}) = \mathcal{A}$. We can also consider $\mathcal{A} \odot \mathcal{A}$, the algebraic tensor product, and its multiplier algebra $M(\mathcal{A} \odot \mathcal{A})$.

1.2. **Comultiplication.** Let $\mathcal{A}$ be a non-degenerate idempotent *-algebra. By a *comultiplication* on $\mathcal{A}$, we mean a *-homomorphism Δ from $\mathcal{A}$ into $M(\mathcal{A} \odot \mathcal{A})$ such that

$$(\Delta a)(1 \otimes b) \in \mathcal{A} \odot \mathcal{A}, \quad \text{and} \quad (c \otimes 1)(\Delta a) \in \mathcal{A} \odot \mathcal{A}, \quad \text{for all } a, b, c \in \mathcal{A}, \tag{1.1}$$

which also satisfy the following "weak coassociativity" condition:

$$(c \otimes 1 \otimes 1)(\Delta \otimes \mathrm{id})\big((\Delta a)(1 \otimes b)\big) = (\mathrm{id} \otimes \Delta)\big((c \otimes 1)(\Delta a)\big)(1 \otimes 1 \otimes b), \quad \text{for } a, b, c \in \mathcal{A}. \tag{1.2}$$

Note that condition (1.1) is needed to formulate the weak coassociativity (1.2).

As $\mathcal{A}$ is a *-algebra and Δ is a *-homomorphism, it automatically follows from condition (1.1) that we also have:

$$(\Delta a)(c \otimes 1) \in \mathcal{A} \odot \mathcal{A}, \quad \text{and} \quad (1 \otimes b)(\Delta a) \in \mathcal{A} \odot \mathcal{A}, \quad \text{for all } a, b, c \in \mathcal{A}. \tag{1.3}$$

There is also another version of the weak coassociativity, as follows:

$$(\Delta \otimes \mathrm{id})\big((1 \otimes b)(\Delta a)\big)(c \otimes 1 \otimes 1) = (1 \otimes 1 \otimes b)(\mathrm{id} \otimes \Delta)\big((\Delta a)(c \otimes 1)\big), \quad \text{for } a, b, c \in \mathcal{A}. \tag{1.4}$$

Remark. Having $(\mathcal{A}, \Delta)$ further satisfy (1.3) and (1.4) means that our comultiplication is "regular", in the sense of [31] (see Definition 1.1 of that paper).

The comultiplication is also assumed to be "full". This means that the left and the right legs of $\Delta(\mathcal{A})$ are all of $\mathcal{A}$ (see [30], [31], also see [2]). A way to characterize the fullness of Δ is as follows:

$$\mathrm{span}\big\{(\mathrm{id} \otimes \omega)((\Delta a)(1 \otimes b)) : a, b \in \mathcal{A}, \omega \in \mathcal{A}^*\big\} = \mathcal{A},$$
$$\mathrm{span}\big\{(\omega \otimes \mathrm{id})((b \otimes 1)(\Delta a)) : a, b \in \mathcal{A}, \omega \in \mathcal{A}^*\big\} = \mathcal{A}.$$

For $(\mathcal{A}, \Delta)$, we have the following result:

Lemma 1.1. *Suppose there exists a self-adjoint idempotent element $E \in M(\mathcal{A} \odot \mathcal{A})$ such that*

$$\Delta(\mathcal{A})(\mathcal{A} \odot \mathcal{A}) = E(\mathcal{A} \odot \mathcal{A}), \qquad \textit{and} \qquad (\mathcal{A} \odot \mathcal{A})\Delta(\mathcal{A}) = (\mathcal{A} \odot \mathcal{A})E.$$

Then this idempotent is unique. It is the smallest idempotent $E \in M(\mathcal{A} \odot \mathcal{A})$ satisfying

$$E(\Delta a) = \Delta a, \quad (\Delta a)E = \Delta a, \quad \forall a \in \mathcal{A}.$$

Proof. See Lemmas 3.3 and 3.5 of [30].

□

The requirement to have E self-adjoint is natural, because we are working with *-algebras. If such an idempotent E exists (called the *canonical idempotent*), then it can be shown that Δ has a unique extension to a *-homomorphism $\widetilde{\Delta} : M(\mathcal{A}) \to M(\mathcal{A} \odot \mathcal{A})$, such that $\widetilde{\Delta}(m) = E\widetilde{\Delta}(m) = \widetilde{\Delta}(m)E$, for all $m \in M(\mathcal{A})$. For proof of this result and more details, see Appendix of [30]. For convenience, we will just denote the extension map also by Δ.

As above, we can make sense of the maps $\Delta \otimes \mathrm{id}$ and $\mathrm{id} \otimes \Delta$ as *-homomorphisms naturally extended to $M(\mathcal{A} \odot \mathcal{A})$, such that

$$(\Delta \otimes \mathrm{id})(m) = (E \otimes 1)\big((\Delta \otimes \mathrm{id})(m)\big) = \big((\Delta \otimes \mathrm{id})(m)\big)(E \otimes 1), \quad \forall m \in M(\mathcal{A} \odot \mathcal{A}),$$

and similarly for $\mathrm{id} \otimes \Delta$. These results mean that when extended to the multiplier algebra level, we have $\Delta(1) = E$, and that $(\Delta \otimes \mathrm{id})(1 \otimes 1) = E \otimes 1$ and $(\mathrm{id} \otimes \Delta)(1 \otimes 1) = 1 \otimes E$.

As a consequence of the weak coassociativity, namely Equations (1.2) and (1.4), now knowing that the maps $\Delta \otimes \mathrm{id}$ and $\mathrm{id} \otimes \Delta$ are extended, we obtain the following *coassociativity* property:

$$(\Delta \otimes \mathrm{id})(\Delta a) = (\mathrm{id} \otimes \Delta)(\Delta a), \quad \forall a \in \mathcal{A}. \tag{1.5}$$

See again Appendix of [30].

The existence of the canonical idempotent $E \in M(\mathcal{A} \odot \mathcal{A})$ as above is referred to as Δ being *weakly non-degenerate*. Note that when $E = 1 \otimes 1$, we would indeed have the non-degeneracy of the comultiplication.

We will require one more condition on Δ, which is also a part of the axioms for a weak multiplier Hopf algebra (see Definition 1.14 of [31]). The following condition is referred to as the *weak comultiplicativity of the unit*:

$$(\mathrm{id}\otimes\Delta)(E) = (E\otimes 1)(1\otimes E) = (1\otimes E)(E\otimes 1). \tag{1.6}$$

This condition already appeared in the theory of weak Hopf algebras (see Definition 2.1 in [3]).

1.3. **Separability idempotent.** The canonical idempotent is further required to be a *separability idempotent*, in the sense of [28]. See also Appendix B of [8]. In the below is a short summary.

Suppose $\mathcal{B}$ and $\mathcal{C}$ are non-degenerate *-algebras. Consider a self-adjoint idempotent $E \in M(\mathcal{B}\odot\mathcal{C})$ such that

$$E(1\otimes c)\in\mathcal{B}\odot\mathcal{C},\ (b\otimes 1)E\in\mathcal{B}\odot\mathcal{C},\quad \forall b\in\mathcal{B},\forall c\in\mathcal{C}.$$

As E is self-adjoint, we also have $(1\otimes c)E\in\mathcal{B}\odot\mathcal{C}$, $E(b\otimes 1)\in\mathcal{B}\odot\mathcal{C}$, for $b\in\mathcal{B}, c\in\mathcal{C}$. Assume also that E is "full", which means that its left and the right legs are all of $\mathcal{B}$ and $\mathcal{C}$, respectively. Or equivalently, we have

$$\operatorname{span}\bigl\{(\mathrm{id}\otimes\omega)(E(1\otimes c)) : c\in\mathcal{C},\omega\in\mathcal{C}^*\bigr\} = \mathcal{B},$$

$$\operatorname{span}\bigl\{(\omega\otimes\mathrm{id})((b\otimes 1)E) : b\in\mathcal{B},\omega\in\mathcal{B}^*\bigr\} = \mathcal{C}.$$

Then, it can be shown that $E\in M(\mathcal{B}\odot\mathcal{C})$ automatically becomes a *regular* separability idempotent, which means that we have:

$$E(\mathcal{B}\odot 1) = E(1\odot\mathcal{C}) \quad\text{and}\quad (\mathcal{B}\odot 1)E = (1\odot\mathcal{C})E. \tag{1.7}$$

See Proposition 3.7 of [28]. The regularity condition implies $\mathcal{B}$ and $\mathcal{C}$ have local units (see [28], v1). The other aspect of the regularity of E is that there exist two anti-isomorphisms $S_{\mathcal{B}}:\mathcal{B}\to\mathcal{C}$ and $S_{\mathcal{C}}:\mathcal{C}\to\mathcal{B}$, characterized by

$$E(b\otimes 1) = E\bigl(1\otimes S_{\mathcal{B}}(b)\bigr),\quad b\in\mathcal{B}, \tag{1.8}$$

$$(1\otimes c)E = \bigl(S_{\mathcal{C}}(c)\otimes 1\bigr)E,\quad c\in\mathcal{C}. \tag{1.9}$$

It can be also shown that $(S_{\mathcal{B}}\otimes S_{\mathcal{C}})(E) = \varsigma E$, where ς is the flip map between $\mathcal{B}\odot\mathcal{C}$ and $\mathcal{C}\odot\mathcal{B}$. The maps $S_{\mathcal{B}}$ and $S_{\mathcal{C}}$ are in general not involutive, but they satisfy the following relations:

$$S_{\mathcal{C}}\bigl(S_{\mathcal{B}}(b)^*\bigr)^* = b \quad\text{and}\quad S_{\mathcal{B}}\bigl(S_{\mathcal{C}}(c)^*\bigr)^* = c,\quad \forall b\in\mathcal{B},\forall c\in\mathcal{C}. \tag{1.10}$$

The existence of such a self-adjoint separability idempotent element $E\in M(\mathcal{B}\odot\mathcal{C})$ restricts the possible structure of the algebras $\mathcal{B}$ and $\mathcal{C}$, which are typically direct sums of finite-dimensional matrix algebras. We will not go too deep into this discussion here. See section 4 of [28].

By the general theory of regular separability idempotents [28], we have the existence of the following *distinguished linear functionals*, namely ν on $\mathcal{B}$ and μ on $\mathcal{C}$, such that

$$(\nu\otimes\mathrm{id})(E) = 1, \quad\text{and}\quad (\mathrm{id}\otimes\mu)(E) = 1.$$

[The notations used in [28] for the distinguished linear functionals are actually $\varphi_{\mathcal{B}}$ and $\varphi_{\mathcal{C}}$, but we are denoting them as ν and μ here, mainly to avoid a future confusion with the total linear functional φ on $\mathcal{A}$.]

The distinguished functionals ν and μ are uniquely determined and are faithful. Being a faithful functional means that we have $[\nu(bk) = 0, \forall k \in \mathcal{B}] \Rightarrow [b = 0]$, and the like. Meanwhile, with the algebras $\mathcal{B}$ and $\mathcal{C}$ being *-algebras, and E being self-adjoint, it also turns out that ν and μ become positive linear functionals (see section 4 of [28]).

The general theory also tells us that the functional ν is equipped with a KMS-type automorphism σ^ν on $\mathcal{B}$, such that $\nu \circ \sigma^\nu = \nu$ and

$$\nu(bb') = \nu\big(b'\sigma^\nu(b)\big), \quad \forall b, b' \in \mathcal{B},$$

given by $\sigma^\nu = {S_{\mathcal{B}}}^{-1} \circ {S_{\mathcal{C}}}^{-1}$. As for μ also, there is a KMS-type automorphism σ^μ on $\mathcal{C}$, such that $\mu \circ \sigma^\mu = \mu$ and

$$\mu(cc') = \mu\big(c'\sigma^\mu(c)\big), \quad \forall c, c' \in \mathcal{C},$$

given by $\sigma^\mu = S_{\mathcal{B}} \circ S_{\mathcal{C}}$. The existence of such automorphisms is referred to as the *weak KMS-property*. See Proposition 2.8 of [28] for more details.

In addition, we can show that

$$\mu = \nu \circ S_{\mathcal{C}} \quad \text{and} \quad \nu = \mu \circ S_{\mathcal{B}},$$

by using the result $(S_{\mathcal{B}} \otimes S_{\mathcal{C}})E = \varsigma E$ and the uniqueness of the functionals.

Here is a Radon–Nikodym type theorem on the functionals on the algebras $\mathcal{B}$ and $\mathcal{C}$:

Proposition 1.2. (1) *Consider the distinguished linear functional ν on $\mathcal{B}$. For any other linear functional g on $\mathcal{B}$, there is a unique element $y \in M(\mathcal{B})$ such that $g(b) = \nu(by)$ for all $b \in \mathcal{B}$. Moreover, the functional g is faithful if and only if y is invertible in $M(\mathcal{B})$.*
(2) *Similar results hold true for linear functionals on $\mathcal{C}$.*

Proof. This is a consequence of the regularity property of E. See Proposition 1.2 of [29], which is actually more general than the result given here. □

As a consequence of Proposition 1.2 above, we can show that any faithful linear functional on $\mathcal{B}$ (or on $\mathcal{C}$) is equipped with a KMS-type automorphism:

Proposition 1.3. (1) *Any faithful linear functional on $\mathcal{B}$ has the weak KMS property.*
(2) *Any faithful linear functional on $\mathcal{C}$ has the weak KMS property.*

Proof. See Proposition 1.3 of [29]. The case for $\mathcal{B}$ is below. The case for $\mathcal{C}$ is similar.

If f is a faithful linear functional on $\mathcal{B}$, then by Proposition 1.2 we can write it as $f(\ ) : b \mapsto \nu(by)$, where y is invertible in $M(\mathcal{B})$. Then for any $b, b' \in \mathcal{B}$, we have:

$$f(bb') = \nu(bb'y) = \nu\big(b'y\sigma^\nu(b)\big) = f\big(b'y\sigma^\nu(b)y^{-1}\big).$$

This shows that $b \mapsto y\sigma^\nu(b)y^{-1}$ determines a KMS-type automorphism for f. □

1.4. **Algebraic quantum groupoid.** Let $\mathcal{A}$ be a non-degenerate idempotent *-algebra, and Δ a regular, full comultiplication that is weakly non-degenerate (See §1.2). This means the existence of a unique canonical idempotent element $E \in M(\mathcal{A} \odot \mathcal{A})$, which is also assumed to satisfy the weak comultiplicativity of the unit, or Equation (1.6).

We will further assume that E is a regular, full separability idempotent (See §1.3), such that there exist two *-subalgebras $\mathcal{B}$ and $\mathcal{C}$ of $M(\mathcal{A})$ sitting in a non-degenerate way, and we have $E \in M(\mathcal{B} \odot \mathcal{C})$. When we say the subalgebras $\mathcal{B}$ and $\mathcal{C}$ sit in a non-degenerate way inside $M(\mathcal{A})$, it means that $\mathcal{B}\mathcal{A} = \mathcal{A}\mathcal{B} = \mathcal{A}$, and $\mathcal{C}\mathcal{A} = \mathcal{A}\mathcal{C} = \mathcal{A}$. Then it is easy to see that $\mathcal{B}$ and $\mathcal{C}$ are non-generate algebras, and that $M(\mathcal{B})$ and $M(\mathcal{C})$ can be regarded as subalgebras in $M(\mathcal{A})$ as well.

Remark. It turns out that the subalgebras $\mathcal{B}$ and $\mathcal{C}$ are completely determined by the conditions given above. See Proposition 3.1 of [8]. As such, it would be all right to just say that "E is a regular separability idempotent", without having to specify $\mathcal{B}$ and $\mathcal{C}$ explicitly.

On $M(\mathcal{B})$ and $M(\mathcal{C})$, regarded as subalgebras in $M(\mathcal{A})$, it turns out that the comultiplication acts the following way (which tuns out to characterize the subalgebras):

Proposition 1.4. *If $x \in M(\mathcal{B})$, then we have $\Delta x = E(1 \otimes x) = (1 \otimes x)E$.*

If $y \in M(\mathcal{C})$, then we have $\Delta y = (y \otimes 1)E = E(y \otimes 1)$.

Proof. See Proposition 3.4 in [8]. □

In this setting, let us next give the definition for left and right integrals. See Definition 3.5 of [8]. To be precise, a linear functional φ on $\mathcal{A}$ is said to be *left invariant*, if

$$(\mathrm{id} \otimes \varphi)(\Delta a) \in M(\mathcal{C}), \quad \forall a \in \mathcal{A}.$$

Similarly, a linear functional ψ on A is said to be *right invariant*, if

$$(\psi \otimes \mathrm{id})(\Delta a) \in M(\mathcal{B}), \quad \forall a \in \mathcal{A}.$$

Any non-zero left-invariant linear functional on $\mathcal{A}$ is called a *left integral*, and any non-zero right-invariant linear functional on $\mathcal{A}$ is called a *right integral*.

Also, as we are working in the *-algebra setting, it seems reasonable to further require that the integrals are positive linear functionals. (It may be possible to prove the positivity from the *-structure and the self-adjointness of E, but at present that is not known.)

Here are some additional consequences of the left/right invariance of φ and ψ:

Proposition 1.5. *Denote*

$$F_1 = (\mathrm{id} \otimes S)(E), \quad F_2 = (S \otimes \mathrm{id})(E), \quad F_3 = (\mathrm{id} \otimes S^{-1})(E), \quad F_4 = (S^{-1} \otimes \mathrm{id})(E).$$

Then if φ is a left integral and if ψ is a right integral, we have:

$$(\mathrm{id} \otimes \varphi)(\Delta a) = (\mathrm{id} \otimes \varphi)\big(F_2(1 \otimes a)\big) = (\mathrm{id} \otimes \varphi)\big((1 \otimes a)F_4\big),$$

$$(\psi \otimes \mathrm{id})(\Delta a) = (\psi \otimes \mathrm{id})\big((a \otimes 1)F_1\big) = (\psi \otimes \mathrm{id})\big(F_3(a \otimes 1)\big),$$

for all $a \in \mathcal{A}$.

Proof. See Proposition 1.4 of [33] and Proposition 3.7 of [8]. □

In general, while we can have a faithful left integral and a faithful right integral on $\mathcal{A}$, it may be possible to have neither. Instead, it may be possible to have a faithful set of left integrals $\{\varphi_\alpha\}$, in the sense that if $a \in \mathcal{A}$ is such that $\varphi_\alpha(ax) = 0$ for all $x \in \mathcal{A}$ and for all left integrals ϕ_α, then we must have $a = 0$. Similarly, if $\varphi_\alpha(xa) = 0$ for all $x \in \mathcal{A}$ and for all ϕ_α, then we must have $a = 0$. We can make sense of a faithful set of right integrals in a similar way.

The main result of [8] is that given data $(\mathcal{A}, \Delta, E)$ as above, if there is a faithful set of left integrals and a faithful set of right integrals, then we have a regular weak multiplier Hopf algebra, in the sense of [31]. In particular, the existence of the counit, ε, and the antipode, S, can be proved. It then becomes equivalent to the situation of a regular weak multiplier Hopf algebra equipped with a faithful set of left integrals, whose right integrals can be obtained using the antipode map. In [33], [29], regular weak Hopf algebras with a faithful set of integrals is referred to as *algebraic quantum groupoids*. It is shown there that they form a self-dual category. Note, however, that this notion is different from Timmermann's notion of an algebraic quantum groupoid (see [23], [22]), which is based on the framework of multiplier Hopf algebroids. Some discussion on the relationship between these two frameworks can be found in [24].

1.5. **Weak multiplier Hopf *-algebra with a single faithful integral.** According to the general theory on weak multiplier Hopf algebras, the existence of a faithful family of (left) integrals is required for the duality picture to be complete. See [33]. Unlike in the case of multiplier Hopf algebras (see [26], [27]), having a left or right integral does not necessarily mean that it is also faithful. There are known examples of weak (multiplier) Hopf algebras where enough integrals exist to form a faithful family, but not a single faithful one [6].

Having said this, in many examples there exists a single faithful integral. In particular, it has been observed that for finite-dimensional weak Hopf algebras, a single faithful integral exists if and only if the underlying algebra is a Frobenius algebra, for instance a finite-dimensional C^*-algebra (See Theorem 3.16 in [3].). Infinite-dimensional case is not fully understood. Nonetheless, considering that our aim is to eventually construct a C^*-algebraic version, this observation seems to suggest that it may not be too restrictive to require the existence of a single faithful (positive) integral. We will do so here:

Definition 1.6. Let $(\mathcal{A}, \Delta, E)$ be as above, and assume that there exists a single positive faithful left integral φ and a single positive faithful right integral ψ. We will call (A, Δ) a *weak multiplier Hopf *-algebra with a faithful integral*.

As noted earlier, having the *-structure means this is a regular weak multiplier Hopf algebra. Meanwhile, having a single faithful left integral and a single faithful right integral just means that we have one-element families of left/right integrals, so the main results outlined in the previous subsection still hold, including the existence of the counit and the antipode.

Here are some additional results before we end this subsection, which we will use down the road.

Proposition 1.7 below gives a relationship between the integrals φ, ψ and the expressions $(\psi\otimes\mathrm{id})(\Delta x)\in M(\mathcal{B})$ and $(\mathrm{id}\otimes\varphi)(\Delta x)\in M(\mathcal{C})$, for $x\in\mathcal{A}$. The proof is fundamentally no different than the one given in Proposition 4.9 of [10], but now done at the *-algebra level.

Proposition 1.7. *We have:*

- $\nu\big((\psi\otimes\mathrm{id})(\Delta x)\big)=\psi(x)$, *for* $x\in\mathcal{A}$.
- $\mu\big((\mathrm{id}\otimes\varphi)(\Delta x)\big)=\varphi(x)$, *for* $x\in\mathcal{A}$.

Proof. Let $a\in\mathcal{A}$. Then by the right invariance of ψ, we know $(\psi\otimes\mathrm{id})(\Delta a)\in M(\mathcal{B})$. Apply here Δ. On the one hand, we have:

$$\Delta\big((\psi\otimes\mathrm{id})(\Delta a)\big)=(\psi\otimes\mathrm{id}\otimes\mathrm{id})\big((\mathrm{id}\otimes\Delta)(\Delta a)\big)=(\psi\otimes\mathrm{id}\otimes\mathrm{id})\big((\Delta\otimes\mathrm{id})(\Delta a)\big),$$

where we used the coassociativity of Δ. Meanwhile, by Proposition 1.4, we have:

$$\Delta\big((\psi\otimes\mathrm{id})(\Delta a)\big)=E\big(1\otimes(\psi\otimes\mathrm{id})(\Delta a)\big)=(\psi\otimes\mathrm{id}\otimes\mathrm{id})\big((1\otimes E)\Delta_{13}(a)\big).$$

It thus follows that

$$(\psi\otimes\mathrm{id}\otimes\mathrm{id})\big((\Delta\otimes\mathrm{id})(\Delta a)\big)=(\psi\otimes\mathrm{id}\otimes\mathrm{id})\big((1\otimes E)\Delta_{13}(a)\big).\tag{1.11}$$

Let $y=\tilde{y}c$, where $\tilde{y}\in\mathcal{A}$ and $c\in\mathcal{C}$. Recall that $\mathcal{AC}=\mathcal{A}$, so such elements span $\mathcal{A}$. Multiply $1\otimes y=1\otimes\tilde{y}c$ to both sides of Equation (1.11), from left. Then the equation becomes:

$$(\psi\otimes\mathrm{id}\otimes\mathrm{id})\big((\Delta\otimes\mathrm{id})((1\otimes\tilde{y}c)(\Delta a))\big)=(\psi\otimes\mathrm{id}\otimes\mathrm{id})\big((1\otimes1\otimes\tilde{y}c)(1\otimes E)\Delta_{13}(a)\big).$$

Let $\omega\in\mathcal{A}^*$, and apply $\mathrm{id}\otimes\omega$ to the equation above. Then it becomes:

$$\begin{aligned}&(\psi\otimes\mathrm{id})\big(\Delta((\mathrm{id}\otimes\omega)[(1\otimes\tilde{y}c)(\Delta a)])\big)\\&=(\psi\otimes\mathrm{id})\big((\mathrm{id}\otimes\mathrm{id}\otimes\omega)[(1\otimes1\otimes\tilde{y})(1\otimes1\otimes c)(1\otimes E)\Delta_{13}(a)]\big).\end{aligned}$$

Apply ν to both sides. By the property of ν, we know $(\nu\otimes\mathrm{id})(E)=1$. So we have:

$$\begin{aligned}&\nu\big((\psi\otimes\mathrm{id})(\Delta((\mathrm{id}\otimes\omega)[(1\otimes\tilde{y}c)(\Delta a)]))\big)\\&=\psi\big((\mathrm{id}\otimes\omega)[(1\otimes\tilde{y})(1\otimes(\nu\otimes\mathrm{id})((1\otimes c)E))(\Delta a)]\big)\\&=\psi\big((\mathrm{id}\otimes\omega)[(1\otimes\tilde{y}c)(\Delta a)]\big).\end{aligned}\tag{1.12}$$

For convenience, write: $x=(\mathrm{id}\otimes\omega)[(1\otimes\tilde{y}c)(\Delta a)]$. By the "fullness" of the comultiplication, such elements span all of $\mathcal{A}$. Then Equation (1.12) can be expressed as $\nu\big((\psi\otimes\mathrm{id})(\Delta x)\big)=\psi(x)$, which would be true for all $x\in\mathcal{A}$.

Similarly, we can show that for any $x\in\mathcal{A}$, we have $\mu\big((\mathrm{id}\otimes\varphi)(\Delta x)\big)=\varphi(x)$. □

The next result provides a characterization of the antipode map:

Proposition 1.8. (1) *We have:* $\mathrm{span}\big\{(\mathrm{id}\otimes\varphi)((\Delta a)(1\otimes b)):a,b\in\mathcal{A}\big\}=\mathcal{A}$.

(2) *The antipode map, S, is anti-multiplicative and bijective on A, and can be characterized as follows:*

$$S:(\mathrm{id}\otimes\varphi)\big((\Delta a)(1\otimes b)\big)\mapsto(\mathrm{id}\otimes\varphi)\big((1\otimes a)(\Delta b)\big),\quad\forall a,b\in\mathcal{A}.$$

(3) *Similarly, the antipode S can be also characterized as follows:*

$$S:(\psi\otimes\mathrm{id})\big((a\otimes1)(\Delta b)\big)\mapsto(\psi\otimes\mathrm{id})\big((\Delta a)(b\otimes1)\big),\quad\forall a,b\in\mathcal{A}.$$

(4) *The antipode S of $(\mathcal{A},\Delta)$ extended to the multiplier algebra $M(\mathcal{A})$ coincides with the maps $S_{\mathcal{B}}$ on $\mathcal{B}$ and $S_{\mathcal{C}}$ on $\mathcal{C}$.*
(5) *We have: $\Delta\big(S(x)\big)=(S\otimes S)\big(\Delta^{\rm cop}(x)\big)=(S\otimes S)\big(\varsigma(\Delta x)\big)$, for $x\in\mathcal{A}$, where ς is the flip map on $M(\mathcal{A}\odot\mathcal{A})$.*
(6) *With respect to the *-structure, we have: $S\big(S(x)^*\big)^*=x$, for $x\in\mathcal{A}$.*

Proof. See Propositions 3.16 and 3.18 of [8], and see also Proposition 1.5 of [33]. For more results on the antipode map on a regular multiplier Hopf algebra, see Section 4 of [31]. □

Even though a characterization for the antipode, S, is given above in terms of the left integral φ, it is actually intrinsic, in the sense that it is independent of any particular choice of the integrals φ and ψ. Once the antipode map is clarified, as $\varphi\circ S$ naturally determines a faithful right integral, it becomes convenient to take $\psi=\varphi\circ S$. We will do that here.

Remark. For a weak multiplier Hopf algebra, unlike the case of a multiplier Hopf algebra, there is no uniqueness result for the right integrals. The functional ψ we used to construct S and the functional $\varphi\circ S$ do not necessarily have to agree. However, the underlying algebra and its structure maps remain the same, including the antipode, so taking $\psi=\varphi\circ S$ will not fundamentally change anything.

The following result is about the existence of a KMS-type automorphism for the left integral φ, which will be referred to as the *modular automorphism for φ*.

Proposition 1.9. *There exists an automorphism σ of $\mathcal{A}$ such that*
$$\varphi(ab)=\varphi\big(b\sigma(a)\big),\quad \forall a,b\in\mathcal{A}.$$
We also have $\varphi\circ\sigma=\varphi$.

Also for the right integral $\psi=\varphi\circ S$, there exists an automorphism σ' of $\mathcal{A}$ such that $\psi\circ\sigma'=\psi$ and that
$$\psi(ab)=\psi\big(b\sigma'(a)\big),\quad \forall a,b\in\mathcal{A}.$$
It is easy to see $\sigma'=S^{-1}\circ\sigma^{-1}\circ S$.

Proof. See Proposition 1.7 of [33] for the existence of σ. Since $\psi=\varphi\circ S$, it is straightforward to verify that $\sigma'=S\circ\sigma^{-1}\circ S^{-1}$ will give us the modular automorphism for ψ. Note that for any $a,b\in\mathcal{A}$, we have:
$$\begin{aligned}\psi(ab)&=\varphi\big(S(ab)\big)=\varphi\big(S(b)S(a)\big)\\&=\varphi\big(\sigma^{-1}(S(a))S(b)\big)=\varphi\big(S(b[S^{-1}(\sigma^{-1}(S(a)))])\big)\\&=\psi\big(b(S^{-1}\circ\sigma^{-1}\circ S)(a)\big).\end{aligned}$$
□

If there is any other left integral, φ_1, then it turns out that it can be written in the form $\varphi_1(\,\cdot\,)=\varphi(\,\cdot\,y)$, for some $y\in M(\mathcal{B})$. Conversely, if $y\in M(\mathcal{B})$, then $\varphi(\,\cdot\,y)$ as well as $\varphi(y\,\cdot\,)$, are left invariant functionals. As a consequence, we can see that σ leaves $M(\mathcal{B})$ invariant. For more details, see Proposition 1.8 of [33] and the paragraph following it.

By a similar argument used to prove the above result, we obtain the following, relating our faithful right integral ψ in terms of the faithful left integral φ:

Proposition 1.10. *There exists an invertible element $\delta \in M(\mathcal{A})$ such that $\psi(x) = \varphi(x\delta)$ for all $x \in \mathcal{A}$.*

Proof. See Proposition 1.9 of [33]. □

Remark. We call δ the *modular element*. By the faithfulness of φ, it is evident that δ is uniquely determined. In fact, a result like above holds true for any right integral ψ_1, which may not necessarily be faithful. If ψ_1 is not faithful, then the corresponding element δ_1 is not invertible. It is an open question whether it is possible to make sense of a modular element in the general case when there is no single faithful left integral φ but a faithful family. We will not pursue this question here, however, as we are requiring the existence of a single faithful integral.

In later sections, the modular element δ will play a central role. So some more facts on the modular element are gathered in Appendix (Section 5).

1.6. **The dual algebra.** Let $(\mathcal{A}, \Delta)$ be a (regular) weak multiplier Hopf *-algebra, with its counit ε and the antipode S, as in [31]. Suppose there exists a faithful left integral φ. Then by the general theory, we can construct its dual object. For details, see Section 2 of [33]. First, we consider $\widehat{\mathcal{A}}$, the space of linear functionals on $\mathcal{A}$ spanned by the elements of the form $\varphi(\,\cdot\, a)$, for $a \in \mathcal{A}$. It can be given a weak multiplier Hopf *-algebra structure, as follows.

For $\omega, \omega' \in \widehat{\mathcal{A}}$ and $x \in \mathcal{A}$, define the multiplication $\omega\omega' \in \widehat{\mathcal{A}}$ by

$$(\omega\omega')(x) := (\omega \otimes \omega')(\Delta x).$$

For $\omega \in \widehat{\mathcal{A}}$ and $x \in \mathcal{A}$, define the involution on $\widehat{\mathcal{A}}$ by $\omega^*(x) := \overline{\omega\big(S(x)^*\big)}$. One can show that $\widehat{\mathcal{A}}$ becomes a non-degenerate idempotent *-algebra.

As for the comultiplication, we define $\widehat{\Delta}$ on $\widehat{\mathcal{A}}$ in such a way that

$$\widehat{\Delta}(\omega)(x \otimes y) = \omega(xy), \quad \text{for } \omega \in \widehat{\mathcal{A}},\ x, y \in \mathcal{A}.$$

It becomes a full coassociative comultiplication. However, making sense of this needs some care, as we need to consider $M(\widehat{\mathcal{A}} \odot \widehat{\mathcal{A}})$ inside the dual space $(\mathcal{A} \odot \mathcal{A})^*$ in a proper way. See Propositions 2.7, 2.8, 2.9 of [33].

The antipode map, $\widehat{S} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$, is given by

$$\widehat{S}(\omega)(x) = \omega\big(S(x)\big), \quad \text{for } \omega \in \widehat{\mathcal{A}},\ x \in \mathcal{A}.$$

Meanwhile, the canonical idempotent $\widehat{E} \in M(\mathcal{A} \odot \mathcal{A})$ should be such that $\widehat{E} = \widehat{\Delta}\big(1_{M(\widehat{A})}\big)$, so it is defined by

$$\widehat{E}(x \otimes y) = \varepsilon(xy), \quad \text{for } x, y \in \mathcal{A},$$

where ε is the counit. It can be shown that these structure maps make $(\widehat{\mathcal{A}}, \widehat{\Delta})$ a regular weak multiplier Hopf *-algebra. See Theorem 2.15 of [33].

Finally, as we are assuming that $(\mathcal{A}, \Delta)$ is equipped with a single faithful integral, it can be shown that $(\widehat{\mathcal{A}}, \widehat{\Delta})$ is also equipped with a single faithful integral. (See Theorem 2.21 of [33].)

We will skip the details, but if φ is a faithful left integral for $(\mathcal{A}, \Delta)$ and $\omega = \varphi(\,\cdot\, c) \in \widehat{\mathcal{A}}$, $c \in \mathcal{A}$, then one can consider the functional $\widehat{\psi}$ on $\widehat{\mathcal{A}}$, such that $\widehat{\psi}(\omega) := \varepsilon(c)$. It can be shown that such a functional $\widehat{\psi}$ becomes a faithful right integral on $(\widehat{\mathcal{A}}, \widehat{\Delta})$, from which we can obtain a faithful left integral $\widehat{\varphi} = \widehat{\psi} \circ \widehat{S}^{-1}$.

To summarize, from a regular multiplier Hopf algebra $(\mathcal{A}, \Delta)$ with a faithful left integral φ, one can construct its dual $(\widehat{\mathcal{A}}, \widehat{\Delta})$, which is also a regular multiplier Hopf algebra, with a faithful left integral $\widehat{\varphi}$. A main theorem is that if we consider the dual of $(\widehat{\mathcal{A}}, \widehat{\Delta})$, then the resulting object is canonically isomorphic to the original $(\mathcal{A}, \Delta)$, which is a generalized Pontryagin duality. See Section 2 of [33].

2. The base C^*-algebras

From this point on, we are going to systematically construct a *C^*-algebraic quantum groupoid of separable type* in the sense of [10], [11], out of our purely algebraic object of a weak multiplier Hopf *-algebra $(\mathcal{A}, \Delta)$ equipped with a faithful left integral φ. We will use the notations and properties summarized in Section 1, including the subalgebras $\mathcal{B}$, $\mathcal{C}$, the distinguished linear functionals ν and μ on them, the canonical idempotent E, and the antipode S.

2.1. **Construction of the C^*-algebras B and C.** We will begin with the construction of the *base* C^*-algebras B and C. This will be done by completing the algebras $\mathcal{B}$ and $\mathcal{C}$ in an appropriate sense.

First, recall (from Section 1.3) that there exists a distinguished linear functional ν on the *-algebra $\mathcal{B}$, which is positive and faithful. Using ν, we can provide $\mathcal{B}$ with an inner product, as follows:
$$\langle x_1, x_2\rangle := \nu(x_2^* x_1), \qquad \text{for } x_1, x_2 \in \mathcal{B}.$$
Form the completion of $\mathcal{B}$ with respect to the induced norm, and obtain a Hilbert space $\mathcal{H}_B$ with the natural inclusion $\Lambda_B : \mathcal{B} \to \mathcal{H}_B$.

In a similar way, using the distinguished linear functional μ on the *-algebra $\mathcal{C}$, we can equip $\mathcal{C}$ with an inner product, by $\langle y_1, y_2\rangle := \mu(y_2^* y_1)$, for $y_1, y_2 \in \mathcal{C}$. As above, we obtain a Hilbert space $\mathcal{H}_C$ with the natural inclusion $\Lambda_C : \mathcal{C} \to \mathcal{H}_C$.

Between the Hilbert spaces $\mathcal{H}_B$ and $\mathcal{H}_C$, there exists a unitary map given by the anti-homomorphism $S_{\mathcal{B}}$. More precisely, note that for any $b_1, b_2 \in \mathcal{B}$, we have:
$$\begin{aligned}\mu\big(S_{\mathcal{B}}(b_1)^* S_{\mathcal{B}}(b_2)\big) &= \mu\big(S_{\mathcal{C}}^{-1}(b_1^*) S_{\mathcal{B}}(b_2)\big) = \mu\big(S_{\mathcal{B}}(b_2) S_{\mathcal{B}}(b_1^*)\big) \\ &= (\mu \circ S_{\mathcal{B}})(b_1^* b_2) = \nu(b_1^* b_2),\end{aligned}$$
because $S_{\mathcal{C}}\big(S_{\mathcal{B}}(b_1)^*\big)^* = b_1$ and $\sigma_{\mathcal{C}} = S_{\mathcal{B}} \circ S_{\mathcal{C}}$, while $\mu \circ S_{\mathcal{B}} = \nu$. This shows that $S_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ provides an isometry with respect to the inner products on $\mathcal{H}_B$ and $\mathcal{H}_C$. So it lifts to a unitary $\widehat{J}_B : \mathcal{H}_B \to \mathcal{H}_C$, by $\widehat{J}_B \Lambda_B(x) = \Lambda_C\big(S_{\mathcal{B}}(x)\big)$, $\forall x \in \mathcal{B}$. This operator will be useful later.

We may consider the Hilbert space tensor product $\mathcal{H}_B \otimes \mathcal{H}_C$. Here, note that our canonical idempotent $E \in M(\mathcal{B} \odot \mathcal{C})$ naturally defines an operator, $\Pi(E)$, by
$$\Pi(E)\big(\Lambda_B(x) \otimes \Lambda_C(y)\big) = (\Lambda_B \otimes \Lambda_C)\big(E(x \otimes y)\big), \qquad \text{for } x \in \mathcal{B},\ y \in \mathcal{C}.$$

As $\mathcal{B}$ and $\mathcal{C}$ are dense in $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively, it is clear that $\Pi(E)$ is a densely-defined operator acting on $\mathcal{H}_B \otimes \mathcal{H}_C$. By the property of E, it is also clear that $\Pi(E)$ is self-adjoint and idempotent. This means $\Pi(E)$ extends to an orthogonal projection, so it is a bounded self-adjoint operator in $\mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_C)$.

Consider an element of the form $x = (\mathrm{id} \otimes \omega)(E) \in \mathcal{B}$, where $\omega \in \mathcal{C}^*$ is defined by $\omega = \mu(c_1^* \cdot c_2)$, for $c_1, c_2 \in \mathcal{C}$. We saw in §1.3 that such elements span the algebra $\mathcal{B}$. By definition of the operator $\Pi(E)$ above, it is clear that for this x, we can consider the operator $(\mathrm{id} \otimes \omega_{\Lambda_C(c_2),\Lambda_C(c_1)})\big(\Pi(E)\big) \in \mathcal{B}(\mathcal{H}_B)$, such that

$$(\mathrm{id} \otimes \omega_{\Lambda_C(c_2),\Lambda_C(c_1)})\big(\Pi(E)\big)\Lambda_B(b) = \Lambda_B(xb), \quad \text{for all } b \in \mathcal{B}.$$

Here, we are using the standard notation that $\omega_{\xi,\eta}(T) = \langle T\xi, \eta\rangle$, for $T \in \mathcal{H}_C$ and $\xi, \eta \in \mathcal{H}_C$. In other words, any element of the form $x = (\mathrm{id} \otimes \omega)(E) \in \mathcal{B}$ can be regarded as an operator on $\mathcal{H}_B$ by left multiplication. Since such elements span all of $\mathcal{B}$, the same can be said true for any element in $\mathcal{B}$. This allows us to define the GNS-representation π_B of ν:

Definition 2.1. Define π_B from $\mathcal{B}$ into $\mathcal{B}(\mathcal{H}_B)$, by

$$\pi_B(x)\Lambda_B(b) = \Lambda(xb), \quad \text{for all } x, b \in \mathcal{B}.$$

Then π_B is an injective *-homomorphism, which is the GNS-representation of $\mathcal{B}$, such that $\pi_B(\mathcal{B})\mathcal{H}_B$ is dense in $\mathcal{H}_B$.

Remark. In general, even if we have a positive linear functional on a *-algebra $\mathcal{B}$, resulting in an inner product and a Hilbert space $\mathcal{H}_B$, it is not always possible to represent the algebra as an algebra of left multiplication operators. Some elements may become unbounded operators. Observe that in our case, the existence of our self-adjoint idempotent E allowed the construction of the GNS-representation. Meanwhile, note that the density statement in the last sentence of the definition is a quick consequence of the fact that $\mathcal{B}$ is a non-degenerate idempotent algebra.

By a similar argument, using E and considering its other leg, we can also define the GNS-representation π_C of μ:

Definition 2.2. Define π_C from $\mathcal{C}$ into $\mathcal{B}(\mathcal{H}_C)$, by

$$\pi_C(y)\Lambda_C(c) = \Lambda(yc), \quad \text{for all } y, c \in \mathcal{C}.$$

Then π_C is an injective *-homomorphism, which is the GNS-representation of $\mathcal{C}$, such that $\pi_C(\mathcal{C})\mathcal{H}_C$ is dense in $\mathcal{H}_C$.

The GNS-representations allow us to properly define our C^*-algebras B and C:

Definition 2.3. (1) Define $B := \overline{\pi_B(\mathcal{B})}^{\|\ \|}$, as a non-degenerate C^*-subalgebra of $\mathcal{B}(\mathcal{H}_B)$. Or, equivalently,

$$B = \overline{\big\{(\mathrm{id} \otimes \omega)(E) : \omega \in \mathcal{B}(\mathcal{H}_C)_*\big\}}^{\|\ \|} \big(\subseteq \mathcal{B}(\mathcal{H}_B)\big),$$

where we wrote $E = \Pi(E) \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_C)$, for convenience.

(2) Define $C := \overline{\pi_C(\mathcal{C})}^{\|\ \|}$, as a non-degenerate C^*-subalgebra of $\mathcal{B}(\mathcal{H}_C)$. Or, equivalently,

$$C = \overline{\big\{(\omega \otimes \mathrm{id})(E) : \omega \in \mathcal{B}(\mathcal{H}_B)_*\big\}}^{\|\ \|} \big(\subseteq \mathcal{B}(\mathcal{H}_C)\big),$$

We may also get to work with their enveloping von Neumann algebras, namely,

$$N := \pi_B(\mathcal{B})'' \subseteq \mathcal{B}(\mathcal{H}_B) \quad \text{and} \quad L := \pi_C(\mathcal{C})'' \subseteq \mathcal{B}(\mathcal{H}_C).$$

2.2. **The left Hilbert algebras.** Eventually, we wish to lift the functionals ν and μ to the C^*-algebra level. Observe first that $\Lambda_B(\mathcal{B}) \subseteq \mathcal{H}_B$ and $\Lambda_C(\mathcal{C}) \subseteq \mathcal{H}_C$ are left Hilbert algebras, as in the Tomita–Takesaki modular theory (see [21]):

Proposition 2.4. *The subspaces $\Lambda_B(\mathcal{B}) \subseteq \mathcal{H}_B$ and $\Lambda_C(\mathcal{C}) \subseteq \mathcal{H}_C$ are left Hilbert algebras, with respect to the multiplications and the *-structures inherited from $\mathcal{B}$ and $\mathcal{C}$, respectively.*

Proof. For any $x \in \mathcal{B}$, we have already shown that $\pi_B(x) : \Lambda_B(b) \mapsto \Lambda_B(xb)$, $b \in \mathcal{B}$, given by the multiplication, determines a bounded operator. The involution, $x \mapsto x^*$, $x \in \mathcal{B}$, is such that

$$\big\langle \Lambda_B(xb), \Lambda_B(b') \big\rangle = \nu({b'}^* xb) = \nu({b'}^*(x^*)^* b) = \nu(x^* {b'}^* b) = \big\langle \Lambda_B(b), \Lambda(x^* b') \big\rangle,$$

for $b, b' \in \mathcal{B}$. This shows that for any $\xi, \eta \in \Lambda_B(\mathcal{B})$, we have $\big\langle \Lambda_B(x)\xi, \eta \big\rangle = \big\langle \xi, \Lambda_B(x^*)\eta \big\rangle$.

To see that the involution is pre-closed, note that for any fixed $b \in \mathcal{B}$ and any $x_n \in \mathcal{B}$, we have:

$$\big\langle \Lambda_B(b), \Lambda_B(x_n^*) \big\rangle = \nu(x_n b) = \nu\big([\sigma^\nu]^{-1}(b) x_n\big) = \big\langle \Lambda_B(x_n), \Lambda_B([\sigma^\nu]^{-1}(b)^*) \big\rangle.$$

Since $\Lambda_B(\mathcal{B})$ is dense in $\mathcal{H}_B$, we can quickly see that if $x_n \to 0$ and $x_n^* \to z$ in $\mathcal{B}$, then $z = 0$. So the involution is pre-closed. We already know that $\mathcal{B}^2 = \mathcal{B}$, and $\Lambda_B(\mathcal{B})^2$ is dense in $\mathcal{H}_B$.

In this way, we showed that $\Lambda_B(\mathcal{B})$ becomes a left Hilbert algebra (see Definition 1.1 in [21]). Similarly, we can show that $\Lambda_C(\mathcal{C})$ is also a left Hilbert algebra. □

We can associate to the left Hilbert algebra $\Lambda_B(\mathcal{B})$ a von Neumann algebra, which should be none other than $N = \pi_B(\mathcal{B})''$. Also by the general theory on left Hilbert algebras, we obtain a normal semi-finite faithful (n.s.f.) weight $\tilde{\nu}$ on N. Consider the associated spaces $\mathfrak{N}_{\tilde{\nu}} = \big\{x \in N : \tilde{\nu}(x^*x) < \infty\big\}$ and $\mathfrak{M}_{\tilde{\nu}} = \mathfrak{N}_{\tilde{\nu}}^* \mathfrak{N}_{\tilde{\nu}}$. The general theory provides us with the following properties:

We have a closed linear map $\Lambda_{\tilde{\nu}} : \mathfrak{N}_{\tilde{\nu}} \to \mathcal{H}_B$ (the same Hilbert space), which is the GNS-map for the weight $\tilde{\nu}$. The map $\Lambda_{\tilde{\nu}}$ extends Λ_B, such that $\pi_B(\mathcal{B}) \subseteq \mathfrak{N}_{\tilde{\nu}}$ and $\Lambda_{\tilde{\nu}} \circ \pi_B = \Lambda_B$. Note that the weight $\tilde{\nu}$ extends the functional ν. In particular, we have $\tilde{\nu}\big(\pi_B(x)^*\pi_B(x)\big) = \nu(x^*x)$, for all $x \in \mathcal{B}$. For any $b \in \mathfrak{N}_{\tilde{\nu}}$, there exists a sequence $(x_n)_n$ in $\mathcal{B}$ such that $\Lambda_B(x_n) \xrightarrow{(\text{in } \mathcal{H}_B)} \Lambda_{\tilde{\nu}}(b)$ and $\pi_B(x_n) \xrightarrow{(\sigma\text{-strong-}^*)} b$.

Denote by $T_{\tilde{\nu}}$ the closure of the involution $\Lambda_B(x) \mapsto \Lambda_B(x^*)$ on $\Lambda_B(\mathcal{B})$. There exists a polar decomposition, $T_{\tilde{\nu}} = J_{\tilde{\nu}} \nabla_{\tilde{\nu}}^{\frac{1}{2}}$, where $\nabla_{\tilde{\nu}}$ is the modular operator, given by $\nabla_{\tilde{\nu}} = T_{\tilde{\nu}}^* T_{\tilde{\nu}}$, and $J_{\tilde{\nu}}$ is the modular conjugation, which is anti-unitary.

According to the modular theory in the von Neumann algebra setting, the modular operator defines a strongly continuous one-parameter group of automorphisms $\sigma^{\tilde{\nu}}$, by $\sigma_t^{\tilde{\nu}}(b) = \nabla_{\tilde{\nu}}^{it} b \nabla_{\tilde{\nu}}^{-it}$, for $b \in \mathcal{B}$, $t \in \mathbb{R}$, leaving the von Neumann algebra N invariant. We have $\tilde{\nu} \circ \sigma_t^{\tilde{\nu}} = \tilde{\nu}$, $t \in \mathbb{R}$, and $\tilde{\nu}$ satisfies a certain KMS boundary condition. In particular, the weak KMS property at the *-algebra level, $\nu(bb') = \nu\big(b'\sigma^\nu(b)\big)$, $b, b' \in \mathcal{B}$, extends to the von Neumann algebra as

$\tilde{\nu}(xx') = \tilde{\nu}\big(x'\sigma^{\tilde{\nu}}_{-i}(x)\big)$, $x \in \mathfrak{M}_{\tilde{\nu}}$, $x' \in \mathcal{D}(\sigma^{\tilde{\nu}}_{-i})$. Meanwhile, the modular conjugation J_ν can be characterized by $J_\nu \Lambda_\nu(x) = \Lambda_\nu\big(\sigma_{\frac{i}{2}}(x)^*\big)$, for $x \in \mathfrak{N}_\nu$.

There exists a useful "Tomita *-algebra", $\mathcal{T}_{\tilde{\nu}}$, which is a strongly *-dense subalgebra of $N = \pi_B(\mathcal{B})''$ consisting of certain elements that are analytic with respect to $\sigma^{\tilde{\nu}}$. In particular, it is known that for any analytic generator $\sigma^{\tilde{\nu}}_z$, $z \in \mathbb{C}$, we have $\sigma^{\tilde{\nu}}_z(\mathcal{T}_{\tilde{\nu}}) = \mathcal{T}_{\tilde{\nu}}$. For the properties and more details on the modular automorphism group, modular conjugation, Tomita *-algebra, please refer to the standard textbooks on modular theory [20], [21].

Meanwhile, we can also do the same with the left Hilbert algebra $\Lambda_C(\mathcal{C})$, obtaining another n.s.f. weight $\tilde{\mu}$ on $L = \pi_C(\mathcal{C})$, as well as the modular operator $\nabla_{\tilde{\mu}}$, the modular conjugation $J_{\tilde{\mu}}$, the modular automorphism group $\sigma^{\tilde{\mu}}$, and also the Tomita *-algebra $\mathcal{T}_{\tilde{\mu}}$.

However, having gathered all these results from the left Hilbert algebra theory and the modular theory, we have to point out that they are not quite sufficient for our purposes. As we wish to develop a C^*-algebraic framework, two main issues arise: The modular automorphism group $\sigma^{\tilde{\nu}}$ leaves the von Neumann algebra N invariant, but we also want it to leave the C^*-algebra B invariant; and while $(\sigma^{\tilde{\nu}}_t)_{t\in\mathbb{R}}$ is strongly continuous, we want it to be norm continuous as well. These are not automatic consequences of the modular theory, so some more work is needed. To remedy this situation, let us gather below some additional results on the canonical idempotent E.

2.3. **The idempotent E.** We saw earlier that we may regard our canonical idempotent $E \in M(\mathcal{B}\odot\mathcal{C})$ as an operator $\Pi(E) \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_C)$. In fact, we can now see that $\Pi = \pi_B \otimes \pi_C$, and that $(\pi_B \otimes \pi_C)(E)$ is an element of the tensor product von Neumann algebra $N \otimes L$. It is also apparent that $(\pi_B \otimes \pi_C)(E) \in M(B \otimes C)$, where $\otimes$ is now a (spatial) C^*-tensor product. For convenience, we will regard $E = (\pi_B \otimes \pi_C)(E)$ in what follows.

Also for convenience, we may regard $x \in \mathcal{B}$ as $x = \pi_B(x) \in B \subseteq N \subseteq \mathcal{B}(\mathcal{H}_B)$, and regard $y \in \mathcal{C}$ as $y = \pi_C(y) \in C \subseteq L \subseteq \mathcal{B}(\mathcal{H}_C)$.

Recall from §1.3 that for $b \in \mathcal{B}$, we have:

$$(\nu \otimes \mathrm{id})\big(E(b \otimes 1)\big) = (\nu \otimes \mathrm{id})\big(E(1 \otimes S_{\mathcal{B}}(b))\big) = (\nu \otimes \mathrm{id})(E)S_{\mathcal{B}}(b) = S_{\mathcal{B}}(b).$$

As the weight $\tilde{\nu}$ extends the functional ν, we can use the above observation to define the following map $\gamma_N : N \to L$, by

$$\gamma_N(b) := (\tilde{\nu} \otimes \mathrm{id})\big(E(b \otimes 1)\big), \quad b \in \mathcal{B}. \tag{2.1}$$

This map may be unbounded, but as $\mathcal{B} = \pi_B(\mathcal{B})$ is dense in N and $S_{\mathcal{B}}(\mathcal{B}) = \mathcal{C}$ is dense in L, we see that $\gamma_N : N \to L$ is a densely-defined map having a dense range, which is an injective anti-homomorphism because $S_{\mathcal{B}}$ is. However it is not a *-map, as $S_{\mathcal{B}}$ is not.

Consider instead $\widetilde{R} := \gamma_N \circ \sigma^{\tilde{\nu}}_{-\frac{i}{2}}$, where $\sigma^{\tilde{\nu}}_{-\frac{i}{2}}$ is the analytic generator for $(\sigma^{\tilde{\nu}}_t)_{t\in\mathbb{R}}$, at $z = -\frac{i}{2}$. Since $\sigma^{\tilde{\nu}}_t$ is an automorphism and since γ_N is an anti-homomorphism, we can see quickly that $\widetilde{R}$ is anti-multiplicative. As $\mathcal{D}(\sigma^{\tilde{\nu}}_{-\frac{i}{2}})$ contains $\mathcal{T}_{\tilde{\nu}}$, we see that $\widetilde{R}$ is densely-defined. In addition, the following observation shows that $\widetilde{R}$ is involutive. Note that for $b \in \mathcal{T}_{\tilde{\nu}}$, we have:

$$\begin{aligned}\widetilde{R}(b^*) &= (\tilde{\nu} \otimes \mathrm{id})\big(E(\sigma^{\tilde{\nu}}_{-\frac{i}{2}}(b^*) \otimes 1)\big) = (\tilde{\nu} \otimes \mathrm{id})\big((\sigma^{\tilde{\nu}}_{i}(\sigma^{\tilde{\nu}}_{-\frac{i}{2}}(b^*)) \otimes 1)E\big) \\ &= (\tilde{\nu} \otimes \mathrm{id})\big(\sigma^{\tilde{\nu}}_{\frac{i}{2}}(b^*) \otimes 1)E\big),\end{aligned} \tag{2.2}$$

because of the KMS property of $\tilde{\nu}$, while

$$\widetilde{R}(b)^* = \big[(\tilde{\nu} \otimes \mathrm{id})(E(\sigma^{\tilde{\nu}}_{-\frac{i}{2}}(b) \otimes 1))\big]^* = (\tilde{\nu} \otimes \mathrm{id})\big(\sigma^{\tilde{\nu}}_{\frac{i}{2}}(b^*) \otimes 1)E\big),$$

because E is self-adjoint. Comparing, we see that $\widetilde{R}(b^*) = \widetilde{R}(b)^*$. This shows that $\widetilde{R}$ is a *-map, which means that it is actually a *-anti-homomorphism, so bounded. Therefore, we can extend $\widetilde{R}$ to all of N. In fact, as $\widetilde{R}$ is a bounded map from N to L, injective, densely-defined, having a dense range, it extends to a *-anti-isomorphism $\widetilde{R} : N \to L$. Meanwhile, from the definition of $\tilde{R}$, it is immediate that we have $\gamma_N = \tilde{R} \circ \sigma^{\tilde{\nu}}_{\frac{i}{2}}$, which is essentially like a polar decomposition.

Next, consider the n.s.f. weight $\tilde{\mu}$ on L, extending the functional μ on $\mathcal{C}$. In an analogous way as above, we can consider the extension of the map $S_{\mathcal{C}} : \mathcal{C} \to \mathcal{B}$ to the von Neumann algebra level, namely the densely-defined anti-homomorphism $\gamma_L : L \to N$. Analogous to Equation (2.1) for γ_N, we can characterize it by

$$\gamma_L(c) = (\mathrm{id} \otimes \tilde{\mu})\big((1 \otimes c)E\big), \quad c \in \mathcal{C}. \tag{2.3}$$

In the following proposition, we gather some useful relationships between the weights $\tilde{\nu}$, $\tilde{\mu}$, and the maps γ_N and γ_L.

Proposition 2.5. *Let the weights $\tilde{\nu}$ on N and $\tilde{\mu}$ on L be the extensions of the functionals ν and μ, and let $\gamma_N : N \to L$ and $\gamma_L : L \to N$ be the densely-defined anti-homomorphisms as in Equations* (2.1) *and* (2.3)*, extending the maps $S_{\mathcal{B}}$ and $S_{\mathcal{C}}$. Also let $\widetilde{R} = \gamma_N \circ \sigma^{\tilde{\nu}}_{-\frac{i}{2}}$ be the *-anti-isomorphism from N to L obtained above. Then*

(1) $\tilde{\nu} = \tilde{\mu} \circ \gamma_N$ *and* $\tilde{\nu} = \tilde{\mu} \circ \widetilde{R}$.
(2) $\gamma_N = \widetilde{R} \circ \sigma^{\tilde{\nu}}_{\frac{i}{2}} = \sigma^{\tilde{\mu}}_{-\frac{i}{2}} \circ \widetilde{R}$.
(3) $\gamma_L = \sigma^{\tilde{\nu}}_{\frac{i}{2}} \circ \widetilde{R}^{-1} = \widetilde{R}^{-1} \circ \sigma^{\tilde{\mu}}_{-\frac{i}{2}}$.
(4) *For any $t \in \mathbb{R}$, we have* $(\sigma^{\tilde{\nu}}_{t} \otimes \sigma^{\tilde{\mu}}_{-t})(E) = E$.
(5) $(\gamma_N \otimes \gamma_L)(E) = \varsigma E$ *and* $(\gamma_L \otimes \gamma_N)(\varsigma E) = E$.
(6) $(\widetilde{R} \otimes \widetilde{R}^{-1})(E) = \varsigma E$ *and* $(\widetilde{R}^{-1} \otimes \widetilde{R})(\varsigma E) = E$.

Proof. (1). Recall that at the *-algebra level, we have $\nu = \mu \circ S_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$. Extending this to the von Neumann algebra level, we have $\tilde{\nu} = \tilde{\mu} \circ \gamma_N$, or equivalently, $\tilde{\nu} \circ \gamma_N^{-1} = \tilde{\mu}$.

From $\widetilde{R} = \gamma_N \circ \sigma^{\tilde{\nu}}_{-\frac{i}{2}}$, we can write $\gamma_N^{-1} = \big(\widetilde{R} \circ \sigma^{\tilde{\nu}}_{\frac{i}{2}}\big)^{-1} = \sigma^{\tilde{\nu}}_{-\frac{i}{2}} \circ \widetilde{R}^{-1}$. So from $\tilde{\nu} \circ \gamma_N^{-1} = \tilde{\mu}$, we have $\tilde{\nu} \circ \sigma^{\tilde{\nu}}_{-\frac{i}{2}} \circ \widetilde{R}^{-1} = \tilde{\mu}$. Since we know $\tilde{\nu} \circ \sigma^{\tilde{\nu}}_{t} = \tilde{\nu}$, $\forall t$, it follows that $\tilde{\nu} \circ \widetilde{R}^{-1} = \tilde{\mu}$. Or equivalently, $\tilde{\nu} = \tilde{\mu} \circ \widetilde{R}$.

(2). Since $\tilde{\nu} = \tilde{\mu} \circ \widetilde{R}$, it is easy to see that the modular automorphism groups have the following relation:

$$\sigma_t^{\tilde{\nu}} = \widetilde{R}^{-1} \circ \sigma_{-t}^{\tilde{\mu}} \circ \widetilde{R}, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

We already know $\gamma_N = \widetilde{R} \circ \sigma_{\frac{i}{2}}^{\tilde{\nu}}$, which is immediate from the definition of $\widetilde{R}$. Moreover, by Equation (2.4), we have $\gamma_N = \widetilde{R} \circ \big(\widetilde{R}^{-1} \circ \sigma_{-\frac{i}{2}}^{\tilde{\mu}} \circ \widetilde{R}\big) = \sigma_{-\frac{i}{2}}^{\tilde{\mu}} \circ \widetilde{R}$.

(3). At the *-algebra level, we know about the KMS-type automorphism $\sigma^{\nu} = S_{\mathcal{B}}^{-1} \circ S_{\mathcal{C}}^{-1}$. At the von Neumann algebra level, this extends to $\sigma_{-i}^{\tilde{\nu}} = \gamma_N^{-1} \circ \gamma_L^{-1}$. We thus have

$$\gamma_L = \big(\gamma_N \circ \sigma_{-i}^{\tilde{\nu}}\big)^{-1} = \sigma_i^{\tilde{\nu}} \circ \gamma_N^{-1} = \sigma_i^{\tilde{\nu}} \circ \big(\widetilde{R} \circ \sigma_{\frac{i}{2}}^{\tilde{\nu}}\big)^{-1} = \sigma_i^{\tilde{\nu}} \circ \sigma_{-\frac{i}{2}}^{\tilde{\nu}} \circ \widetilde{R}^{-1} = \sigma_{\frac{i}{2}}^{\tilde{\nu}} \circ \widetilde{R}^{-1}.$$

Alternatively, by Equation (2.4), we have $\gamma_L = \big(\widetilde{R}^{-1} \circ \sigma_{-\frac{i}{2}}^{\tilde{\mu}} \circ \widetilde{R}\big) \circ \widetilde{R}^{-1} = \widetilde{R}^{-1} \circ \sigma_{-\frac{i}{2}}^{\tilde{\mu}}$.

(4). For arbitrary $b \in \mathcal{D}(\sigma_{\frac{i}{2}}^{\tilde{\nu}})$ and $t \in \mathbb{R}$, observe that

$$\begin{aligned}
&(\tilde{\nu} \otimes \mathrm{id})\big((\sigma_t^{\tilde{\nu}} \otimes \sigma_{-t}^{\tilde{\mu}})(E)(b \otimes 1)\big) \\
&= (\tilde{\nu} \otimes \mathrm{id})\big((\sigma_t^{\tilde{\nu}} \otimes \sigma_{-t}^{\tilde{\mu}})(E(\sigma_{-t}^{\tilde{\nu}}(b) \otimes 1))\big) = (\tilde{\nu} \otimes \mathrm{id})\big((\mathrm{id} \otimes \sigma_{-t}^{\tilde{\mu}})(E(\sigma_{-t}^{\tilde{\nu}}(b) \otimes 1))\big) \\
&= \sigma_{-t}^{\tilde{\mu}}\big(\gamma_N(\sigma_{-t}^{\tilde{\nu}}(b))\big) = \big(\sigma_{-t}^{\tilde{\mu}} \circ \widetilde{R} \circ \sigma_{\frac{i}{2}}^{\tilde{\nu}} \circ \sigma_{-t}^{\tilde{\nu}}\big)(b) = \big(\sigma_{-t}^{\tilde{\mu}} \circ \widetilde{R} \circ \sigma_{-t}^{\tilde{\nu}} \circ \sigma_{\frac{i}{2}}^{\tilde{\nu}}\big)(b) \\
&= \big((\widetilde{R} \circ \sigma_t^{\tilde{\nu}} \circ \widetilde{R}^{-1}) \circ \widetilde{R} \circ \sigma_{-t}^{\tilde{\nu}} \circ \sigma_{\frac{i}{2}}^{\tilde{\nu}}\big)(b) = (\widetilde{R} \circ \sigma_{\frac{i}{2}}^{\tilde{\nu}})(b) \\
&= \gamma_N(b) = (\tilde{\nu} \otimes \mathrm{id})\big(E(b \otimes 1)\big).
\end{aligned}$$

We used the fact that $\sigma_t^{\tilde{\nu}}$ is an automorphism for the first equality; for the second, we used $\tilde{\nu} \circ \sigma_t^{\tilde{\nu}} = \tilde{\nu}$; and in the rest, we used the definition of γ_N and Equation (2.4). This is true for any $b \in \mathcal{D}(\sigma_{\frac{i}{2}}^{\tilde{\nu}})$ and $\tilde{\nu}$ is faithful, so we see that $(\sigma_t^{\tilde{\nu}} \otimes \sigma_{-t}^{\tilde{\mu}})(E) = E$, for any $t \in \mathbb{R}$.

(5). At the *-algebra level, it is known that $(S_{\mathcal{B}} \otimes S_{\mathcal{C}})(E) = \varsigma E$, where ς is the flip map between $\mathcal{B} \odot \mathcal{C}$ and $\mathcal{C} \odot \mathcal{B}$. It thus follows that at the von Neumann algebra level, we have $(\gamma_N \otimes \gamma_L)(E) = \varsigma E$. Also $(\gamma_L \otimes \gamma_N)(\varsigma E) = E$.

(6). Combine the results of (4) and (5). Since $\widetilde{R} = \gamma_N \circ \sigma_{-\frac{i}{2}}^{\tilde{\nu}}$ and $\widetilde{R}^{-1} = \gamma_L \circ \sigma_{\frac{i}{2}}^{\tilde{\mu}}$, we have:

$$(\widetilde{R} \otimes \widetilde{R}^{-1})(E) = (\gamma_N \otimes \gamma_L)\big((\sigma_{-\frac{i}{2}}^{\tilde{\nu}} \otimes \sigma_{\frac{i}{2}}^{\tilde{\mu}})(E)\big) = (\gamma_N \otimes \gamma_L)(E) = \varsigma E.$$

We also have $(\widetilde{R}^{-1} \otimes \widetilde{R})(\varsigma E) = E$. $\square$

2.4. **The KMS weights on the C^*-algebras B and C.** Note that by restricting the weight $\tilde{\nu}$ on $N = \pi_B(\mathcal{B})''$ to the C^*-algebra $B = \overline{\pi_B(\mathcal{B})}^{\|\ \|}$, represented on the same Hilbert space $\mathcal{H}_B$, we obtain a faithful lower semi-continuous weight. For convenience, we will denote this weight by ν, as it is also an extension of the functional ν at the *-algebra level. We can consider the associated spaces $\mathfrak{N}_\nu = \big\{x \in B : \nu(x^*x) < \infty\big\}$ and $\mathfrak{M}_\nu = \mathfrak{N}_\nu^* \mathfrak{N}_\nu$.

We can also consider the operator T_ν, the closure of $\Lambda_B(x) \mapsto \Lambda_B(x^*)$, $x \in \mathcal{B}$. It is apparent that it will exactly coincide with $T_{\tilde{\nu}}$ earlier, and the polar decomposition will also remain exactly same, $T_\nu = J_\nu \nabla_\nu^{\frac{1}{2}}$, with $\nabla_\nu = \nabla_{\tilde{\nu}}$ and $J_{\tilde{\nu}} = J_\nu$. However, the stumbling issue in §2.2 was the question whether the associate modular automorphism group $\sigma_t^\nu : x \mapsto \nabla_\nu^{it} x \nabla_\nu^{-it}$ leaves the C^*-algebra B invariant, and whether $\left(\sigma_t^\nu\right)_{t\in\mathbb{R}}$ forms a norm-continuous one-parameter group. With the results gathered in §2.3, we are now in a position to resolve this issue in the affirmative.

Proposition 2.6. *Consider the weight ν on B, restricted from $\tilde{\nu}$ on N. Then*

(1) *The automorphism group $(\sigma_t^{\tilde{\nu}})_{t\in\mathbb{R}}$ leaves B invariant. So we can consider $\sigma_t^\nu := \sigma_t^{\tilde{\nu}}|_B$, for $t \in \mathbb{R}$.*
(2) *ν becomes a KMS weight on B, equipped with the automorphism group $(\sigma_t^\nu)_{t\in\mathbb{R}}$, which is norm continuous.*

Proof. (1). Consider $\omega \in \mathcal{B}(\mathcal{H}_C)_*$ and consider $(\mathrm{id} \otimes \omega)(E) \in B$. Such elements are dense in B. For any $t \in \mathbb{R}$ we know from Proposition 2.5 (4) that $(\sigma_{-t}^{\tilde{\nu}} \otimes \sigma_t^{\tilde{\mu}})(E) = E$. we thus have

$$\sigma_t^{\tilde{\nu}}\big((\mathrm{id} \otimes \omega)(E)\big) = \sigma_t^{\tilde{\nu}}\big((\mathrm{id} \otimes \omega)[(\sigma_{-t}^{\tilde{\nu}} \otimes \sigma_t^{\tilde{\mu}})(E)]\big) = \big(\mathrm{id} \otimes (\omega \circ \sigma_t^{\tilde{\mu}})\big)(E) \in B. \tag{2.5}$$

This shows that $\sigma_t^{\tilde{\nu}}(B) = B$, for all $t \in \mathbb{R}$. We will just write $\sigma_t^\nu := \sigma_t^{\tilde{\nu}}|_B$.

(2). As we noted above, it is clear that ν is a faithful lower semi-continuous weight because $\tilde{\nu}$ is an n.s.f. weight. In addition, we know that ν is semi-finite because it extends the distinguished functional ν, which is defined on a dense subalgebra $\mathcal{B} \subseteq B$.

Meanwhile, since $\sigma_t^{\tilde{\nu}}(B) = B$, we can consider the one-parameter group of automorphisms $(\sigma_t^\nu)_{t\in\mathbb{R}}$. At present, we only know that it is strongly continuous. But, the strong continuity together with Equation (2.5) show us that $t \mapsto \sigma_t^\nu(b)$ is indeed norm-continuous.

Finally, it is evident that ν satisfies appropriate KMS properties, by inheriting the properties of the n.s.f. weight $\tilde{\nu}$ and the automorphism group $(\sigma_t^{\tilde{\nu}})_{t\in\mathbb{R}}$. In particular, it is obvious that $\nu \circ \sigma_t^\nu = \nu$, and that for any $x \in \mathcal{D}(\sigma_{\frac{i}{2}}^\nu)$, we have

$$\nu(x^*x) = \nu\big(\sigma_{\frac{i}{2}}^\nu \sigma_{\frac{i}{2}}^\nu(x)^*\big).$$

In this way, we show that ν is a KMS weight (see [21], [14], [13]). □

Note, by the way, that the *-anti-isomorphism $\widetilde{R} : N \to L$ can be restricted to the C^*-algebra level. So consider $R := \widetilde{R}|_B$. Then for $(\mathrm{id} \otimes \omega)(E) \in B$, we have:

$$\begin{aligned} R\big((\mathrm{id} \otimes \omega)(E)\big) &= \widetilde{R}\big((\omega \otimes \mathrm{id})(\varsigma E)\big) \\ &= \widetilde{R}\big((\omega \otimes \mathrm{id})[(\widetilde{R} \otimes \widetilde{R}^{-1})(E)]\big) \\ &= \big((\omega \circ \widetilde{R}) \otimes \mathrm{id}\big)(E) \in C, \end{aligned}$$

where we used the result of Proposition 2.5 (6). This shows that $\widetilde{R} : N \to L$ restricts to $R : B \to C$. It is apparent that this is a C^*-anti-isomorphism.

This means that together with the *-anti-isomorphism $R : B \to C$ and the KMS weight ν on B, it turns out that (E, B, ν) forms a *separability triple*, in the sense of [9]:

Proposition 2.7. *The restriction $R = \widetilde{R}|_B : B \to C$ is a C^*-anti-isomorhism. The self-adjoint idempotent $E \in M(B \otimes C)$ is a separability idempotent, in the sense that*

(1) $(\nu \otimes \mathrm{id})(E) = 1$
(2) *For* $b \in \mathcal{D}(\sigma^\nu_{\frac{i}{2}})$ *we have:* $(\nu \otimes \mathrm{id})\big(E(b \otimes 1)\big) = (R \circ \sigma^\nu_{\frac{i}{2}})(b)$.

Proof. We showed that R and σ^ν are now valid at the level of the C^*-algebra B. Then (1) is just recognizing the fact that ν extends the distinguished functional ν on $\mathcal{B}$, and (2) is just noting that $R \circ \sigma^\nu_{\frac{i}{2}} = \gamma_B = \gamma_L|_B$, extending $S_{\mathcal{B}}$. □

Also at the C^*-algebra level, we can consider the Tomita subalgebra for our KMS weight ν. Being a restriction of the n.s.f. weight $\tilde{\nu}$, it is known by general theory that $\mathcal{T}_\nu := \mathcal{T}_{\tilde{\nu}} \cap B$ is a norm-dense *-subalgebra in B, and that σ^ν leaves $\mathcal{T}_\nu$ invariant. As such, it is convenient to work with $\mathcal{T}_\nu$ in the C^*-framework that we are in.

Remark. While we gave results only regarding the weight ν on B, it is evident that a very much similar argument can be given for the weight μ on the C^*-algebra C, as a restriction of $\tilde{\mu}$ on L. It would extend the distinguished functional μ, and become a KMS weight on the C^*-algebra C, equipped with the norm-continuous one-parameter group $(\sigma^\mu_t)_{t\in\mathbb{R}}$ given by the modular operator.

Observe that the elements in $\mathcal{B}$ and $\mathcal{C}$ are actually analytic elements for ν and μ, respectively.

Proposition 2.8. *Consider the KMS weights* ν *on* B *and* μ *on* C*.*

- *Any element* $b \in \mathcal{B}\,(\subseteq B)$ *is an analytic element for* ν*.*
- *Any element* $c \in \mathcal{C}\,(\subseteq C)$ *is an analytic element for* μ*.*

Proof. Recall the weak KMS property of ν, such that there exists an automorphism σ^ν on $\mathcal{B}$ satisfying $\nu(bb') = \nu\big(b'\sigma^\nu(b)\big)$, for all $b, b' \in \mathcal{B}$. As such, we can see that for any $b \in \mathcal{B}$, we have $b \in \mathcal{D}(\sigma^\nu_i)$ and that $\sigma^\nu_i(b) = \nabla_\nu^{-1} b \nabla_\nu = \sigma^\nu(b)$. Then we can see quickly that we also have $b \in \mathcal{D}(\sigma^\nu_{mi})$ and that $\sigma^\nu_{mi}(b) = \nabla_\nu^{-m} b \nabla_\nu^m$, for all $m \in \mathbb{Z}$. Continuing, it is not difficult to see that $b \in \mathcal{D}(\sigma^\nu_z)$ and that $\sigma^\nu_z(b) = \nabla_\nu^{iz} b \nabla_\nu^{-iz}$, for any $z \in \mathbb{C}$.

Similarly, for any $c \in \mathcal{C}$, we can show that $c \in \mathcal{D}(\sigma^\mu_z)$ and that $\sigma^\mu_z(c) = \nabla_\mu^{iz} c \nabla_\mu^{-iz}$, for any $z \in \mathbb{C}$. □

This result suggests that the elements in $\Lambda_B(\mathcal{B})$ and $\Lambda_C(\mathcal{C})$ are *right bounded vectors* in $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively. (See Definition 1.7 of [21] for the notion of right bounded vectors.) To make this point clearer, see the proposition below.

Proposition 2.9.
- *For any* $b \in \mathcal{B}$*, the vector* $\Lambda_B(b) \in \mathcal{H}_B$ *is right bounded. This means that the map* $\pi_B^R(b) : \Lambda_B(x) \mapsto \Lambda_B(xb)$ *is a bounded operator on* $\mathcal{H}_B$*.*
- *For any* $c \in \mathcal{C}$*, the vector* $\Lambda_C(c) \in \mathcal{H}_C$ *is right bounded. Or the map* $\pi_C^R(c) : \Lambda_C(y) \mapsto \Lambda_C(yc)$ *is a bounded operator on* $\mathcal{H}_C$*.*

Proof. (1). For any $x \in \mathcal{B}$, we know $\pi_B(x)\Lambda_B(b) = \Lambda_B(xb)$. Recall next the unitary operator $\widehat{J}_B : \mathcal{H}_B \to \mathcal{H}_C$ defined earlier (see §2.1), given by $\widehat{J}_B \Lambda_B(x) = \Lambda_C\big(S_{\mathcal{B}}(x)\big)$. Note that we can write:

$$\begin{aligned}[\widehat{J}_B]^*\pi_C\big(S_{\mathcal{B}}(b)\big)\widehat{J}_B\Lambda_B(x) &= [\widehat{J}_B]^*\pi_C\big(S_{\mathcal{B}}(b)\big)\Lambda_C\big(S_{\mathcal{B}}(x)\big) = [\widehat{J}_B]^*\Lambda_C\big(S_{\mathcal{B}}(b)S_{\mathcal{B}}(x)\big) \\ &= [\widehat{J}_B]^*\Lambda_C\big(S_{\mathcal{B}}(xb)\big) = \Lambda_B\big((S_{\mathcal{B}}^{-1}\circ S_{\mathcal{B}})(xb)\big) \\ &= \Lambda_B(xb).\end{aligned}$$

Combining, we observe that $\pi_B(x)\Lambda_B(b) = \Lambda_B(xb) = \pi_B^R(b)\Lambda_B(x)$, where $\pi_B^R(b)$ is the bounded operator $[\widehat{J}_B]^*\pi_C\big(S_{\mathcal{B}}(b)\big)\widehat{J}_B$. This proves that $\Lambda_B(b)$, $b\in\mathcal{B}$, is right bounded in $\mathcal{H}_B$.

(2). Similarly, we can show that $\pi_C(y)\Lambda_C(c) = \Lambda_C(yc) = \pi_C^R(c)\Lambda_C(y)$, where $\pi_C^R(c) = \widehat{J}_B\pi_B\big(S_{\mathcal{B}}^{-1}(c)\big)[\widehat{J}_B]^*$, a bounded operator. So $\Lambda_C(c)$, $c\in\mathcal{C}$, is right bounded in $\mathcal{H}_C$. □

3. The C^*-bialgebra (A,Δ)

Recall that our weak multiplier Hopf *-algebra $(\mathcal{A},\Delta)$ is equipped with a faithful positive left integral φ. As φ is a positive linear functional, we can equip $\mathcal{A}$ with an inner product:

$$\langle x,y\rangle := \varphi(y^*x), \qquad \text{for } x,y\in\mathcal{A}.$$

As usual, complete $\mathcal{A}$ with respect to the induced norm, and obtain a Hilbert space $\mathcal{H}$ with the natural inclusion $\Lambda:\mathcal{A}\to\mathcal{H}$. (Note that Λ is injective because φ is faithful.) We are planning to represent our C^*-algebra as an operator algebra in $\mathcal{B}(\mathcal{H})$, but at present it is not clear if the left multiplication of the elements of A are bounded. Some work is needed.

3.1. **Representations of B and C on $\mathcal{H}$.** Note that $\mathcal{C}\mathcal{A} = \mathcal{A}\mathcal{C} = \mathcal{A}$. This suggests us to define the map $\rho_C:\mathcal{A}\to\mathcal{L}(\mathcal{H}_C,\mathcal{H})$, by

$$\rho_C(a)\Lambda_C(y) = \Lambda(ya), \qquad \text{for } a\in\mathcal{A},\ y\in\mathcal{C}.$$

The next proposition shows that $\rho_C(a)$, $a\in\mathcal{A}$, is bounded.

Proposition 3.1. *Consider $\rho_C:\mathcal{A}\to\mathcal{L}(\mathcal{H}_C,\mathcal{H})$ above. Then $\rho_C(a)$ is a bounded element in $\mathcal{L}(\mathcal{H}_C,\mathcal{H})$, for any $a\in\mathcal{A}$.*

Proof. Let $a\in\mathcal{A}$ and $y\in\mathcal{C}$. Then

$$\big\|\rho_C(a)\Lambda_C(y)\big\|_{\mathcal{H}}^2 = \big\langle\Lambda(ya),\Lambda(ya)\big\rangle = \varphi(a^*y^*ya) = \varphi\big(y^*ya\sigma(a^*)\big),$$

where σ is the modular automorphism for φ, as noted in Proposition 1.9. Apply here the result of Proposition 1.7, knowing that the weight μ extends the functional μ on $\mathcal{C}$. Then we have:

$$\big\|\rho_C(a)\Lambda_C(y)\big\|_{\mathcal{H}}^2 = \varphi\big(y^*ya\sigma(a^*)\big) = \mu\big((\mathrm{id}\otimes\varphi)(\Delta(y^*ya\sigma(a^*)))\big),$$

Note that by Proposition 1.4, since $y^*y\in\mathcal{C}$, we have $\Delta(y^*y) = (y^*y\otimes 1)E$. So we have

$$\Delta\big(y^*ya\sigma(a^*)\big) = \Delta(y^*y)\Delta\big(a\sigma(a^*)\big) = (y^*y\otimes 1)E\Delta\big(a\sigma(a^*)\big) = (y^*y\otimes 1)\Delta\big(a\sigma(a^*)\big).$$

Putting this in the previous equation, we see that

$$\big\|\rho_C(a)\Lambda_C(y)\big\|_{\mathcal{H}}^2 = \mu\big(y^*y(\mathrm{id}\otimes\varphi)(\Delta(a\sigma(a^*)))\big) = \mu(y^*yc) = \big\langle\Lambda_C(yc),\Lambda_C(y)\big\rangle_{\mathcal{H}_C},$$

where $c = (\mathrm{id}\otimes\varphi)\big(\Delta(a\sigma(a^*))\big)\in M(\mathcal{C})$, by the left invariance property of φ.

By Proposition 2.9, we can write $\Lambda_C(yc) = \pi_C^R(c)\Lambda_C(y)$, where $\pi_C^R(c)$ is a bounded operator. So the previous equation becomes:

$$\left\|\rho_C(a)\Lambda_C(y)\right\|_{\mathcal{H}}^2 = \left\langle \pi_C^R(c)\Lambda_C(y), \Lambda_C(y)\right\rangle_{\mathcal{H}_C} \leq \left\|\pi_C^R(c)\right\|\left\|\Lambda_C(y)\right\|_{\mathcal{H}_C}^2,$$

showing that $\left\|\rho_C(a)\right\| \leq \left\|\pi_C^R(c)\right\|^{\frac{1}{2}}$. □

Corollary. *For any $a, b \in \mathcal{A}$, we have $\rho_C(b)^*\rho_C(a) \in \mathcal{B}(\mathcal{H}_C)$, a bounded operator on $\mathcal{H}_C$.*

Proof. By the previous proposition, we know $\rho_C(a), \rho_C(b) \in \mathcal{L}(\mathcal{H}_C, \mathcal{H})$ are bounded, which also means $\rho_C(b)^*$ is a bounded element in $\mathcal{L}(\mathcal{H}, \mathcal{H}_C)$. So it is immediate that $\rho_C(b)^*\rho_C(a)$ is a bounded operator on $\mathcal{H}_C$. □

Similarly, we can consider $\rho_B : \mathcal{A} \to \mathcal{L}(\mathcal{H}_B, \mathcal{H})$. See below.

Proposition 3.2. *Let $\rho_B : \mathcal{A} \to \mathcal{L}(\mathcal{H}_B, \mathcal{H})$, by*

$$\rho_B(a)\Lambda_B(x) = \Lambda(xa), \qquad \text{for } a \in \mathcal{A},\ x \in \mathcal{B}.$$

Then $\rho_B(a)$ is a bounded element in $\mathcal{L}(\mathcal{H}_B, \mathcal{H})$, for any $a \in \mathcal{A}$.

Proof. The proof is essentially no different from that of Proposition 3.1. Use again Propositions 1.7 and 1.4, but this time use the right invariance of ψ. □

We can show that any operator of the form $\rho_C(b)^*\rho_C(a) \in \mathcal{B}(\mathcal{H}_C)$, $a, b \in \mathcal{A}$, commutes with the elements of the C^*-algebra $C = \overline{\pi_C(\mathcal{C})}^{\|\ \|}\ \big(\subseteq \mathcal{B}(\mathcal{H}_C)\big)$.

Proposition 3.3. *For $a, b \in \mathcal{A}$, consider $\rho_C(b)^*\rho_C(a) \in \mathcal{B}(\mathcal{H}_C)$ as above. It commutes with the elements of the C^*-algebra C, regarded as an operator algebra contained in $\mathcal{B}(\mathcal{H}_C)$.*

Proof. For any $y_1, y_2 \in \mathcal{C}$, we have

$$\begin{aligned}\left\langle \rho_C(b)^*\rho_C(a)\Lambda_C(y_1), \Lambda_C(y_2)\right\rangle_{\mathcal{H}_C} &= \left\langle \rho_C(a)\Lambda_C(y_1), \rho_C(b)\Lambda_C(y_2)\right\rangle_{\mathcal{H}} = \left\langle \Lambda(y_1a), \Lambda(y_2b)\right\rangle_{\mathcal{H}} \\ &= \varphi\big(b^*y_2^*y_1a\big) = \varphi\big(y_2^*y_1a\sigma(b^*)\big).\end{aligned}$$

By the same argument as in the proof of Proposition 3.1, where we used Propositions 1.7 and 1.4, we can write

$$\varphi\big(y_2^*y_1a\sigma(b^*)\big) = \cdots = \mu(y_2^*y_1\tilde{c}),$$

where $\tilde{c} = (\mathrm{id}\otimes\varphi)\big(\Delta(a\sigma(b^*))\big) \in M(\mathcal{C})$. So we can write $\rho_C(b)^*\rho_C(a)\Lambda_C(y_1) = \Lambda_C(y_1\tilde{c})$, which means that $\rho_C(b)^*\rho_C(a) \in \mathcal{B}(\mathcal{H}_C)$ is none other than $\pi_C^R(\tilde{c})$.

As such, for $\pi_C(c) \in C$ and any $y \in \mathcal{C}$, we have:

$$\rho_C(b)^*\rho_C(a)\pi_C(c)\Lambda_C(y) = \rho_C(b)^*\rho_C(a)\Lambda_C(cy) = \pi_C^R(\tilde{c})\Lambda_C(cy) = \Lambda_c(cy\tilde{c}),$$

$$\pi_C(c)\rho_C(b)^*\rho_C(a)\Lambda_C(y) = \pi_C(c)\pi_C^R(\tilde{c})\Lambda_C(y) = \pi_C(c)\Lambda_C(y\tilde{c}) = \Lambda_C(cy\tilde{c}),$$

showing that $\rho_C(b)^*\rho_C(a)$ commutes with any $\pi_C(c) \in C$. □

We are now ready to construct a *-representation of the C^*-algebra C into $\mathcal{B}(\mathcal{H})$. See below:

Proposition 3.4. *Consider any $c \in \mathcal{C}$, which we regard as $c = \pi_C(c)$, an element of the C^*-algebra C. Define $\alpha\big(\pi_C(c)\big) \in \mathcal{L}(\mathcal{H})$, by*

$$\alpha\big(\pi_C(c)\big)\Lambda(a) = \Lambda(ca), \quad a \in \mathcal{A}.$$

Then

1. *$\alpha\big(\pi_C(c)\big)$, $c \in \mathcal{C}$, is a bounded operator on $\mathcal{H}$.*
2. *α extends to a (bounded) C^*-representation $\alpha : C \to \mathcal{B}(\mathcal{H})$.*
3. *$\alpha : C \to \mathcal{B}(\mathcal{H})$ becomes a non-degenerate *-representation. It also extends to the *-representation at the level of the multiplier algebra $M(C)$.*

Proof. (1). Without loss of generality, we may consider the vectors of the type $\Lambda(ya) \in \mathcal{H}$, where $y \in \mathcal{C}$, $a \in \mathcal{A}$, because $\mathcal{C}\mathcal{A} = \mathcal{A}$. Note that we can write

$$\alpha(\pi_C(c))\Lambda(ya) = \Lambda(cya) = \rho_C(a)\Lambda_C(cy) = \rho_C(a)\pi_C(c)\Lambda_C(y).$$

We know that $\rho_C(a)$ is bounded. We thus have:

$$\begin{aligned}
&\big\|\alpha(\pi_C(c))\Lambda(ya)\big\|^2 \\
&= \big\langle \rho_C(a)\pi_C(c)\Lambda_C(y), \rho_C(a)\pi_C(c)\Lambda_C(y)\big\rangle_{\mathcal{H}} = \big\langle \rho_C(a)^*\rho_C(a)\pi_C(c)\Lambda_C(y), \pi_C(c)\Lambda_C(y)\big\rangle_{\mathcal{H}_C} \\
&= \big\langle \pi_C(c)\rho_C(a)^*\rho_C(a)\Lambda_C(y), \pi_C(c)\Lambda_C(y)\big\rangle_{\mathcal{H}_C} \\
&\le \big\|\pi_C(c)\big\|^2\big\langle \rho_C(a)^*\rho_C(a)\Lambda_C(y), \Lambda_C(y)\big\rangle_{\mathcal{H}_C} = \big\|\pi_C(c)\big\|^2\big\langle \rho_C(a)\Lambda_C(y), \rho_C(a)\Lambda_C(y)\big\rangle_{\mathcal{H}} \\
&= \big\|\pi_C(c)\big\|^2\big\langle \Lambda(ya), \Lambda(ya)\big\rangle_{\mathcal{H}} = \big\|\pi_C(c)\big\|^2\big\|\Lambda(ya)\big\|^2.
\end{aligned}$$

Note that the third equality is because $\rho_C(a)^*\rho_C(a)$ commutes with $\pi_C(c) \in C$ (see Proposition 3.3). This observation shows that $\alpha(\pi_C(c))$ is bounded, with $\big\|\alpha(\pi_C(c))\big\| \le \big\|\pi_C(c)\big\|$.

(2). It is not difficult to see that α preserves multiplication. Note that for any $c_1, c_2 \in \mathcal{C}$ and for any $a \in \mathcal{A}$, we have

$$\alpha\big(\pi_C(c_1)\big)\alpha\big(\pi_C(c_2)\big)\Lambda(a) = \alpha\big(\pi_C(c_1)\big)\Lambda(c_2a) = \Lambda(c_1c_2a) = \alpha\big(\pi_C(c_1c_2)\big)\Lambda(a).$$

As $\pi_C(c_1c_2) = \pi_C(c_1)\pi_C(c_2)$, and since the vectors $\Lambda(a)$, $a \in \mathcal{A}$, are dense in $\mathcal{H}$, it is evident that $\alpha\big(\pi_C(c_1)\big)\alpha\big(\pi_C(c_2)\big) = \alpha\big(\pi_C(c_1)\pi_C(c_2)\big)$.

To see if α preserves the involution, note that for any $c \in \mathcal{C}$ and any $a_1, a_2 \in \mathcal{A}$, we have

$$\begin{aligned}
\big\langle \alpha(\pi_C(c))\Lambda(a_1), \Lambda(a_2)\big\rangle &= \big\langle \Lambda(ca_1), \Lambda(a_2)\big\rangle = \varphi(a_2^*ca_1) = \varphi\big((c^*a_2)^*a_1\big) \\
&= \big\langle \Lambda(a_1), \alpha(\pi_C(c)^*)\Lambda(a_2)\big\rangle,
\end{aligned}$$

since $\pi_C(c^*) = \pi_C(c)^*$. Since $a_1, a_2 \in \mathcal{A}$ are arbitrary, we see that $\alpha\big(\pi_C(c)\big)^* = \alpha\big(\pi_C(c)^*\big)$.

This means $\alpha : \pi_C(\mathcal{C}) \to \mathcal{B}(\mathcal{H})$ is a *-representation, which is automatically bounded. It follows that α extends to $C = \overline{\pi_C(\mathcal{C})}^{\|\ \|}$, giving us the *-representation of the C^*-algebra C.

(3). The non-degeneracy of α is easy to see, using the fact that $\mathcal{C}\mathcal{A} = \mathcal{A}$. As a result, it is clear that α naturally extends to the level of the multiplier algebra $M(C)$ $\square$

Proposition 3.5. *For any $b \in \mathcal{B}$, regarded as an element of the C^*-algebra B by $b = \pi_B(b)$, define $\beta\big(\pi_B(b)\big) \in \mathcal{L}(\mathcal{H})$, by*

$$\beta\big(\pi_B(b)\big)\Lambda(a) = \Lambda(ba), \quad a \in \mathcal{A}.$$

Then

1. *$\beta\big(\pi_B(b)\big) \in \mathcal{B}(\mathcal{H})$, for any $b \in \mathcal{B}$.*
2. *β extends to a (bounded) C^*-representation $\beta : B \to \mathcal{B}(\mathcal{H})$.*
3. *$\beta : B \to \mathcal{B}(\mathcal{H})$ becomes a non-degenerate *-representation. It also extends to the *-representation at the level of the multiplier algebra $M(B)$.*

Proof. Proof can be carried out using essentially the same argument as in Propositions 3.3 and 3.4 above, in the construction of the C^*-representation $\alpha : C \to \mathcal{B}(\mathcal{H})$. □

The representations α and β are naturally extended to the multiplier algebra level of $M(C)$ and $M(B)$, respectively. As for the total algebra $\mathcal{A}$, however, it is not clear at this stage whether the elements of $\mathcal{A}$ can be similarly all regarded as bounded operators on $\mathcal{H}$.

3.2. **The C^*-algebra A.** Using the left invariance of φ, we can construct the following operator W, which will help us construct the left regular representation of $\mathcal{A}$:

Proposition 3.6. (1) *There exists a bounded operator $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ satisfying*

$$\big((\theta \otimes \mathrm{id})(W^*)\big)\Lambda(a) = \Lambda\big((\theta \otimes \mathrm{id})(\Delta a)\big), \qquad \text{for } a \in \mathcal{A},\ \theta \in \mathcal{B}(\mathcal{H})_*.$$

(2) *For $a, p \in \mathcal{A}$, we have*

$$W^*\big(\Lambda(p) \otimes \Lambda(a)\big) = (\Lambda \otimes \Lambda)\big((\Delta a)(p \otimes 1)\big).$$

Proof. (1). Let $\xi \in \mathcal{H}$ be arbitrary. We need to show that $W^* : \xi \otimes \Lambda(a) \mapsto W\big(\xi \otimes \Lambda(a)\big)$ is bounded, for any $a \in \mathcal{A}$. Let $(e_j)_{j\in J}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$, and consider $\omega_{\xi,e_j} \in \mathcal{B}(\mathcal{H})_*$. Note here that for $\eta, \zeta \in \mathcal{H}$, the functional $\omega_{\eta,\zeta} \in \mathcal{B}(\mathcal{H})$ is given by $\omega_{\eta,\zeta}(T) = \langle T\eta, \zeta\rangle$, $\forall T \in \mathcal{B}(\mathcal{H})$, similar as before.

Note first that using the property $\omega_{\xi,e_j}^* = \omega_{e_j,\xi}$, we can write:

$$\begin{aligned}\sum_{j\in J}\big\|\Lambda((\omega_{\xi,e_j} \otimes \mathrm{id})(\Delta a))\big\|^2 &= \sum_{j\in J}\varphi\big((\omega_{\xi,e_j} \otimes \mathrm{id})(\Delta a)^*(\omega_{\xi,e_j} \otimes \mathrm{id})(\Delta a)\big)\\ &= \sum_{j\in J}\varphi\big((\omega_{e_j,\xi} \otimes \mathrm{id})(\Delta(a^*))(\omega_{\xi,e_j} \otimes \mathrm{id})(\Delta a)\big)\end{aligned} \tag{3.1}$$

The terms in the sum are all positive, so this is a monotone increasing sequence of partial sums. Meanwhile, by the property of the functionals ω_{ξ,e_j} (basically Linear Algebra; see Lemma 3.1 in [11]), we have:

$$\sum_{j\in J}(\omega_{e_j,\xi} \otimes \mathrm{id})\big(\Delta(a^*)\big)(\omega_{\xi,e_j} \otimes \mathrm{id})(\Delta a) = (\omega_{\xi,\xi} \otimes \mathrm{id})\big(\Delta(a^*)\Delta(a)\big) = (\omega_{\xi,\xi} \otimes \mathrm{id})\big(\Delta(a^*a)\big).$$

What this implies is that the sum in the right hand side of Equation (3.1) gives a monotone increasing sequence of partial sums, which is bounded above by $\varphi\big((\omega_{\xi,\xi}\otimes\mathrm{id})(\Delta(a^*a))\big)$. So the series converges, and we have:

$$\sum_{j\in J}\big\|\Lambda((\omega_{\xi,e_j}\otimes\mathrm{id})(\Delta a))\big\|^2 \le \varphi\big((\omega_{\xi,\xi}\otimes\mathrm{id})(\Delta(a^*a))\big)$$
$$= \big\langle(\mathrm{id}\otimes\varphi)(\Delta(a^*a))\xi,\xi\big\rangle \le \big\|(\mathrm{id}\otimes\varphi)(\Delta(a^*a))\big\|\|\xi\|^2. \tag{3.2}$$

The left side of the inequality (3.2) is actually $\sum_{j\in J}\big\|\langle W^*(\xi\otimes\Lambda(a)),e_j\otimes\cdot\rangle\big\|^2$, by the way W^* was defined. Meanwhile, $(\mathrm{id}\otimes\varphi)(\Delta(a^*a))$ in the right side is an element in $M(C)$, due to the left invariance of φ, and is being considered as the bounded operator $\alpha\big((\mathrm{id}\otimes\varphi)(\Delta(a^*a))\big)\in\mathcal{B}(\mathcal{H})$. In this way, we can see that $W^*:\xi\otimes\Lambda(a)\mapsto W^*\big(\xi\otimes\Lambda(a)\big)$ is a bounded operator. As W^* is bounded in $\mathcal{B}(\mathcal{H})$, so is W.

(2). Let $a,p,b,q\in\mathcal{A}$ be arbitrary, and consider $\theta=\omega_{\Lambda(p),\Lambda(q)}$. Then,

$$\begin{aligned}&\big\langle W^*(\Lambda(p)\otimes\Lambda(a)),\Lambda(q)\otimes\Lambda(b)\big\rangle\\ &=\big\langle(\omega_{\Lambda(p),\Lambda(q)}\otimes\mathrm{id})(W^*)\Lambda(a),\Lambda(b)\big\rangle=\big\langle\Lambda((\omega_{\Lambda(p),\Lambda(q)}\otimes\mathrm{id})(\Delta a),\Lambda(b)\big\rangle\\ &=(\varphi\otimes\varphi)\big((q^*\otimes b^*)(\Delta a)(p\otimes 1)\big)=\big\langle(\Lambda\otimes\Lambda)((\Delta a)((p\otimes 1)),\Lambda(q)\otimes\Lambda(b)\big\rangle.\end{aligned}$$

This shows that $W^*\big(\Lambda(p)\otimes\Lambda(a)\big)=(\Lambda\otimes\Lambda)\big((\Delta a)(p\otimes 1)\big)$. □

Another way to characterize the operator W is given below:

Proposition 3.7. *For any $a,b\in\mathcal{A}$, we have*

$$W\big(\Lambda(a)\otimes\Lambda(b)\big)=(\Lambda\otimes\Lambda)\big((S^{-1}\otimes\mathrm{id})(\Delta b)(a\otimes 1)\big),$$

where S denotes the antipode on $(\mathcal{A},\Delta)$.

Proof. Let $c,d\in\mathcal{A}$ be arbitrary. Then

$$\begin{aligned}&\big\langle W(\Lambda(a)\otimes\Lambda(b)),\Lambda(c)\otimes\Lambda(d)\big\rangle\\ &=\big\langle\Lambda(a)\otimes\Lambda(b),W^*(\Lambda(c)\otimes\Lambda(d))\big\rangle=\big\langle\Lambda(a)\otimes\Lambda(b),(\Lambda\otimes\Lambda)((\Delta d)(c\otimes 1))\big\rangle\\ &=(\varphi\otimes\varphi)\big((c^*\otimes 1)\Delta(d^*)(a\otimes b)\big)=\varphi\big(c^*(\mathrm{id}\otimes\varphi)(\Delta(d^*)(1\otimes b))a\big).\end{aligned}$$

Here, use the characterization of the antipode map S, given in Proposition 1.8. Then this becomes

$$\begin{aligned}&=\varphi\big(c^*S^{-1}((\mathrm{id}\otimes\varphi)((1\otimes d^*)(\Delta b)))a\big)=(\varphi\otimes\varphi)\big((c^*\otimes d^*)(S^{-1}\otimes\mathrm{id})(\Delta b)(a\otimes 1)\big)\\ &=\big\langle(\Lambda\otimes\Lambda)((S^{-1}\otimes\mathrm{id})(\Delta b)(a\otimes 1)),\Lambda(c)\otimes\Lambda(d)\big\rangle.\end{aligned}$$

As $c,d\in\mathcal{A}$ are arbitrary, this shows $W\big(\Lambda(a)\otimes\Lambda(b)\big)=(\Lambda\otimes\Lambda)\big((S^{-1}\otimes\mathrm{id})(\Delta b)(a\otimes 1))\big)$. □

The operator W is essentially like the multiplicative unitary operator (in the sense of [1], [34]) in the framework of locally compact quantum groups [15]. In our setting, however, it is no longer a unitary and will turn out to be only a partial isometry (see [4], for finite-dimensional weak Hopf algebras). Here is one more result regarding W, which will be useful in defining the GNS representation of A in $\mathcal{H}$:

Proposition 3.8. *For any $a, p, q \in \mathcal{A}$, we have:*

$$(\mathrm{id} \otimes \omega_{\Lambda(p),\Lambda(q)})(W)\Lambda(a) = \Lambda\big((\mathrm{id} \otimes \varphi)(\Delta(q^*)(1 \otimes p))a\big).$$

Proof. For any $d \in \mathcal{A}$, we have:

$$\begin{aligned}
&\big\langle(\mathrm{id} \otimes \omega_{\Lambda(p),\Lambda(q)})(W)\Lambda(a), \Lambda(d)\big\rangle \\
&= \big\langle W(\Lambda(a) \otimes \Lambda(p)), \Lambda(d) \otimes \Lambda(q)\big\rangle = \big\langle \Lambda(a) \otimes \Lambda(p), W^*(\Lambda(d) \otimes \Lambda(q))\big\rangle \\
&= \big\langle \Lambda(a) \otimes \Lambda(p), (\Lambda \otimes \Lambda)(\Delta(q)(d \otimes 1))\big\rangle = (\varphi \otimes \varphi)\big((d^* \otimes 1)\Delta(q^*)(a \otimes p)\big) \\
&= \varphi\big(d^*(\mathrm{id} \otimes \varphi)(\Delta(q^*)(1 \otimes p))a\big) = \big\langle \Lambda((\mathrm{id} \otimes \varphi)(\Delta(q^*)(1 \otimes p))a), \Lambda(d)\big\rangle.
\end{aligned}$$

As $d \in \mathcal{A}$ was arbitrary, this proves the result. □

Recall that by the fullness assumption of Δ, we know that the elements of the form $x = (\mathrm{id} \otimes \varphi)(\Delta(q^*)(1 \otimes p))$, where $p, q \in \mathcal{A}$, span all of $\mathcal{A}$. Therefore what Proposition 3.8 is saying is that for any $x \in \mathcal{A}$, we can find a bounded operator $X \in \mathcal{B}(\mathcal{H})$ such that $X\Lambda(a) = \Lambda(xa)$, for all $a \in \mathcal{A}$. In this way, we can define the GNS-representation π of φ:

Definition 3.9. Define π from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ by

$$\pi(x)\Lambda(a) = \Lambda(xa), \quad \text{for all } x, a \in \mathcal{A}.$$

Then π is an injective $*$-homomorphism, which is the GNS representation of $\mathcal{A}$, such that $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$.

The last statement on the non-degeneracy of π is a consequence of $\mathcal{A}^2 = \mathcal{A}$. The GNS-representation allows us to properly define our C^*-algebra A:

Definition 3.10. Define $A := \overline{\pi(\mathcal{A})}^{\|\ \|}$, as a non-degenerate C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. It can be also characterized as

$$A = \overline{\big\{(\mathrm{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\big\}}^{\|\ \|}.$$

The alternative characterization of A is obtained by noting from Proposition 3.8 that for $x = (\mathrm{id} \otimes \varphi)\big(\Delta(q^*)(1 \otimes p)\big)$, for any $p, q \in \mathcal{A}$, we have

$$\pi(x) = \pi\big((\mathrm{id} \otimes \varphi)(\Delta(q^*)(1 \otimes p))\big) = (\mathrm{id} \otimes \omega_{\Lambda(p),\Lambda(q)})(W).$$

As π is a non-generate *-representation, it can be naturally extended to the level of the multiplier algebra $M(A)$. We will often regard $A = \pi(A)$ and $M(A) = \pi\big(M(A)\big)$.

At the *-algebra level, we saw that $\mathcal{B}$ and $\mathcal{C}$ are subalgebras in $M(\mathcal{A})$. As these algebras are now all represented on $\mathcal{H}$ by left multiplications, and in turn completed to the C^*-algebras $\beta(B)$, $\alpha(C)$, $\pi\big(M(A)\big)$, respectively, it is apparent that $\alpha = \pi|_{\mathcal{C}}$, $\beta = \pi|_{\mathcal{B}}$. It is thus natural to regard $B = \beta(B) \subset M(A)$ and $C = \alpha(C) \subset M(A)$, as operator algebras in $\mathcal{B}(\mathcal{H})$. We also have $M(B) \subset M(A)$ and $M(C) \subset M(A)$. While it is true that in Section 2 we considered the C^*-algebras B and C as represented on $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively, and such aspect will still be needed down the road, we will be able to tell from the context on which space they are represented.

3.3. **The comultiplication on A.** We next wish to define the comultiplication at the C^*-algebra level, extending the comultiplication on $\mathcal{A}$. We have our candidate below using the operator W, analogous to the quantum group case. We still need some work to clarify that this is indeed a correct definition.

Definition 3.11. Define the map $\widetilde{\Delta}$ from the C^*-algebra $A = \pi(A)$ to $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, by $\widetilde{\Delta}(x) = W^*(1\otimes x)W$, for all $x \in A$

The next proposition shows that $\widetilde{\Delta}$ is an extension of the comultiplication on $\mathcal{A}$.

Proposition 3.12. *For any $a \in \mathcal{A}$, we have $\widetilde{\Delta}\big(\pi(a)\big) = (\pi\otimes\pi)(\Delta a)$.*

Proof. Let $c, d \in \mathcal{A}$ be arbitrary. Using the characterization of W given in Proposition 3.7, we have

$$\begin{aligned} W^*\big(1\otimes\pi(a)\big)W\big(\Lambda(c)\otimes\Lambda(d)\big) &= W^*\big(1\otimes\pi(a)\big)\big((\Lambda\otimes\Lambda)((S^{-1}\otimes\mathrm{id})(\Delta d)(c\otimes 1))\big) \\ &= W^*\big((\Lambda\otimes\Lambda)((1\otimes a)(S^{-1}\otimes\mathrm{id})(\Delta d)(c\otimes 1))\big). \end{aligned}$$

For convenience, we can use the Sweedler notation to write $(1\otimes a)(S^{-1}\otimes\mathrm{id})(\Delta d)(c\otimes 1)) = \sum_{(d)}\big[S^{-1}(d_{(1)})c\otimes ad_{(2)}\big]$. Thus, by applying the characterization of W^* given in Proposition 3.6, the above becomes:

$$W^*\big(1\otimes\pi(a)\big)W\big(\Lambda(c)\otimes\Lambda(d)\big) = (\Lambda\otimes\Lambda)\left((\Delta a)\sum_{(d)}\big[d_{(2)}S^{-1}(d_{(1)})c\otimes d_{(3)}\big]\right).$$

Apply here a result from the algebraic framework, namely, Proposition 4.3 of [31], which says that $\sum_{(d)}\big[d_{(2)}S^{-1}(d_{(1)})c\otimes d_{(3)}\big] = E(c\otimes d)$, so we have $(\Delta a)\sum_{(d)}\big[d_{(2)}S^{-1}(d_{(1)})c\otimes d_{(3)}\big] = (\Delta a)E(c\otimes d) = (\Delta a)(c\otimes d)$. It follows that

$$W^*\big(1\otimes\pi(a)\big)W\big(\Lambda(c)\otimes\Lambda(d)\big) = (\Lambda\otimes\Lambda)\big((\Delta a)(c\otimes d)\big) = (\pi\otimes\pi)(\Delta a)\big(\Lambda(c)\otimes\Lambda(d)\big).$$

As $c, d \in \mathcal{A}$ can be arbitrary, this shows $\widetilde{\Delta}\big(\pi(a)\big) = W^*\big(1\otimes\pi(a)\big)W = (\pi\otimes\pi)(\Delta a)$. □

As it is the case that $\widetilde{\Delta}$ extends Δ at the *-algebra level, let us from now on denote by the same notation Δ for the map at the C^*-algebra level. Moreover, from the way the map is defined, and from the proof of the proposition, it is also evident that Δ extends further to the multiplier algebra $M(A)$, such that $\Delta\big(\pi(m)\big) = W^*\big(1\otimes\pi(m)\big)W = (\pi\otimes\pi)(\Delta m)$, for all $m \in M(\mathcal{A})$.

Consider next the canonical idempotent E at the *-algebra level. Since it is an element contained in $M(\mathcal{A}\odot\mathcal{A})$, we can regard it as an operator $E = (\pi\otimes\pi)(E) \in \mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, by the GNS-representation π. The operator W allows us find an alternative expression for E, as follows:

Proposition 3.13. *We have:*

$$E = (\pi\otimes\pi)(E) = W^*W \in \mathcal{B}(\mathcal{H}\otimes\mathcal{H}).$$

Proof. As an element of $M(\mathcal{A}\odot\mathcal{A})$, we know that $E=\Delta(1_{M(\mathcal{A})})$. So, by Proposition 3.12, we have:

$$E=(\pi\otimes\pi)(E)=\widetilde{\Delta}\big(\pi(1_{M(\mathcal{A})})\big)=W^*\big(1\otimes\pi(1_{M(\mathcal{A})})\big)W.$$

But, note that $W^*\big(1\otimes\pi(1_{M(\mathcal{A})})\big)=W^*$, because for any $c,d\in\mathcal{A}$ we have

$$\big(1\otimes\pi(1_{M(\mathcal{A})})\big)\big(\Lambda(c)\otimes\Lambda(d)\big)=\Lambda(c)\otimes\Lambda(1_{M(\mathcal{A})}d)=\Lambda(c)\otimes\Lambda(d).$$

It follows that $E=(\pi\otimes\pi)(E)=W^*W$. □

Note that W is not a unitary, unless $E=1\otimes 1$. It is a partial isometry, though. See below:

Proposition 3.14. *We have*

1. $W^*(1\otimes x)=(\Delta x)W^*$, *for any* $x\in A$.
2. $EW^*=W^*$.
3. W *is a partial isometry, satisfying* $W^*WW^*=W^*$ *and* $WW^*W=W$.

Proof. (1). For $a\in\mathcal{A}$, and for any $c,d\in\mathcal{A}$, note that

$$\begin{aligned}W^*\big(1\otimes\pi(a)\big)\big(\Lambda(c)\otimes\Lambda(d)\big)&=W^*\big(\Lambda(c)\otimes\Lambda(ad)\big)=(\Lambda\otimes\Lambda)\big(\Delta(ad)(c\otimes 1)\big)\\&=(\pi\otimes\pi)(\Delta a)(\Lambda\otimes\Lambda)\big(\Delta(d)(c\otimes 1)\big)\\&=\Delta\big(\pi(a)\big)W^*\big(\Lambda(c)\otimes\Lambda(d)\big).\end{aligned}$$

As $c,d\in\mathcal{A}$ are arbitrary, this shows that $W^*\big(1\otimes\pi(a)\big)=\Delta\big(\pi(a)\big)W^*$. Since the $\pi(a)$, $a\in\mathcal{A}$, are dense in A, it follows that we have $W^*(1\otimes x)=(\Delta x)W^*$, for any $x\in A$.

(2). It is evident that the result of (1) will hold true also for all $m\in M(A)$. So, in particular, if $m=1_{M(A)}=\pi\big(1_{M(\mathcal{A})}\big)$, we have:

$$W^*=W^*\big(1\otimes 1_{M(A)}\big)=\Delta(1_{M(A)})W^*=EW^*.$$

(3). We know from Proposition 3.13 that $E=W^*W$. Combining this with $EW^*=W^*$, we see that $W^*WW^*=W^*$. Also $WW^*W=W$. So W is a partial isometry. □

We are now ready to show that $\Delta:x\mapsto W^*(1\otimes x)W$ determines a valid comultiplication on the C^*-algebra A.

Proposition 3.15. *The map* Δ *defined in Definition 3.11 is a* **-representation of* A *into* $M(A\otimes A)$, *which also extends to a* **-representation from* $M(A)$ *into* $M(A\otimes A)$. *We have:*

1. $(\Delta x)(1\otimes y)\in A\otimes A$ *and* $(x\otimes 1)(\Delta y)\in A\otimes A$, *for all* $x,y\in A$.
2. *The following spaces are norm-dense in* A:
$$\operatorname{span}\big\{(\mathrm{id}\otimes\omega)((\Delta x)(1\otimes y)):\omega\in A^*,x,y\in A\big\},$$
$$\operatorname{span}\big\{(\omega\otimes\mathrm{id})((x\otimes 1)(\Delta y)):\omega\in A^*,x,y\in A\big\}.$$
3. *The coassociativity condition holds:*
$$(\Delta\otimes\mathrm{id})(\Delta x)=(\mathrm{id}\otimes\Delta)(\Delta x),\quad\forall x\in A.$$

Proof. It is easy to see that $\Delta(x^*) = W^*(1\otimes x^*)W = \bigl(W^*(1\otimes x)W\bigr)^* = \Delta(x)^*$. In addition, for $x, y \in A$, by Proposition 3.14 we have,

$$\Delta(x)\Delta(y) = (\Delta x)W^*(1\otimes y)W = W^*(1\otimes x)(1\otimes y)W = W^*(1\otimes xy)W = \Delta(xy).$$

This shows that Δ is a *-representation.

Next, let $a, b \in \mathcal{A}$ be arbitrary. Then

$$\bigl(\pi(a)\otimes 1\bigr)\Delta\bigl(\pi(b)\bigr) = \bigl(\pi(a)\otimes 1\bigr)(\pi\otimes\pi)(\Delta b) = (\pi\otimes\pi)\bigl((a\otimes 1)(\Delta b)\bigr) \in (\pi\otimes\pi)(\mathcal{A}\odot\mathcal{A}) \subset A\otimes A,$$

because we know from Equation (1.1) that $(a\otimes 1)(\Delta b) \in \mathcal{A}\odot\mathcal{A}$ at the *-algebra level. As $\pi(\mathcal{A})$ is dense in A, this shows that $(x\otimes 1)(\Delta y) \in A\otimes A$, for all $x, y \in A$. Similarly, we can also show that $(\Delta x)(1\otimes y) \in A\otimes A$ for all $x, y, \in A$. As an immediate consequence, we can see that for any $x, y, z \in A$, we have$(\Delta x)(y\otimes z) \in A\otimes A$ and $(y\otimes z)(\Delta x) \in A\otimes A$, showing that $\Delta(A) \subseteq M(A\otimes A)$.

The "fullness" property, given in (2), is also a consequence of the fullness of Δ at the *-algebra level, as the spanned space is exactly $\mathcal{A}$, which is norm-dense in A.

We have already seen that Δ is well established at the level of both A and $M(A)$, and that it extends the comultiplication at the dense *-algebra level. So, we expect that it indeed satisfies the coassociativity at the C^*-algebra level. To see this, consider $a, b, c \in \mathcal{A}$, we have:

$$\begin{aligned}
&(\Delta\otimes\mathrm{id})\bigl(\Delta(\pi(a))\bigr)\bigl(\pi(b)\otimes 1\otimes\pi(c)\bigr) = (\Delta\otimes\mathrm{id})\bigl(\Delta(\pi(a))(1\otimes\pi(c))\bigr)\bigl(\pi(b)\otimes 1\otimes 1\bigr)\\
&= (\pi\otimes\pi\otimes\pi)\bigl((\Delta\otimes\mathrm{id})((\Delta a)(1\otimes c))(b\otimes 1\otimes 1)\bigr)\\
&= (\pi\otimes\pi\otimes\pi)\bigl((\mathrm{id}\otimes\Delta)((\Delta a)(b\otimes 1))(1\otimes 1\otimes c)\bigr)\\
&= (\mathrm{id}\otimes\Delta)\bigl(\Delta(\pi(a))(\pi(b)\otimes 1)\bigr)\bigl(1\otimes 1\otimes\pi(c)\bigr)\\
&= (\mathrm{id}\otimes\Delta)\bigl(\Delta(\pi(a))\bigr)\bigl(\pi(b)\otimes 1\otimes\pi(c)\bigr).
\end{aligned}$$

As A acts non-degenerately, this shows that $(\Delta\otimes\mathrm{id})\bigl(\Delta(\pi(a))\bigr) = (\mathrm{id}\otimes\Delta)\bigl(\Delta(\pi(a))\bigr)$, $\forall a \in \mathcal{A}$. Since $\pi(\mathcal{A})$ is dense in A, we see that $(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$ on A. ☐

In this way, we have shown that (A, Δ) is a C^*-bialgebra, with the comultiplication Δ satisfying all the conditions prescribed in Definition 3.1 of [10].

3.4. **The idempotent E.** We already noted that the canonical idempotent element E at the *-algebra level can be considered as the operator $E = (\pi\otimes\pi)(E) \in M(A\otimes A) \subset \mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, by the GNS-representation π. As such, its properties will be inherited from those at the *-algebra level, which we gather below.

Proposition 3.16. *Consider the canonical idempotent E, regarded as $E = (\pi\otimes\pi)(E) \in M(A\otimes A) \subset \mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, such that $E^* = E$ and $E^2 = E$. We have:*

1. $\overline{\Delta(A)(A\otimes A)}^{\|\ \|} = E(A\otimes A)$ *and* $\overline{(A\otimes A)\Delta(A)}^{\|\ \|} = (A\otimes A)E$
2. $E(\Delta x) = \Delta x = (\Delta x)E$, *for all* $x \in A$
3. $E\otimes 1$ *and* $1\otimes E$ *commute, and we also have*

$$(\mathrm{id}\otimes\Delta)(E) = (E\otimes 1)(1\otimes E) = (1\otimes E)(E\otimes 1) = (\Delta\otimes\mathrm{id})(E).$$

(4) *There exists a* **-anti-isomorphism* $R = R_{BC} : B \to C$*, and together with the KMS weight* ν *on* B *and* $E \in M(B \otimes C)$*, we obtain a separability triple* (E, B, ν)*, in the sense of* [9].

Proof. (1), (2) are the consequences of Lemma 1.1 at the *-algebra level. Since $\mathcal{A}$ is norm dense in A, the results follow immediately. As shown in Proposition 3.3 in [10], this uniquely determines E.

(3) is the weak comultiplicativity of the unit, noted already in Equation (1.6).

(4). As $E \in M(\mathcal{B} \odot \mathcal{C})$, it can be also considered as an element in $M(B \otimes C)$, where B and C are the C^*-subalgebras of $M(A)$ we saw earlier. We saw in Proposition 2.7 that E satisfies the properties of being a *separability idempotent*, in the C^*-algebraic sense (see [9]).

□

In this way, we have shown that $E \in M(B \otimes C) \subseteq M(A \otimes A)$ satisfies all the conditions for being the *canonical idempotent* for (A, Δ), as prescribed in Definition 3.7 of [10].

4. Left and right invariant weights

We have so far established our C^*-algebra A; the comultiplication Δ; the base C^*-algebra B (and C) as a C^*-subalgebra of $M(A)$; a KMS weight ν on B; and the canonical idempotent E. In view of the definition of a C^*-algebraic quantum groupoid of separable type (Definition 4.8 of [10] and Definition 1.2 of [11]), what remains is showing the existence of a suitable left invariant weight φ and a right invariant weight ψ. We will construct them here, as extensions of the left and right integrals at the *-algebra level.

The overall idea for the construction of the KMS weights φ and ψ on A is similar to what we did in §2.2 – §2.4, though technically a bit more challenging. The left/right invariance properties will then carry over from the *-algebra level.

4.1. Quasi-invariance assumption.

Observe first that $\Lambda(\mathcal{A}) \subseteq \mathcal{H}$ obtained from the functional φ at the *-algebra level, as given in the beginning part of Section 3, is a left Hilbert algebra, with respect to the multiplication and the $*$-structure inherited from $\mathcal{A}$. We will skip the proof, which is essentially no different in nature from that of Proposition 2.4 in Section 2.

By the general theory on left Hilbert algebras (see [21]), we can associate to $\Lambda(\mathcal{A})$ a von Neumann algebra M. In our case, in terms of the GNS-representation π, it would be exactly $M = \pi(A)''$. Also by the general theory on left Hilbert algebras, we obtain a normal semi-finite faithful weight $\tilde{\varphi}$ on M.

We can consider the associated spaces $\mathfrak{N}_{\tilde{\varphi}} = \{x \in M : \tilde{\varphi}(x^*x) < \infty\}$ and $\mathfrak{M}_{\tilde{\varphi}} = \mathfrak{N}_{\tilde{\varphi}}^* \mathfrak{N}_{\tilde{\varphi}}$. The associated GNS map $\Lambda_{\tilde{\varphi}}$ is an injective map from $\mathfrak{N}_{\tilde{\varphi}}$ to $\mathcal{H}$ (same Hilbert space), which extends Λ. The weight $\tilde{\varphi}$ extends the functional φ.

Denote by T the closure of the involution $\Lambda(x) \mapsto \Lambda(x^*)$ on $\Lambda(\mathcal{A})$. As before, there exists a polar decomposition, $T = J\nabla^{\frac{1}{2}}$, where $\nabla = T^*T$ is the modular operator, and J is the modular conjugation, which is anti-unitary.

According to the modular theory in the von Neumann algebra setting ([21], [20]), the modular operator defines a strongly continuous one-parameter group of automorphisms $\tilde{\sigma} = (\tilde{\sigma}_t)_{t\in\mathbb{R}}$, by $\tilde{\sigma}_t(a) = \nabla^{it} a \nabla^{-it}$, for $a \in M$, $t \in \mathbb{R}$. We have $\tilde{\varphi} \circ \tilde{\sigma}_t = \tilde{\varphi}$, $t \in \mathbb{R}$, and $(\tilde{\sigma}_t)$ satisfies a certain KMS boundary condition. In particular, the weak KMS property at the *-algebra level, $\varphi(ab) = \varphi\big(b\sigma(a)\big)$, $a, b \in \mathcal{A}$, extends to the von Neumann algebra as $\tilde{\varphi}(xy) = \tilde{\varphi}\big(y\tilde{\sigma}_{-i}(x)\big)$, $x \in \mathfrak{M}_{\tilde{\varphi}}$, $y \in \mathcal{D}(\tilde{\sigma}_{-i})$. Meanwhile, the modular conjugation J can be characterized by $J\Lambda(x) = \Lambda\big(\tilde{\sigma}_{\frac{i}{2}}(x)^*\big)$, for $x \in \mathfrak{N}_{\tilde{\varphi}}$.

Next, consider the functional $\psi = \varphi \circ S$ on $\mathcal{A}$, which can be now regarded as defined on a dense subalgebra of M. We should be able to carry out essentially the same procedure, to obtain an n.s.f. weight, $\tilde{\psi}$, on M. We can also consider its associated modular automorphism group, $\tilde{\sigma}' = (\tilde{\sigma}'_t)_{t\in\mathbb{R}}$, such that the modular automorphism σ' at the *-algebra level is again none other than the restriction of $\tilde{\sigma}'_{-i}$ to the *-algebra $\mathcal{A}$.

For the operator algebraic theory to work properly, we not only need suitable extension weights φ and $\varphi \circ S$, but we actually need a *quasi-invariance* condition, such that we are able to find a Radon–Nikodym derivative between the extended weights. This was already the case even in the setting of classical locally compact groupoids, where the quasi-invariance condition is assumed as a part of the definition (see [19], [18]).

In our case, to allow this development, we introduce the following *quasi-invariance Assumption* at the purely algebraic level. This is the one small (but necessary) additional condition we are going to require:

Assumption (Quasi-invariance)**.** *We will assume that $\sigma|_{\mathcal{B}}$, the restriction of σ to the base algebra $\mathcal{B}$, leaves $\mathcal{B}$ invariant, and that $\nu \circ \sigma_{\mathcal{B}} = \nu$.*

See §5.3 in Appendix, where some of the consequences of this purely algebraic assumption are collected. Among the consequences of this Assumption is that the modular automorphisms σ and σ' (for φ and ψ, respectively) commute: See Proposition 5.11 (3).

As $\mathcal{A}$ is dense in M, what this implies is that if we were to require this quasi-invariance Assumption, the modular automorphism groups at the von Neumann algebraic level, namely $(\tilde{\sigma}_t)_{t\in\mathbb{R}}$ and $(\tilde{\sigma}'_t)_{t\in\mathbb{R}}$, must commute. Note here that σ and σ' are essentially no different from the analytic generators $\tilde{\sigma}_{-i}$ and $\tilde{\sigma}'_{-i}$. But then, the commutativity of the modular automorphism groups in turn should imply the existence of the Radon–Nikodym derivative. We quote here a result by Vaes:

Proposition 4.1. *(Vaes* [25]*): Let φ and ψ be two n.s.f. weights on a von Neumann algebra M. Then the following are equivalent:*

(i). *The modular automorphism groups σ^{ψ} and σ^{φ} commute.*
(ii). *There exist a strictly positive operator δ affiliated with M and a strictly positive operator λ affiliated with the center of M such that $\sigma^{\varphi}_s(\delta^{it}) = \lambda^{ist}\delta^{it}$ for all $s, t \in \mathbb{R}$ and such that $\psi = \varphi_\delta = \varphi(\delta^{\frac12} \cdot \delta^{\frac12})$.*
(iii). *There exist a strictly positive operator δ affiliated with M and a strictly positive operator λ affiliated with the center of M such that $[D\psi : D\varphi]_t = \lambda^{\frac12 it^2}\delta^{it}$ for all $t \in \mathbb{R}$.*

Proof. See Proposition 5.2 in [25]. □

In our case, we should consider the weights $\tilde{\varphi}$, $\tilde{\psi}$, and the commuting modular automorphism groups $(\tilde{\sigma}_t)$, $(\tilde{\sigma}'_t)$. The commutativity is a consequence of our (purely algebraic) quasi-invariance Assumption. See Proposition 5.11 (3). As Vaes's result (Proposition 4.1) indicates, this assures us the existence of a suitable Radon–Nikodym derivative, justifying the term "quasi-invariance".

Note also the existence of a positive operator δ, affiliated with M. It is evident that this is an extension of the modular element at the purely algebraic level, and what Proposition 4.1 implies is that δ must be positive (so self-adjoint). As such, the purely algebraic results we gathered in §5.3 can all be used. This observation indicates that there is possibly a way to give a direct, purely algebraic proof of the self-adjointness of the modular element δ from the quasi-invariance Assumption, but we will not pursue that here.

Meanwhile, Proposition 4.1 indicates an existence of another positive operator λ, which would be a generalization of the "scaling constant" in the quantum group theory (see Proposition 6.8 in [15]). For our current purposes, its role will be downplayed. In a future paper (such as [12], when we study the duality theory for C^*-algebraic quantum groupoids), we will have more occasions to discuss further implications of having δ and λ.

4.2. **The KMS weight φ.** Now that we have an n.s.f. weight $\tilde{\varphi}$ at the von Neumann algebra level, a natural next step is to consider its restriction to the C^*-algebra level. By restricting the weight $\tilde{\varphi}$ on the von Neumann algebra $M = \pi(\mathcal{A})''$ to the level of the C^*-algebra $A = \overline{\pi(\mathcal{A})}^{\|\ \|}$, we obtain a faithful lower semi-continuous weight φ on A. For convenience, we choose to use the same notation as the linear functional at the *-algebra level. Denote the associated spaces by $\mathfrak{N}_\varphi = \{x \in A : \varphi(x^*x) < \infty\}$ and $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi$.

We can consider the operators T, ∇, J as before, because the Hilbert space remains the same. However, as was the case for the weights $\tilde{\nu}$ and $\tilde{\mu}$ earlier (Section 2), the main issues are whether the restriction of the modular automorphism group $(\tilde{\sigma}_t)$ to the C^*-algebra level would leave A invariant, and whether the restriction is norm-continuous. These are not automatic consequences of the modular theory. Earlier, for the weights at the base C^*-algebra level, we were benefitted by the existence of the canonical idempotent E. However, that is not possible this time. We need a different approach.

Let us return back down to the *-algebra level, and consider the modular element $\delta \in M(\mathcal{A})$. For its properties, see Propositions 1.10 and Appendix (Section 5). Note that due to our *quasi-invariance* Assumption (see discussion given in §4.1), we can assume that δ is positive. This in turn means that we can use the results obtained in §5.3 (Appendix), such as Proposition 5.13 and Proposition 5.14.

Using δ, define a new Hilbert space $\mathcal{H}_\delta$, as follows:

Proposition 4.2. *Let $a, b \in \mathcal{A}$. Then as δ is a positive element, we can define a positive sesquilinear form:*

$$(a, b) \mapsto \varphi(b^*\delta a).$$

In this way, we can define a Hilbert space $\mathcal{H}_\delta$, together with an injective linear map $\Lambda_\delta : \mathcal{A} \to \mathcal{H}_\delta$, having a dense range in $\mathcal{H}_\delta$, such that

$$\langle \Lambda_\delta(a), \Lambda_\delta(b) \rangle = \varphi(b^*\delta a), \qquad \textit{for all } a, b \in \mathcal{A}.$$

Proof. As δ is positive, it is clear that $\varphi(a^*\delta a)$ is positive, for any $a \in \mathcal{A}$.

Assume that $\varphi(a^*\delta a) = 0$. By the Schwarz inequality, we have, for any $b \in \mathcal{A}$,

$$\big|\varphi(b^*\delta a)\big|^2 \leq \varphi(b^*\delta b)\varphi(a^*\delta a),$$

so $\varphi(b^*\delta a) = 0$. Since φ is faithful and since this is true for any $b \in \mathcal{A}$, this means that $\delta a = 0$. As δ is invertible, we must have $a = 0$. We see that $\varphi(a^*\delta a) = 0$ if and only if $a = 0$.

From the positive definiteness, we thus have an inner product on $\mathcal{A}$. By completing $\mathcal{A}$ with respect to the induced norm, we thereby obtain the Hilbert space $\mathcal{H}_\delta$, with the natural inclusion $\Lambda_\delta : \mathcal{A} \to \mathcal{H}_\delta$. □

We define an anti-linear, closed (unbounded) operator Z from $\mathcal{H}$ to $\mathcal{H}_\delta$, in the following proposition:

Proposition 4.3. *For $a \in \mathcal{A}$, define:*

$$Z\Lambda(a) := \Lambda_\delta\big(S(a^*)\big).$$

Then:

1. *Z is a well-defined map from $\Lambda(\mathcal{A})$ into $\mathcal{H}_\delta$.*
2. *Z is a closed, densely-defined, injective operator from $\mathcal{H}$ into $\mathcal{H}_\delta$, such that $\Lambda(\mathcal{A})$ forms a core and has a dense range.*
3. *Z is anti-linear.*
4. *$\Lambda_\delta(\mathcal{A})$ forms a core for Z^*, which is also a densely-defined, injective, and has dense range, and given by*

$$Z^*\Lambda_\delta(a) = \Lambda\big(\delta^{-1}S(a)^*\delta\big), \qquad \text{for } a \in \mathcal{A}.$$

Proof. As S is well-defined from $\mathcal{A}$ onto itself, and since $\Lambda(\mathcal{A})$ is dense in $\mathcal{H}$ as well as in $\mathcal{H}_\delta$, with respect to the relevant norms, it is clear that Z is well-defined, densely-defined, and has a dense range. Meanwhile, for $a, b \in \mathcal{A}$, note that

$$\big\langle \Lambda_\delta(S(a^*)), \Lambda_\delta(b)\big\rangle = \varphi\big(b^*\delta S(a^*)\big) = \varphi\big(S(a^*\delta^{-1}S^{-1}(b^*)\big) = \varphi\big(S(a^*\delta^{-1}S(b)^*)\big),$$

because $S(\delta) = \delta^{-1}$ (see Proposition 5.13 in Appendix) and $S\big(S(b)^*\big)^* = b$. Since $\varphi \circ S = \varphi(\,\cdot\,\delta)$, this becomes:

$$\big\langle \Lambda_\delta(S(a^*)), \Lambda_\delta(b)\big\rangle = \varphi\big(a^*\delta^{-1}S(b)^*\delta\big) = \big\langle \Lambda(\delta^{-1}S(b)^*\delta), \Lambda(a)\big\rangle.$$

With $Z^*\Lambda_\delta(b) = \Lambda\big(\delta^{-1}S(b)^*\delta\big)$, for $b \in \mathcal{A}$, this result can be expressed as

$$\big\langle Z\Lambda(a), \Lambda_\delta(b)\big\rangle = \big\langle Z^*\Lambda_\delta(b), \Lambda(a)\big\rangle = \overline{\big\langle \Lambda(a), Z^*\Lambda_\delta(b)\big\rangle}.$$

From this, we can see quickly that Z is closed, injective, anti-linear. It is also also apparent that Z^* is also a closed, injective, anti-linear operator, that is densely-defined and has a dense range. □

Define $P := Z^*Z$. As a consequence of Proposition 4.3, we see that P is a closed, positive, injective operator on $\mathcal{H}$, which is densely-defined and has a dense range. It is clear that $\Lambda(\mathcal{A})$ forms a core for P. Moreover, we have:

$$P\Lambda(a) = Z^*Z\Lambda(a) = Z^*\Lambda_\delta\big(S(a^*)\big) = \Lambda\big(\delta^{-1}S(S(a^*))^*\delta\big) = \Lambda\big(\delta^{-1}S^{-2}(a)\delta\big), \tag{4.1}$$

because $S(x^*) = S^{-1}(x)^*$, which is applied twice. Next proposition gives a useful relationship between the operators W, $\nabla(= T^*T)$, and $P(= Z^*Z)$:

Proposition 4.4. *For any $a, b \in \mathcal{A}$, we have:*

$$W(\nabla \otimes P)\big(\Lambda(a) \otimes \Lambda(b)\big) = (\nabla \otimes \nabla)W\big(\Lambda(a) \otimes \Lambda(b)\big).$$

Proof. Let $a, b, c, d \in \mathcal{A}$ be arbitrary. Then

$$\begin{aligned}
&\big\langle W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d)\big\rangle \\
&= \big\langle T^*T\Lambda(a) \otimes Z^*Z\Lambda(b), W^*(\Lambda(c) \otimes \Lambda(d))\big\rangle \\
&= \overline{\big\langle T\Lambda(a) \otimes Z\Lambda(b), (T \otimes Z)(\Lambda \otimes \Lambda)((\Delta d)(c \otimes 1))\big\rangle} \\
&= \big\langle (\Lambda \otimes \Lambda_\delta)((c^* \otimes 1)(\mathrm{id} \otimes S)(\Delta(d^*))), \Lambda(a^*) \otimes \Lambda_\delta(S(b^*))\big\rangle,
\end{aligned}$$

using the characterization of W^* as in Proposition 3.6, the fact that $S(b^*)^* = S^{-1}(b)$, and using the definitions of T and Z. Note that now the inner product is in $\mathcal{H} \otimes \mathcal{H}_\delta$. Continuing, this becomes:

$$\begin{aligned}
(RHS) &= (\varphi \otimes \varphi)\big((a \otimes S^{-1}(b))(1 \otimes \delta)(c^* \otimes 1)(\mathrm{id} \otimes S)(\Delta(d^*))\big) \\
&= (\varphi \otimes \varphi)\big((ac^* \otimes S^{-1}(b)\delta)(\mathrm{id} \otimes S)(\Delta(d^*))\big).
\end{aligned}$$

We thus have, so far:

$$\big\langle W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d)\big\rangle = (\varphi \otimes \varphi)\big((ac^* \otimes S^{-1}(b)\delta)(\mathrm{id} \otimes S)(\Delta(d^*))\big). \tag{4.2}$$

Meanwhile, we have:

$$\begin{aligned}
&\big\langle (\nabla \otimes \nabla)W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d)\big\rangle \\
&= \big\langle (T^*T \otimes T^*T)W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d)\big\rangle \\
&= \overline{\big\langle (T \otimes T)(\Lambda \otimes \Lambda)((S^{-1} \otimes \mathrm{id})(\Delta b)(a \otimes 1)), T\Lambda(c) \otimes T\Lambda(d)\big\rangle} \\
&= \big\langle \Lambda(c^*) \otimes \Lambda(d^*), (\Lambda \otimes \Lambda)((a^* \otimes 1)(S \otimes \mathrm{id})(\Delta(b^*)))\big\rangle \\
&= (\varphi \otimes \varphi)\big((S^{-1} \otimes \mathrm{id})(\Delta b)(a \otimes 1)(c^* \otimes d^*)\big) = \varphi\big(S^{-1}[(\mathrm{id} \otimes \varphi)((\Delta b)(1 \otimes d^*))]ac^*\big),
\end{aligned}$$

by the characterization of W given in Proposition 3.7, and the fact that $S^{-1}(x)^* = S(x^*)$. Using a characterization of S given in Proposition 1.8, we can go further:

$$\begin{aligned}
(RHS) &= \varphi\big(S^{-2}[(\mathrm{id} \otimes \varphi)((1 \otimes b)\Delta(d^*))]ac^*\big) \\
&= (\varphi \otimes \varphi)\big((1 \otimes b)(S^{-2} \otimes \mathrm{id})(\Delta(d^*))(ac^* \otimes 1)\big) \\
&= (\varphi \otimes \varphi)\big((ac^* \otimes b)(\sigma \otimes \mathrm{id})[(S^{-2} \otimes \mathrm{id})(\Delta(d^*))]\big).
\end{aligned} \tag{4.3}$$

Note here that we used the modular automorphism σ.

In Proposition 5.14, we prove that

$$(\sigma^{-1} \otimes \sigma')(\Delta x) = \Delta\big(S^{-2}(x)\big) = (S^{-2} \otimes S^{-2})(\Delta x), \quad \forall x \in \mathcal{A}.$$

As a consequence, it follows that for all $x \in \mathcal{A}$,

$$(\sigma \otimes \mathrm{id})\big[(S^{-2} \otimes \mathrm{id})(\Delta x)\big] = \big(\mathrm{id} \otimes (S^2 \circ \sigma')\big)(\Delta x) = \big(\mathrm{id} \otimes (S \circ \sigma^{-1})\big)\big[(\mathrm{id} \otimes S)(\Delta x)\big],$$

because $\sigma' = S^{-1} \circ \sigma^{-1} \circ S$ (Proposition 1.9). Using this result in Equation (4.3), we have:

$$\begin{aligned}
&\big\langle (\nabla \otimes \nabla) W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \big\rangle \\
&= \cdots = (\varphi \otimes \varphi)\big((ac^* \otimes b)(\mathrm{id} \otimes (S \circ \sigma^{-1}))[(\mathrm{id} \otimes S)(\Delta(d^*))]\big) \\
&= \big(\varphi \otimes (\varphi \circ S)\big)\big((ac^* \otimes 1)(\mathrm{id} \otimes \sigma^{-1})[(\mathrm{id} \otimes S)(\Delta(d^*))](1 \otimes S^{-1}(b))\big) \\
&= (\varphi \otimes \varphi)\big((ac^* \otimes 1)(\mathrm{id} \otimes \sigma^{-1})[(\mathrm{id} \otimes S)(\Delta(d^*))](1 \otimes S^{-1}(b)\delta)\big) \\
&= (\varphi \otimes \varphi)\big((ac^* \otimes S^{-1}(b)\delta)(\mathrm{id} \otimes S)(\Delta(d^*))\big).
\end{aligned}$$

We thus have:

$$\big\langle (\nabla \otimes \nabla) W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \big\rangle = (\varphi \otimes \varphi)\big((ac^* \otimes S^{-1}(b)\delta)(\mathrm{id} \otimes S)(\Delta(d^*))\big). \tag{4.4}$$

Compare Equations (4.2) and (4.4). From the observation, we conclude that

$$\big\langle W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \big\rangle = \big\langle (\nabla \otimes \nabla) W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \big\rangle,$$

true for any $c, d \in \mathcal{A}$, so we have:

$$W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)) = (\nabla \otimes \nabla) W(\Lambda(a) \otimes \Lambda(b)),$$

for any $a, b \in \mathcal{A}$. □

Note, by the way that since we are working with unbounded operators P and ∇, which are only densely-defined, the above result does not necessarily mean $W(\nabla \otimes P) = (\nabla \otimes \nabla)W$. As it is possible that some of the elements contained in $\mathrm{Ker}(W)$ may not be contained in $\mathcal{D}(\nabla \otimes P)$, we would have $\mathcal{D}\big(W(\nabla \otimes P)\big) \subsetneq \mathcal{D}\big((\nabla \otimes \nabla)W\big)$. So, to be precise, this should be written as $W(\nabla \otimes P) \subseteq (\nabla \otimes \nabla)W$.

If W was a unitary operator, then there exists a clever method one can use to quickly establish $W(\nabla \otimes P) = (\nabla \otimes \nabla)W$ (see, for instance, Lemma 5.9 in [15]), which helps us proceed along. That, however, is not the case here. We need a more roundabout approach, as in the below (see similar discussions carried out in Propositions 4.12 – 4.17 in [11].).

Recall that the operator W is a partial isometry, such that $W^*W = E = (\pi \otimes \pi)(E)$, regarded as a projection operator in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. See Propositions 3.13 and 3.14. As W is a partial isometry, we can also consider another projection, $G = WW^*$. Note the standard results, such as $\mathrm{Ran}(E) = \mathrm{Ran}(W^*) = \mathrm{Ker}(W)^{\perp}$ and $\mathrm{Ran}(G) = \mathrm{Ran}(W) = \mathrm{Ker}(W^*)^{\perp}$, as subspaces in $\mathcal{H} \otimes \mathcal{H}$. Here are some consequences of $W(\nabla \otimes P) \subseteq (\nabla \otimes \nabla)W$, as observed in Proposition 4.4.

Proposition 4.5. *We have:*

(1) $(\nabla \otimes P)E = E(\nabla \otimes P)E$ *and* $(\nabla \otimes \nabla)G = G(\nabla \otimes \nabla)G$

(2) *The restrictions* $(\nabla\otimes P)|_{\mathrm{Ran}(E)}$, $(\nabla\otimes P)|_{\mathrm{Ker}(W)}$, $(\nabla\otimes\nabla)|_{\mathrm{Ran}(G)}$, $(\nabla\otimes\nabla)|_{\mathrm{Ker}(W^*)}$ *become valid operators on the subspaces* $\mathrm{Ran}(E)$, $\mathrm{Ker}(W)$, $\mathrm{Ran}(G)$, $\mathrm{Ker}(W^*)$, *respectively.*

Proof. (1). From $W(\nabla\otimes P)\subseteq(\nabla\otimes\nabla)W$, take the adjoint. Then $W^*(\nabla\otimes\nabla)\subseteq(\nabla\otimes P)W^*$. By using these two results and by using the partial isometry property ($WW^*W=W$), we see that

$$\begin{aligned} W^*W(\nabla\otimes P)W^*W &\subseteq W^*(\nabla\otimes\nabla)WW^*W = W^*(\nabla\otimes\nabla)W = W^*WW^*(\nabla\otimes\nabla)W \\ &\subseteq W^*W(\nabla\otimes P)W^*W. \end{aligned}$$

As a result, we obtain the following observation:

$$W^*(\nabla\otimes\nabla)W = E(\nabla\otimes P)E, \tag{4.5}$$

since $E=W^*W$. As a consequence, we also have:

$$E(\nabla\otimes P)E = W^*(\nabla\otimes\nabla)W \subseteq (\nabla\otimes P)W^*W = (\nabla\otimes P)E. \tag{4.6}$$

Note that $E(\nabla\otimes P)E$ and $(\nabla\otimes P)E$ in Equation (4.6) may be regarded as operators restricted to the subspace $\mathrm{Ran}(E)$, with the common domain $\mathrm{Ran}(E)\cap\mathcal{D}(\nabla\otimes P)$. Since the domains are same, we can say that in fact,

$$E(\nabla\otimes P)E = (\nabla\otimes P)E. \tag{4.7}$$

By using a similar argument, we can also show that

$$W(\nabla\otimes P)W^* = G(\nabla\otimes\nabla)G. \tag{4.8}$$

and that

$$G(\nabla\otimes\nabla)G = (\nabla\otimes\nabla)G. \tag{4.9}$$

(2). As an immediate consequence of Equation (4.7), we see that $(\nabla\otimes P)|_{\mathrm{Ran}(E)}$ is an operator on the subspace $\mathrm{Ran}(E)$. Similarly, as a consequence of Equation (4.9), we see that $(\nabla\otimes\nabla)|_{\mathrm{Ran}(G)}$ is an operator on the subspace $\mathrm{Ran}(G)$.

Meanwhile, consider an arbitrary $\zeta\in\mathrm{Ran}(1-E)\cap\mathcal{D}(\nabla\otimes P)$, which is actually the common domain for $(\nabla\otimes P)(1-E)$ and $(1-E)(\nabla\otimes P)(1-E)$, if they are regarded as defined on the subspace $\mathrm{Ran}(1-E)$. Then, again by Proposition 4.4, we have

$$E(\nabla\otimes P)\zeta = W^*W(\nabla\otimes P)\zeta = W^*(\nabla\otimes\nabla)W\zeta = (\nabla\otimes P)W^*W\zeta = (\nabla\otimes P)E\zeta = 0.$$

As a consequence, we can see that

$$(\nabla\otimes P)(1-E)\zeta = (1-E)(\nabla\otimes P)(1-E)\zeta, \tag{4.10}$$

for any $\zeta\in\mathrm{Ran}(1-E)\cap\mathcal{D}(\nabla\otimes P)$. Since $\mathrm{Ker}(W)=\mathrm{Ran}(1-E)$, we see in this way that $(\nabla\otimes P)|_{\mathrm{Ker}(W)}$ is an operator on the subspace $\mathrm{Ker}(W)$.

Similarly, we can show that $(\nabla\otimes\nabla)|_{\mathrm{Ker}(W^*)}$ is an operator on the subspace $\mathrm{Ker}(W^*)=\mathrm{Ran}(1-G)$. □

A useful thing to remember about W being a partial isometry is that when restricted to the relevant subspaces, the maps $W|_{\mathrm{Ran}(E)}$ and $W^*|_{\mathrm{Ran}(G)}$ become onto isometries between $\mathrm{Ran}(E)$ and $\mathrm{Ran}(G)$. We also have $W^*|_{\mathrm{Ran}(G)}W|_{\mathrm{Ran}(E)} = \mathrm{Id}\,|_{\mathrm{Ran}(E)}$ and $W|_{\mathrm{Ran}(E)}W^*|_{\mathrm{Ran}(G)} = \mathrm{Id}\,|_{\mathrm{Ran}(G)}$. As such, the result of Equations (4.5) and (4.8) can be re-interpreted as follows:

Proposition 4.6. *We have:*

1. $(\nabla \otimes P)|_{\mathrm{Ran}(E)} = W^*(\nabla \otimes \nabla)|_{\mathrm{Ran}(G)}W$*, as operators on* $\mathrm{Ran}(E)$*.*
2. $(\nabla \otimes \nabla)|_{\mathrm{Ran}(G)} = W(\nabla \otimes P)|_{\mathrm{Ran}(E)}W^*$*, as operators on* $\mathrm{Ran}(G)$*.*

Since ∇ and P are self-adjoint operators, we can perform functional calculus and consider ∇^z and P^z, for any $z \in \mathbb{C}$. Using the result of Proposition 4.6 and the fact that $W|_{\mathrm{Ran}(E)}$ and $W^*|_{\mathrm{Ran}(G)}$ are inverses of each other, we can establish the following:

$$(\nabla^z \otimes P^z)|_{\mathrm{Ran}(E)} = W^*(\nabla^z \otimes \nabla^z)|_{\mathrm{Ran}(G)}W, \tag{4.11}$$

as operators on $\mathrm{Ran}(E)$, and similarly,

$$(\nabla^z \otimes \nabla^z)|_{\mathrm{Ran}(G)} = W(\nabla^z \otimes P^z)|_{\mathrm{Ran}(E)}W^*, \tag{4.12}$$

as operators on $\mathrm{Ran}(G)$.

We do not expect a result like $W^*(\nabla^z \otimes \nabla^z)W = \nabla^z \otimes P^z$ on the level of the whole space $\mathcal{H} \otimes \mathcal{H}$. As before, the domains won't agree. The best we can have is the following:

Proposition 4.7. *For any $z \in \mathbb{C}$, we have:*

$$W(\nabla^z \otimes P^z) \subseteq (\nabla^z \otimes \nabla^z)W \quad \textit{and} \quad W^*(\nabla^z \otimes \nabla^z) \subseteq (\nabla^z \otimes P^z)W^*.$$

Proof. Suppose $\zeta \in \mathcal{D}(\nabla^z \otimes P^z)$, and write it as $\zeta = \zeta_0 \oplus \zeta_1$, where $\zeta_0 \in \mathrm{Ker}(W)$, $\zeta_1 \in \mathrm{Ran}(E)$. Note that ζ_0 and ζ_1 must also be contained in $\mathcal{D}(\nabla^z \otimes P^z)$. This is because the restrictions of $\nabla^z \otimes P^z$ to the subspaces $\mathrm{Ker}(W)$ and $\mathrm{Ran}(E)$ are operators on them. Also because $(\nabla^z \otimes P^z)|_{\mathrm{Ker}(W)}$ is an operator on $\mathrm{Ker}(W)$, it is easy to see that $W(\nabla^z \otimes P^z)\zeta_0 = 0$. We thus have

$$\begin{aligned} W(\nabla^z \otimes P^z)\zeta &= W(\nabla^z \otimes P^z)(\zeta_0 + \zeta_1) = W(\nabla^z \otimes P^z)\zeta_1 \\ &= WW^*(\nabla^z \otimes \nabla^z)W\zeta_1 = (\nabla^z \otimes \nabla^z)W\zeta_1, \end{aligned}$$

using Equation (4.11). It follows that we have

$$W(\nabla^z \otimes P^z)(\zeta_0 \oplus \zeta_1) = (\nabla^z \otimes \nabla^z)W\zeta_1 = (\nabla^z \otimes \nabla^z)W(\zeta_0 \oplus \zeta_1),$$

true for every $\zeta = \zeta_0 \oplus \zeta_1 \in \mathcal{D}(\nabla^z \otimes P^z)$. This proves the result $W(\nabla^z \otimes P^z) \subseteq (\nabla^z \otimes \nabla^z)W$.

The proof for the second inclusion is similar, using $\mathcal{H} \otimes \mathcal{H} = \mathrm{Ker}(W^*) \oplus \mathrm{Ran}(G)$. □

We only have "$\subseteq$" in general, but the situation is better if $z \in \mathbb{C}$ is purely imaginary. If $z = it$, $t \in \mathbb{R}$, then the operators ∇^{it} and P^{it} become bounded, so the domain $\mathcal{D}(\nabla^{it} \otimes P^{it})$ becomes the whole space. Similar also for $\mathcal{D}(\nabla^{it} \otimes \nabla^{it})$. This gives us the following:

Proposition 4.8. *Let $t \in \mathbb{R}$. The following results hold on the whole space $\mathcal{H} \otimes \mathcal{H}$:*

1. $(\nabla^{it} \otimes \nabla^{it})W(\nabla^{-it} \otimes P^{-it}) = W$*,*
2. $(\nabla^{it} \otimes 1)W(\nabla^{-it} \otimes 1) = (1 \otimes \nabla^{-it})W(1 \otimes P^{it})$*.*

Proof. Since ∇^{it} and P^{it} are bounded operators, there is no issue with their domains. We already know from Proposition 4.7 that $W(\nabla^z \otimes P^z) \subseteq (\nabla^z \otimes \nabla^z)W$, in general. So when z is purely imaginary, the "$\subseteq$" becomes "$=$", and (1) follows.

(2) is an immediate consequence of (1). □

As a consequence of Proposition 4.8, we are now ready to resolve our question on our modular automorphism group. For this, consider $\omega \in \mathcal{B}(\mathcal{H})_*$ and let $t \in \mathbb{R}$. Apply $\mathrm{id} \otimes \omega$ to the result (2) of Proposition 4.8. Then we have

$$\nabla^{it}(\mathrm{id} \otimes \omega)(W)\nabla^{-it} = (\mathrm{id} \otimes \theta)(W),$$

where $\theta \in \mathcal{B}(\mathcal{H})_*$ is such that $\theta(X) = \omega(\nabla^{-it} X P^{-it})$. for $X \in \mathcal{B}(\mathcal{H})$.

As elements of the form $(\mathrm{id} \otimes \omega)(W)$, $\omega \in \mathcal{B}(\mathcal{H})_*$, generate the C^*-algebra A, the observation above shows that for any $t \in \mathbb{R}$, we have $\nabla^{it} a \nabla^{-it} \in A$, for any $a \in A$. We can also observe that $t \mapsto \nabla^{it} a \nabla^{-it}$ is norm-continuous.

We can thus justify the following:

Definition 4.9. Define the norm-continuous one-parameter group $\sigma = (\sigma_t)$ on the C^*-algebra A, by

$$\sigma_t(x) = \nabla^{it} a \nabla^{-it},$$

for all $t \in \mathbb{R}$, $a \in A$.

The one-parameter group (σ_t) is a restriction of the modular automorphism group $(\tilde{\sigma}_t)$ for the n.s.f. weight $\tilde{\varphi}$. With (σ_t), the faithful lower semi-continuous weight φ becomes a KMS weight on A. Its KMS properties are inherited from the properties of $\tilde{\varphi}$. In particular, we have $\varphi \circ \sigma_t = \varphi$; and for any $x \in \mathcal{D}(\sigma_{\frac{i}{2}})$, we have $\varphi(x^* x) = \varphi\big(\sigma_{\frac{i}{2}}(x)\sigma_{\frac{i}{2}}(x)^*\big)$.

With the construction of the KMS weight φ, it is not difficult to construct ψ also as a KMS weight, by letting $\psi = \varphi(\,\cdot\,\delta)$, where δ is the positive operator from Proposition 4.1.

4.3. **The C^*-algebraic locally compact quantum groupoid.** In [10], [11], Van Daele and the author developed a C^*-algebraic framework of a class of C^*-algebraic locally compact quantum groupoids (*quantum groupoids of separable type*). The definition is given below (see Definition 4.8 of [10] and Definition 1.2 of [11]):

Definition 4.10. The data $(A, \Delta, E, B, \nu, \varphi, \psi)$ defines a *locally compact quantum groupoid of separable type*, if

- A is a C^*-algebra.
- $\Delta : A \to M(A \otimes A)$ is a comultiplication on A.
- B is a non-degenerate C^*-subalgebra of $M(A)$.
- ν is a KMS weight on B.
- E is the canonical idempotent of (A, Δ). That is,
  (1) $\Delta(A)(A \otimes A)$ is dense in $E(A \otimes A)$ and $(A \otimes A)\Delta(A)$ is dense in $(A \otimes A)E$;
  (2) there exists a C^*-subalgebra $C \cong B^{\mathrm{op}}$ contained in $M(A)$, with a *-anti-isomorphism $R = R_{BC} : B \to C$, so that $E \in M(B \otimes C)$ and the triple (E, B, ν) forms a separability triple;

(3) $E \otimes 1$ and $1 \otimes E$ commute, and we have:
$$(\mathrm{id} \otimes \Delta)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1) = (\Delta \otimes \mathrm{id})(E).$$

- φ is a KMS weight, and is left invariant.
- ψ is a KMS weight, and is right invariant.
- There exists a (unique) one-parameter group of automorphisms $(\theta_t)_{t \in \mathbb{R}}$ of B such that $\nu \circ \theta_t = \nu$ and that $\sigma_t^{\varphi}|_B = \theta_t$, $\forall t \in \mathbb{R}$.

Remark. We will refer the details to the main papers. For instance, the notion of the canonical idempotent is summarized in Definition 3.7 of [10].

This definition is similar, but different from that of *measured quantum groupoids*, in the von Neumann algebra setting [17], [5]. The von Neumann algebra setting may be a bit more general, which is rather related to the algebraic framework of multiplier Hopf algebroids [23]. There are some subtle differences between the two locally compact frameworks.

Let us verify the constructions we carried so far, starting from a purely algebraic, weak multiplier Hopf *-algebra with a faithful integral (see Definition 1.6), without any additional conditions other than the quasi-invariance Assumption (§4.1), indeed gives us a C^*-algebraic quantum groupoid of separable type, as in Definition 4.10 above.

Our C^*-algebra was defined in Definition 3.10, extending the *-algebra $\mathcal{A}$. The *comultiplication* $\Delta : A \to M(A \otimes A)$ was given in Definition 3.11 and Proposition 3.15.

The base C^*-algebras B and C were defined in Definition 2.3. They are equipped with KMS weights ν and μ, respectively, which actually extends the distinguished linear functionals at the *-algebra level (see Proposition 2.6). There exists a C^*-anti-isomorphism $R : B \to C$, while the canonical idempotent E at the *-algebra level extends to the *separability idempotent* $E \in M(B \otimes C)$. See Proposition 2.7.

The idempotent E was further shown to satisfy additional properties, making it a valid *canonical idempotent* at the C^*-level. See Proposition 3.16.

Extending the left integral φ and the right integral $\psi = \varphi \circ S$ at the *-algebra level to the C^*-algebra level was a bit tricky, but was carried out in §4.2. We first needed to introduce the *quasi-invariance* Assumption (see §4.1), which allowed us to first prove that the modular automorphism groups associated with the left and right integrals commute, and that the modular element is positive. These results helped us establish that a KMS weight φ can be constructed at the C^*-algebra level, extending the left integral. See Definition 4.9. Once φ is constructed, the construction of another KMS weight ψ can be done quickly, knowing $\psi = \varphi(\,\cdot\,\delta)$.

The left invariance property of φ and the right invariance property of ψ are as follows:

- For any $a \in \mathfrak{M}_\varphi$, we have $\Delta a \in \overline{\mathfrak{M}}_{\mathrm{id} \otimes \varphi}$ and $(\mathrm{id} \otimes \varphi)(\Delta a) \in M(C)$.
- For any $a \in \mathfrak{M}_\psi$, we have $\Delta a \in \overline{\mathfrak{M}}_{\psi \otimes \mathrm{id}}$ and $(\psi \otimes \mathrm{id})(\Delta a) \in M(B)$.

As φ and ψ are well-defined on $\mathcal{A} = \pi(\mathcal{A})$, which is dense in A, these properties just immediately carry over from the invariance properties at the *-algebra level.

Meanwhile, one can notice an additional condition about the restriction of σ to the base C^*-algebra B, but that is an immediate consequence of none other than the *quasi-invariance*

Assumption we required. Note that $\sigma|_{\mathcal{B}}$ would play the role of the analytic generator for (θ_t). As pointed out in §4.1, such a condition is needed for a locally compact theory, as has been known already in the classical locally compact groupoid case ([19], [18]). What we are noticing is that for a purely algebraic object of a weak multiplier Hopf *-algebra to allow a construction of a C^*-algebraic quantum groupoid, some form of the quasi-invariance property is required even at the algebra level.

Summarizing, we have the following conclusion:

Theorem 4.11. *Let $(\mathcal{A}, \Delta, E)$ be a weak multiplier Hopf *-algebra with a single faithful integral, as in Definition 1.6. With the quasi-invariance Assumption (as in §4.1), we can construct from it a C^*-algebraic quantum groupoid of separable type, in the sense of* [10]*,* [11]*.*

Now that the construction is done, we can take advantage of the already-developed theory in [11]. Among the results include the description of the antipode map S as a polar decomposition (Definition 4.25 and Proposition 4.26 in [11]), as well as some alternative characterizations of the antipode, more relations between the base algebras B, C and the total algebra A.

On the other hand, we did not pursue the duality aspect in this paper (see §1.6). Ideally, it would make the picture complete if one can confirm that the C^*-extension of the dual weak multiplier Hopf *-algebra $(\widehat{\mathcal{A}}, \widehat{\Delta})$ is the dual in the C^*-context of the C^*-algebraic quantum groupoid. We will postpone that project to a future occasion, after the paper on the duality theory for the C^*-algebraic quantum groupoids ([12], in preparation) is finished.

5. Appendix: The modular element at the *-algebra level

In this Appendix, we gather some purely algebraic results. As in §1.5, we assume the existence of a single faithful positive left integral φ, and take $\psi = \varphi \circ S$, a faithful positive right integral.

In the purely algebraic setting, we noted in Propositions 1.9 and 1.10 the existence of the modular automorphisms σ and σ' for φ and ψ, respectively, and also the existence of an invertible element $\delta \in M(\mathcal{A})$, called the *modular element*. The modular element behaves like a modular function in the classical setting, and acts like a Radon–Nikodym derivative for the functionals φ and ψ, such that $\psi(x) = \varphi(x\delta)$, for $x \in \mathcal{A}$.

In this Appendix, we gather some results on the modular automorphisms σ, σ', and the modular element δ. While what appear below are all purely algebraic results, and some results are likely already known, the author could not find any reference for some of these results (especially regarding δ), and some results here may be new. As such, unlike in §1, all the proofs are given here.

5.1. Relationships between σ, σ' and the antipode S.

Let σ, σ' be the modular automorphisms for φ and ψ, respectively (see Proposition 1.9). Recall also that $\sigma' = S^{-1} \circ \sigma^{-1} \circ S$.

Here are some results regarding their restrictions to the level of the base algebras:

Proposition 5.1. (1) *The restriction of σ to $\mathcal{C}$ leaves $\mathcal{C}$ invariant, and we have:*

$$\sigma|_{\mathcal{C}} = S^2|_{\mathcal{C}} = S_{\mathcal{B}} \circ S_{\mathcal{C}} = \sigma^{\mu}.$$

(2) *The restriction of σ' to $\mathcal{B}$ leaves $\mathcal{B}$ invariant, and we have:*

$$\sigma'|_{\mathcal{B}} = S^{-2}|_{\mathcal{B}} = S_{\mathcal{B}}^{-1} \circ S_{\mathcal{C}}^{-1} = \sigma^{\nu}.$$

(3) *We have: $\mu \circ \sigma|_{\mathcal{C}} = \mu$ and $\nu \circ \sigma'|_{\mathcal{B}} = \nu$.*

Proof. (1). Let $y \in \mathcal{C}$ and let $a \in \mathcal{A}$ be arbitrary. Note that

$$\begin{aligned}\varphi(ya) &= \mu\big((\mathrm{id}\otimes\varphi)(\Delta(ya))\big) = \mu\big((\mathrm{id}\otimes\varphi)((y\otimes 1)(\Delta a))\big) = \mu\big(y(\mathrm{id}\otimes\varphi)(\Delta a)\big) \\ &= \mu\big((\mathrm{id}\otimes\varphi)(\Delta a)\sigma^{\mu}(y)\big) = \mu\big((\mathrm{id}\otimes\varphi)((\Delta a)(\sigma^{\mu}(y)\otimes 1))\big) \\ &= \mu\big((\mathrm{id}\otimes\varphi)(\Delta(a\sigma^{\mu}(y)))\big) = \varphi\big(a\sigma^{\mu}(y)\big).\end{aligned}$$

We used here the result of Proposition 1.7, and the fact that $\Delta y = (y \otimes 1)E = E(y \otimes 1)$ and $\Delta\big(\sigma^{\mu}(y)\big) = \big(\sigma^{\mu}(y) \otimes 1\big)E = E\big(\sigma^{\mu}(y) \otimes 1\big)$, because $y, \sigma^{\mu}(y) \in \mathcal{C}$ (Proposition 1.4).

Meanwhile, note that $\varphi(ya) = \varphi\big(a\sigma(y)\big)$. Combining the two observations, we see that

$$\varphi\big(a\sigma(y)\big) = \varphi\big(a\sigma^{\mu}(y)\big).$$

Since φ is faithful and since the result is true for any $a \in \mathcal{A}$, this shows that $\sigma(y) = \sigma^{\mu}(y)$, for all $y \in \mathcal{C}$, that is, $\sigma|_{\mathcal{C}} = \sigma^{\mu}$. We already know from §1.3 that $\sigma^{\mu} = S_{\mathcal{B}} \circ S_{\mathcal{C}}$. We also know that $S|_{\mathcal{B}} = S_{\mathcal{B}}$ and $S|_{\mathcal{C}} = S_{\mathcal{C}}$ (Proposition 1.6), so we have: $\sigma|_{\mathcal{C}} = \sigma^{\mu} = S^2|_{\mathcal{C}}$.

(2). The proof for the restriction $\sigma'|_{\mathcal{B}} = \sigma^{\nu} = S^{-2}|_{\mathcal{B}}$ is similar.

(3). As it is known that $\mu \circ \sigma^{\mu} = \mu$ and $\nu \circ \sigma^{\nu} = \nu$, the results follow immediately from (1) and (2). □

The following results show how σ and σ' behave when the comultiplication map is applied:

Proposition 5.2. *We have:*

(1) $\Delta\big(\sigma(a)\big) = (S^2 \otimes \sigma)(\Delta a)$, *for all $a \in \mathcal{A}$,*
(2) $\Delta\big(\sigma'(a)\big) = (\sigma' \otimes S^{-2})(\Delta a)$, *for all $a \in \mathcal{A}$.*

Proof. (1). Let $a, x \in \mathcal{A}$ be arbitrary. By a characterization of the antipode S given in Proposition 1.8 (2), we have:

$$\begin{aligned}(\mathrm{id}\otimes\varphi)\big((1\otimes x)\Delta((\sigma(a))\big) &= (\mathrm{id}\otimes\varphi)\big(S((\Delta x)(1\otimes\sigma(a)))\big) = S\big((\mathrm{id}\otimes\varphi)((\Delta x)(1\otimes\sigma(a)))\big) \\ &= S\big((\mathrm{id}\otimes\varphi)((1\otimes a)(\Delta x))\big) = S^2\big((\mathrm{id}\otimes\varphi)((\Delta a)(1\otimes x))\big) \\ &= S^2\big((\mathrm{id}\otimes\varphi)((1\otimes x)(\mathrm{id}\otimes\sigma)(\Delta a))\big) \\ &= (\mathrm{id}\otimes\varphi)\big((1\otimes x)(S^2\otimes\sigma)(\Delta a)\big).\end{aligned}$$

Since φ is faithful and since $x \in \mathcal{A}$ is arbitrary, this shows that $\Delta\big(\sigma(a)\big) = (S^2 \otimes \sigma)(\Delta a)$, for all $a \in \mathcal{A}$.

(2). Proof for $\Delta\big(\sigma'(a)\big) = (\sigma' \otimes S^{-2})(\Delta a)$, $\forall a \in \mathcal{A}$, is similar, using an alternative characterization of S, namely, $S\big((\psi\otimes\mathrm{id})((a\otimes 1)(\Delta x))\big) = (\psi\otimes\mathrm{id})\big((\Delta a)(x\otimes 1)\big)$, given in Proposition 1.8 (3). □

Corollary. (1) *For any $a \in \mathcal{A}$, we have:*
$$\Delta\big(\sigma^{-1}(a)\big) = (S^{-2} \otimes \sigma^{-1})(\Delta a) \quad \text{and} \quad \Delta\big([\sigma']^{-1}(a)\big) = ([\sigma']^{-1} \otimes S^2)(\Delta a),$$
(2) *We have:*
$$\Delta\big(\sigma(S^{-2}(a))\big) = \big(\mathrm{id} \otimes (\sigma \circ S^{-2})\big)(\Delta a), \quad \forall a \in \mathcal{A}.$$

Proof. (1). This is an immediate consequence of the previous proposition. Note that from $\Delta\big(\sigma(x)\big) = (S^2 \otimes \sigma)(\Delta x)$, we have $(S^{-2} \otimes \sigma^{-1})\big(\Delta(\sigma(x))\big) = \Delta x$. Letting $x = \sigma^{-1}(a)$, for $a \in \mathcal{A}$, we have: $(S^{-2} \otimes \sigma^{-1})(\Delta a) = \Delta\big(\sigma^{-1}(a)\big)$. Similar proof for the second result.

(2). By Proposition 1.8 (5), applied twice, we know $\Delta\big(S^{-2}(a)\big) = (S^{-2} \otimes S^{-2})(\Delta a)$. Combine this result with (1) of Proposition 5.2. ☐

5.2. **Some results on the modular element.** In Proposition 1.10, we noted the existence of a unique invertible element $\delta \in M(\mathcal{A})$ such that $\psi(a) = (\varphi \circ S)(a) = \varphi(a\delta)$, for all $a \in \mathcal{A}$. See Proposition 1.9 of [33]. In what follows, we will gather some additional results about δ.

Let us begin with a lemma, which gives a similar result for $\varphi \circ S^{-1}$, also a right invariant functional:

Lemma 5.3. *We have: $(\varphi \circ S^{-1})(a) = \varphi(\delta^* a)$, for all $a \in \mathcal{A}$.*

Proof. Let $a \in \mathcal{A}$ be arbitrary. Recall that $S\big(S(a)^*\big)^* = a$, so in particular, we have: $S^{-1}(a)^* = S(a^*)$. As φ is a positive functional, we thus have:
$$(\varphi \circ S^{-1})(a) = \varphi\big(S^{-1}(a)\big) = \overline{\varphi\big(S^{-1}(a)^*\big)} = \overline{\varphi\big(S(a^*)\big)} = \overline{\varphi(a^*\delta)} = \overline{\varphi\big((\delta^* a)^*\big)} = \varphi(\delta^* a).$$
☐

In the below is a result showing what happens when the antipode map S, naturally extended to the multiplier algebra level, is applied to δ:

Proposition 5.4. *We have: $S(\delta) = (\delta^*)^{-1}$.*

Proof. Let $x \in \mathcal{A}$ be arbitrary. As S is anti-multiplicative, we have:
$$\varphi(x) = (\varphi \circ S)\big(S^{-1}(x)\big) = \varphi\big(S^{-1}(x)\delta\big) = (\varphi \circ S^{-1})\big(S(\delta)x\big) = \varphi\big(\delta^* S(\delta)x\big),$$
where we used the result of Lemma 5.3.

As φ is faithful and since $x \in \mathcal{A}$ is arbitrary, it follows that $\delta^* S(\delta) = 1$. So we have: $S(\delta)^{-1} = \delta^*$ and $S(\delta) = (\delta^*)^{-1}$. ☐

Next, we gather some results under the modular automorphisms σ and σ':

Proposition 5.5. *Under the modular automorphisms σ and σ', which can be naturally extended to the multiplier algebra level, we have:*

(1) $\sigma^{-1}(\delta) = [\sigma']^{-1}(\delta)$;
(2) $\sigma\big([\sigma']^{-1}(\delta)\big) = \delta$ *and* $\sigma'\big(\sigma^{-1}(\delta)\big) = \delta$;
(3) $\sigma'(a) = \delta\sigma(a)\delta^{-1}$ *and* $\sigma(a) = \delta^{-1}\sigma'(a)\delta$, *for any $a \in \mathcal{A}$;*
(4) $\sigma\big([\sigma']^{-1}(a)\big) = \delta^{-1}a\delta$, *for any $a \in \mathcal{A}$;*

Proof. (1). Let $a \in \mathcal{A}$. We have:

$$\psi(a) = (\varphi \circ S)(a) = \varphi(a\delta) = \varphi\big(\sigma^{-1}(\delta)a\big).$$

Meanwhile, we have:

$$\psi(a) = \psi(a\delta^{-1}\delta) = \psi\big([\sigma']^{-1}(\delta)a\delta^{-1}\big) = (\varphi \circ S)\big([\sigma']^{-1}(\delta)a\delta^{-1}\big) = \varphi\big([\sigma']^{-1}(\delta)a\big).$$

Compare the two equations. Since φ is faithful and since $a \in \mathcal{A}$ is arbitrary, this shows that $\sigma^{-1}(\delta) = [\sigma']^{-1}(\delta)$.

(2). As a consequence of (1), we have: $\sigma\big([\sigma']^{-1}(\delta)\big) = \delta$ and $\sigma'\big(\sigma^{-1}(\delta)\big) = \delta$.

(3). Let $a, x \in \mathcal{A}$. We have:

$$\psi(ax) = (\varphi \circ S)(ax) = \varphi(ax\delta) = \varphi\big(x\delta\sigma(a)\big).$$

Meanwhile,

$$\psi(ax) = \psi\big(x\sigma'(a)\big) = \varphi\big(x\sigma'(a)\delta\big).$$

Compare the two expressions. Since φ is faithful and since $x \in \mathcal{A}$ is arbitrary, this shows that $\delta\sigma(a) = \sigma'(a)\delta$. Or, equivalently, $\sigma'(a) = \delta\sigma(a)\delta^{-1}$ and $\sigma(a) = \delta^{-1}\sigma'(a)\delta$, true for any $a \in \mathcal{A}$.

(4). We may consider $x = [\sigma']^{-1}(a)$, for $a \in \mathcal{A}$, and apply (3). Then we have: $\sigma\big([\sigma']^{-1}(a)\big) = \sigma(x) = \delta^{-1}\sigma'(x)\delta = \delta^{-1}a\delta$. □

The following result is a consequences of the left/right invariance of ϕ and ψ. See Proposition 1.5 for the definitions of F_1, F_2, F_3, F_4, which are elements in $M(\mathcal{A} \odot \mathcal{A})$.

Proposition 5.6. *Let $a \in \mathcal{A}$. We have:*

$$(\varphi \otimes \mathrm{id})(\Delta a) = (\mathrm{id} \otimes \varphi)\big(F_1(1 \otimes a)\big)\delta = \delta^*(\mathrm{id} \otimes \varphi)\big((1 \otimes a)F_3\big).$$

Here, $F_1 = (\mathrm{id} \otimes S)(E) \in M(\mathcal{A} \odot \mathcal{A})$ and $F_3 = (\mathrm{id} \otimes S^{-1})(E) \in M(\mathcal{A} \odot \mathcal{A})$.

Proof. Let $a, x \in \mathcal{A}$. Observe:

$$\begin{aligned}
\varphi\big(x(\varphi \otimes \mathrm{id})(\Delta a)\big) &= (\varphi \otimes \varphi)\big((1 \otimes x)(\Delta a)\big) = \varphi\big((\mathrm{id} \otimes \varphi)((1 \otimes x)(\Delta a))\big) \\
&= \varphi\big(S((\mathrm{id} \otimes \varphi)((\Delta x)(1 \otimes a)))\big) = \psi\big((\mathrm{id} \otimes \varphi)((\Delta x)(1 \otimes a))\big) \\
&= \varphi\big((\psi \otimes \mathrm{id})(\Delta x)a\big) = \varphi\big((\psi \otimes \mathrm{id})((x \otimes 1)F_1(1 \otimes a))\big) \\
&= \psi\big(x(\mathrm{id} \otimes \varphi)(F_1(1 \otimes a))\big) \\
&= \varphi\big(x(\mathrm{id} \otimes \varphi)(F_1(1 \otimes a))\delta\big).
\end{aligned}$$

Third equality is using the characterization of the antipode given in Proposition 1.8; We used $\psi = \varphi \circ S$ in the fourth and the eighth equalities; The sixth equity is a consequence of the right invariance of ψ, as in Proposition 1.5.

Since $x \in \mathcal{A}$ is arbitrary and since φ is faithful, this shows that

$$(\varphi \otimes \mathrm{id})(\Delta a) = (\mathrm{id} \otimes \varphi)(F_1(1 \otimes a))\delta, \qquad \text{for all } a \in \mathcal{A}.$$

Taking the adjoint, we have: $(\varphi \otimes \mathrm{id})(\Delta(a^*)) = \delta^*(\mathrm{id} \otimes \varphi)((1 \otimes a^*)F_1^*)$, $\forall a \in \mathcal{A}$, since ϕ is a positive functional. Or, equivalently, $(\varphi \otimes \mathrm{id})(\Delta a) = \delta^*(\mathrm{id} \otimes \varphi)((1 \otimes a)F_1^*)$. Note here that since

$F_1 = (\mathrm{id} \otimes S)(E)$ and since $S(a)^* = S^{-1}(a^*)$ for any $a \in \mathcal{A}$, we have $F_1^* = (\mathrm{id} \otimes S^{-1})(E^*) = (\mathrm{id} \otimes S^{-1})(E) = F_3$. In other words, we have:

$$(\varphi \otimes \mathrm{id})(\Delta a) = \delta^*(\mathrm{id} \otimes \varphi)((1 \otimes a)F_3), \quad \text{for all } a \in \mathcal{A}.$$

□

Eventually, we wish to find what $\Delta(\delta)$ is. At present that is not clear, but the following proposition will help us in that direction.

Proposition 5.7. *Let $p, q \in \mathcal{A}$, and $\delta \in M(\mathcal{A})$ be the modular element. We have:*

$$(\varphi \otimes \varphi)\big((p \otimes q)\Delta(\delta)\big) = (\psi \otimes \psi)\big((p \otimes q)E\big),$$

$$(\varphi \otimes \varphi)\big(\Delta(\delta^*)(p \otimes q)\big) = (\psi \otimes \psi)\big(E(p \otimes q)\big).$$

Proof. The second result can be obtained immediately from the first one by taking the adjoint. So let us just prove the first result.

Let $a, b \in \mathcal{A}$ be arbitrary. By Proposition 5.6, we have:

$$(\varphi \otimes \varphi)\big((1 \otimes a)(\Delta b)\Delta(\delta)\big) = \varphi\big(a(\varphi \otimes \mathrm{id})(\Delta(b\delta))\big) = \varphi\big(a(\mathrm{id} \otimes \varphi)(F_1(1 \otimes b\delta))\delta\big).$$

Since $\varphi(\,\cdot\,\delta) = \varphi \circ S = \psi$, this becomes:

$$(\varphi \otimes \varphi)\big((1 \otimes a)(\Delta b)\Delta(\delta)\big) = (\psi \otimes \psi)\big((a \otimes 1)F_1(1 \otimes b)\big) = (\psi \otimes \psi)\big((1 \otimes a)\varsigma F_1(b \otimes 1)\big),$$

where ς denotes taking the flip on $M(\mathcal{A} \odot \mathcal{A})$.

Note that $\varsigma F_1 = \varsigma\big((\mathrm{id} \otimes S)E\big) = (S \otimes \mathrm{id})(\varsigma E)$. Since $\varsigma E = (S^{-1} \otimes S^{-1})(E)$, we thus have $\varsigma F_1 = (\mathrm{id} \otimes S^{-1})(E) = F_3$. Apply this result to the above, then we have:

$$(\varphi \otimes \varphi)\big((1 \otimes a)(\Delta b)\Delta(\delta)\big) = (\psi \otimes \psi)\big((1 \otimes a)F_3(b \otimes 1)\big) = \psi\big(a(\psi \otimes \mathrm{id})(F_3(b \otimes 1))\big).$$

Use here the right invariance result $(\psi \otimes \mathrm{id})\big(F_3(b \otimes 1)\big) = (\psi \otimes \mathrm{id})(\Delta b)$, as in Proposition 1.5. Then we obtain

$$(\varphi \otimes \varphi)\big((1 \otimes a)(\Delta b)\Delta(\delta)\big) = (\psi \otimes \psi)\big((1 \otimes a)(\Delta b)\big). \tag{5.1}$$

This is true for any $a, b \in \mathcal{A}$. But note that the elements of the form $(1 \otimes a)(\Delta b)$ spans $(\mathcal{A} \odot \mathcal{A})E$. Therefore, the observation made in Equation (5.1) is equivalent to saying

$$(\varphi \otimes \varphi)\big((p \otimes q)\Delta(\delta)\big) = (\psi \otimes \psi)\big((p \otimes q)E\big), \quad \text{for all } p, q \in \mathcal{A}. \tag{5.2}$$

□

Before we find $\Delta(\delta)$, we first prove the following lemma, which is also a consequence of the right invariance of ψ.

Lemma 5.8. *Let $p, q \in \mathcal{A}$. Then we have:*

$$(\psi \otimes \mathrm{id})\big((p \otimes q)E\big) = qS^{-1}\big((\psi \otimes \mathrm{id})(\Delta p)\big).$$

Proof. Let $\omega \in \mathcal{A}^*$ be arbitrary. Then:

$$\omega\big((\psi \otimes \mathrm{id})((p \otimes q)E)\big) = \psi\big(p(\mathrm{id} \otimes \omega)((1 \otimes q)E)\big) = \psi(px),$$

where $x = (\mathrm{id} \otimes \omega)\big((1 \otimes q)E\big)$. As E is full, note that such elements span the base algebra $\mathcal{B}$. By Proposition 1.7, we have:

$$\psi(px) = \nu\big((\psi \otimes \mathrm{id})(\Delta(px))\big) = \nu\big((\psi \otimes \mathrm{id})((\Delta p)(1 \otimes x))\big) = \nu\big((\psi \otimes \mathrm{id})(\Delta p)x\big).$$

Here, we used the fact that since $x \in \mathcal{B}$, we have $\Delta x = E(1 \otimes x)$ by Proposition 1.4, from which it follows that $\Delta(px) = (\Delta p)(\Delta x) = (\Delta p)E(1 \otimes x) = (\Delta p)(1 \otimes x)$.

Note that by the right invariance of ψ, we also have $(\psi \otimes \mathrm{id})(\Delta p) \in M(\mathcal{B})$. Therefore, we have:

$$\omega\big((\psi \otimes \mathrm{id})((p \otimes q)E)\big) = \psi(px) = \nu\big((\psi \otimes \mathrm{id})(\Delta p)x\big) = \omega\big(q(\nu \otimes \mathrm{id})([(\psi \otimes \mathrm{id})(\Delta p) \otimes 1]E)\big).$$

As $\omega \in \mathcal{A}^*$ is arbitrary, this means that

$$(\psi \otimes \mathrm{id})\big((p \otimes q)E\big) = q(\nu \otimes \mathrm{id})\big([(\psi \otimes \mathrm{id})(\Delta p) \otimes 1]E\big).$$

As $(\psi \otimes \mathrm{id})(\Delta p) \in M(\mathcal{B})$, using the map $S_{\mathcal{C}}$, as characterized in Equation (1.9), we have: $\big[(\psi \otimes \mathrm{id})(\Delta p) \otimes 1\big]E = \big[(1 \otimes S_{\mathcal{C}}^{-1}((\psi \otimes \mathrm{id})(\Delta p))\big]E$. Note also that $(\nu \otimes \mathrm{id})(E) = 1$. Putting all these together, we thus obtain:

$$(\psi \otimes \mathrm{id})\big((p \otimes q)E\big) = qS_{\mathcal{C}}^{-1}\big((\psi \otimes \mathrm{id})(\Delta p)\big) = qS^{-1}\big((\psi \otimes \mathrm{id})(\Delta p)\big),$$

as S^{-1} extends $S_{\mathcal{C}}^{-1}$. □

The next proposition provides us with some characterizations of $\Delta(\delta)$ and $\Delta(\delta^*)$:

Proposition 5.9. *Let $\delta \in M(\mathcal{A})$ be the modular element. We have:*

- $\Delta(\delta) = \big(\delta \otimes S^{-1}(\delta^{-1})\big)E = \big(\delta \otimes S^{-2}(\delta^*)\big)E = E\big(\delta \otimes S^{-2}(\delta^*)\big)E.$
- $\Delta(\delta^*) = E\big(\delta^* \otimes S^2(\delta)\big) = E\big(\delta^* \otimes S^2(\delta)\big)E$
- $\Delta(\delta^*) = (\delta \otimes \delta^*)E = E(\delta \otimes \delta^*)E$
- $\Delta(\delta) = E(\delta^* \otimes \delta) = E(\delta^* \otimes \delta)E$
- $\Delta(\delta) = E(\delta \otimes \delta)E$
- $\Delta(\delta^*) = E(\delta^* \otimes \delta^*)E$

Proof. (1). Recall from Proposition 5.7 that $(\varphi \otimes \varphi)\big((p \otimes q)\Delta(\delta)\big) = (\psi \otimes \psi)\big((p \otimes q)E\big)$, for any $p, q \in \mathcal{A}$. By Lemma 5.8, we have:

$$(\varphi \otimes \varphi)\big((p \otimes q)\Delta(\delta)\big) = \psi\big((\psi \otimes \mathrm{id})((p \otimes q)E)\big) = \psi\big(qS^{-1}((\psi \otimes \mathrm{id})(\Delta p))\big). \tag{5.3}$$

Meanwhile, note that

$$\begin{aligned}(\psi \otimes \mathrm{id})(\Delta p) &= (\psi \otimes \mathrm{id})\big(F_3(p \otimes 1)\big) = (\varphi \otimes \mathrm{id})\big(F_3(p\delta \otimes 1)\big) \\ &= (\mathrm{id} \otimes \varphi)\big(\varsigma F_3(1 \otimes p\delta)\big) = (\mathrm{id} \otimes \varphi)\big(F_1(1 \otimes p\delta)\big) \\ &= (\varphi \otimes \mathrm{id})\big(\Delta(p\delta)\big)\delta^{-1} \\ &= \delta^*(\mathrm{id} \otimes \varphi)\big((1 \otimes p\delta)F_3\big)\delta^{-1}.\end{aligned} \tag{5.4}$$

The first equality is the right invariance property of ψ (Proposition 1.5); the second used $\psi = \varphi(\,\cdot\,\delta)$; in the third and fourth, the flip map is applied, together with the observation earlier (see proof of Proposition 5.7) that $\varsigma F_1 = F_3$; and the fifth and sixth used the result of Proposition 5.6.

Insert into Equation (5.3) the result of Equation (5.4). Then we have:

$$\begin{aligned}
&(\varphi\otimes\varphi)\big((p\otimes q)\Delta(\delta)\big) = \psi\big(qS^{-1}((\psi\otimes\mathrm{id})(\Delta p))\big) \\
&= \psi\big(qS^{-1}(\delta^*(\mathrm{id}\otimes\varphi)[(1\otimes p\delta)F_3]\delta^{-1})\big) = \varphi\big(qS^{-1}(\delta^*(\mathrm{id}\otimes\varphi)[(1\otimes p\delta)F_3]\delta^{-1})\delta\big) \\
&= \varphi\big(qS^{-1}(S(\delta)\delta^*(\mathrm{id}\otimes\varphi)[(1\otimes p\delta)F_3]\delta^{-1})\big) = \varphi\big(qS^{-1}((\mathrm{id}\otimes\varphi)[(1\otimes p\delta)F_3]\delta^{-1})\big) \\
&= \varphi\big(qS^{-1}(\delta^{-1})S^{-1}((\mathrm{id}\otimes\varphi)[(1\otimes p\delta)F_3])\big). \qquad (5.5)
\end{aligned}$$

The fourth and sixth equalities used the anti-multiplicativity S^{-1}; and in the fifth, we used the result of Proposition 5.4, namely $S(\delta) = (\delta^*)^{-1}$.

Note that by applying the flip map and using again $\varsigma F_1 = F_3$, we have:

$$S^{-1}\big((\mathrm{id}\otimes\varphi)[(1\otimes p\delta)F_3]\big) = S^{-1}\big((\varphi\otimes\mathrm{id})[(p\delta\otimes 1)F_1]\big) = (\varphi\otimes\mathrm{id})\big[(p\delta\otimes 1)E\big], \qquad (5.6)$$

because $F_1 = (\mathrm{id}\otimes S)(E)$. Insert the result of Equation (5.6) into Equation (5.5), to obtain:

$$\begin{aligned}
(\varphi\otimes\varphi)\big((p\otimes q)\Delta(\delta)\big) &= \varphi\big(qS^{-1}(\delta^{-1})(\varphi\otimes\mathrm{id})[(p\delta\otimes 1)E]\big) \\
&= (\varphi\otimes\varphi)\big((p\delta\otimes qS^{-1}(\delta^{-1}))E\big) \\
&= (\varphi\otimes\varphi)\big((p\otimes q)(\delta\otimes S^{-1}(\delta^{-1}))E\big).
\end{aligned}$$

Here, note that φ is faithful and that $p, q \in \mathcal{A}$ are arbitrary, which means that we have

$$\Delta(\delta) = \big(\delta\otimes S^{-1}(\delta^{-1})\big)E.$$

Or $\Delta(\delta) = \big(\delta\otimes S^{-2}(\delta^*)\big)E$, by using the fact $\delta^* = S(\delta^{-1})$, as in Proposition 5.4. Meanwhile, noting that $\Delta(\delta) = E\Delta(\delta)$, we can also write this as

$$\Delta(\delta) = E\big(\delta\otimes S^{-2}(\delta^*)\big)E.$$

(2). From (1), we know $\Delta(\delta) = \big(\delta\otimes S^{-2}(\delta^*)\big)E$. Take the adjoint, to obtain $\Delta(\delta^*) = E\big(\delta^*\otimes S^2(\delta)\big)$, using the property of S. Also, as we should have $\Delta(\delta^*) = \Delta(\delta^*)E$, we can also write this as $\Delta(\delta^*) = E\big(\delta^*\otimes S^2(\delta)\big)E$.

(3). From (1), we saw $\Delta(\delta) = \big(\delta\otimes S^{-1}(\delta^{-1})\big)E$. As the comultiplication preserves multiplication and since $\Delta(1) = E$, we can see from this quickly that

$$\Delta(\delta^{-1}) = E\big(\delta^{-1}\otimes[S^{-1}(\delta^{-1})]^{-1}\big) = E\big(\delta^{-1}\otimes S^{-1}(\delta)\big).$$

As $S(\delta^{-1}) = \delta^*$, by Proposition 1.8 and by using the result we just obtained on $\Delta(\delta^{-1})$, we have:

$$\begin{aligned}
\Delta(\delta^*) = \Delta\big(S(\delta^{-1})\big) &= (S\otimes S)\Delta^{\mathrm{cop}}(\delta^{-1}) \\
&= (S\otimes S)\big(\varsigma E(S^{-1}(\delta)\otimes\delta^{-1})\big) \\
&= \big(\delta\otimes S(\delta^{-1})\big)(S\otimes S)(\varsigma E) = \big(\delta\otimes\delta^*\big)E.
\end{aligned}$$

Again, as we should have $\Delta(\delta^*) = E\Delta(\delta^*)$, we can also write this as $\Delta(\delta^*) = E\big(\delta\otimes\delta^*\big)E$.

(4). From (3), we saw $\Delta(\delta^*) = \big(\delta \otimes \delta^*\big)E$. Take the adjoint, to obtain: $\Delta(\delta) = E\big(\delta^* \otimes \delta\big)$. As before, this can be also written as $\Delta(\delta) = E\big(\delta^* \otimes \delta\big)E$.

(5). Consider $x = (\mathrm{id} \otimes \varphi)\big(F_1(1 \otimes a)\big) = \big(\mathrm{id} \otimes (\varphi \circ S)(S^{-1}(a) \cdot)\big)(E)$. It is an element in $\mathcal{B}$, and it is evident that such elements span $\mathcal{B}$. Note that since $x \in \mathcal{B}$, we have $\Delta x = E(1 \otimes x) = (1 \otimes x)E$, as noted in Proposition 1.4. We thus have: $(1 \otimes x)\Delta(\delta) = (1 \otimes x)E\Delta(\delta) = \Delta(x\delta)$. But note that by Proposition 5.6, we have:

$$x\delta = (\mathrm{id} \otimes \varphi)\big(F_1(1 \otimes a)\big)\delta = \delta^*(\mathrm{id} \otimes \varphi)\big((1 \otimes a)F_3\big) = \delta^*\tilde{x},$$

where $\tilde{x} = (\mathrm{id} \otimes \varphi)\big((1 \otimes a)F_3\big)$, also an element of $\mathcal{B}$, so we have $\delta(\tilde{x}) = E(1 \otimes \tilde{x}) = (1 \otimes \tilde{x})E$.

Combining these observations, we have:

$$\begin{aligned}(1 \otimes x)\Delta(\delta) &= \Delta(x\delta) = \Delta(\delta^*\tilde{x}) = \Delta(\delta^*)(1 \otimes \tilde{x}) \\ &= E(\delta \otimes \delta^*)E(1 \otimes \tilde{x}) = E(\delta \otimes \delta^*)(1 \otimes \tilde{x})E \\ &= E(\delta \otimes \delta^*\tilde{x})E = E(\delta \otimes x\delta)E = E(1 \otimes x)(\delta \otimes \delta)E \\ &= (1 \otimes x)E(\delta \otimes \delta)E,\end{aligned}$$

where we used $\Delta(\delta^*) = E(\delta \otimes \delta^*)E$, observed in (3) above.

As noted above, the elements $x = (\mathrm{id} \otimes \varphi)\big(F_1(1 \otimes a)\big)$ span $\mathcal{B}$. Note also that $\mathcal{B}$ is non-degenerate. So this result means that

$$\Delta(\delta) = E(\delta \otimes \delta)E.$$

(6). From $\Delta(\delta) = E(\delta \otimes \delta)E$, take the adjoint, to obtain $\Delta(\delta^*) = E(\delta^* \otimes \delta^*)E$. ☐

There is no reason to believe that δ is self-adjoint, so $\Delta(\delta)$ and $\Delta(\delta^*)$ can be different and we made a distinction so far. See next subsection for the case when δ is self-adjoint.

5.3. **Quasi-invariance assumption.** In Proposition 5.1, we observed that the restriction $\sigma|_{\mathcal{C}}$ leaves $\mathcal{C}$ invariant with $\mu \circ \sigma_{\mathcal{C}} = \mu$, and that $\sigma'|_{\mathcal{B}}$ leaves $\mathcal{B}$ invariant with $\nu \circ \sigma'|_{\mathcal{B}} = \nu$. On the other hand, we do not know whether similar properties hold for the restrictions $\sigma|_{\mathcal{B}}$ and $\sigma'|_{\mathcal{C}}$. There is no reason why they should. As such, let us make the following extra assumption:

Assumption (Quasi-invariance)**.** *We will assume that $\sigma|_{\mathcal{B}}$, the restriction of σ to the base algebra $\mathcal{B}$, leaves $\mathcal{B}$ invariant, and that $\nu \circ \sigma_{\mathcal{B}} = \nu$.*

Remark. It is possible to actually prove the invariance of $\mathcal{B}$ under $\sigma|_{\mathcal{B}}$ (see the paragraph following Proposition 1.9), but not the $\nu \circ \sigma|_{\mathcal{B}} = \nu$ result. We termed this assumption as "quasi-invariance", because this assumption indeed gives rise to the quasi-invariance property at the C^*-algebraic framework. See the remarks given later in the subsection, as well as §4.1. If this assumption holds, we can quickly show that analogous results hold for $\sigma'|_{\mathcal{C}}$, using the antipode S.

In the below is a lemma that results from having the above Assumption:

Lemma 5.10. *Under the above quasi-invariance Assumption, we have:*

(1) $\psi\big((\sigma \circ S^{-2})(a)\big) = \psi(a)$, *for all $a \in \mathcal{A}$.*

(2) $(\sigma' \circ \sigma \circ S^{-2})(a) = (\sigma \circ S^{-2} \circ \sigma')(a)$, *for all* $a \in \mathcal{A}$.

Proof. (1). Let $a \in \mathcal{A}$. By Proposition 1.7 and by Corollary of Proposition 5.2, we have:

$$\psi\big((\sigma \circ S^{-2})(a)\big) = \nu\big((\psi \otimes \mathrm{id})(\Delta((\sigma \circ S^{-2})(a)))\big) = \nu\big((\sigma \circ S^{-2})((\psi \otimes \mathrm{id})(\Delta a))\big).$$

Note that $(\psi\otimes\mathrm{id})(\Delta a) \in M(\mathcal{B})$, by the right invariance of ψ, So, we may regard the maps σ and S^{-2} as $\sigma|_{\mathcal{B}}$ and $S^{-2}|_{\mathcal{B}}$, respectively. As $\nu \circ \sigma|_{\mathcal{B}} = \nu$ (by Assumption) and $\nu \circ S^{-2}|_{\mathcal{B}} = \nu \circ \sigma^{\nu} = \nu$ (by Proposition 5.1), the above becomes:

$$\psi\big((\sigma \circ S^{-2})(a)\big) = \nu\big((\psi \otimes \mathrm{id})(\Delta a)\big) = \psi(a),$$

by Proposition 1.7.

(2). Let $a, x \in \mathcal{A}$ be arbitrary. We have:

$$\psi(ax) = \psi\big(x\sigma'(a)\big) = \psi\big((\sigma \circ S^{-2})(x\sigma'(a))\big) = \psi\big((\sigma \circ S^{-2})(x)(\sigma \circ S^{-2} \circ \sigma')(a)\big), \tag{5.7}$$

by (1) above and the fact that σ and S^{-2} are multiplicative.

On the other hand, we also have:

$$\begin{aligned} \psi(ax) &= \psi\big((\sigma \circ S^{-2})(ax)\big) = \psi\big((\sigma \circ S^{-2})(a)(\sigma \circ S^{-2})(x)\big) \\ &= \psi\big((\sigma \circ S^{-2})(x)(\sigma' \circ \sigma \circ S^{-2})(a)\big). \end{aligned} \tag{5.8}$$

Comparing Equations (5.7) and (5.8), since $x \in \mathcal{A}$ is arbitrary and since ψ is faithful, it follows that $(\sigma \circ S^{-2} \circ \sigma')(a) = (\sigma' \circ \sigma \circ S^{-2})(a)$, for all $a \in \mathcal{A}$. □

This lemma helps us prove the following commutativity results:

Proposition 5.11. *Under the quasi-invariance Assumption above, we have:*

(1) $\sigma \circ S^2 = S^2 \circ \sigma$
(2) $\sigma' \circ S^2 = S^2 \circ \sigma'$
(3) $\sigma' \circ \sigma = \sigma \circ \sigma'$

Proof. (1). Let $a \in \mathcal{A}$ be arbitrary. By Corollary following Proposition 5.2, and by using the fact that $\sigma \circ S^{-2} = [\sigma']^{-1} \circ \sigma \circ S^{-2} \circ \sigma'$ (from Lemma 5.10), we have:

$$\big(\mathrm{id} \otimes (\sigma \circ S^{-2})\big)(\Delta a) = \Delta\big(\sigma(S^{-2}(a))\big) = \Delta\big(([\sigma']^{-1} \circ \sigma \circ S^{-2} \circ \sigma')(a)\big).$$

Again by Corollary of Proposition 5.2, this becomes:

$$= \big([\sigma']^{-1} \otimes S^2\big)\big(\Delta((\sigma \circ S^{-2})(\sigma'(a)))\big) = \big([\sigma']^{-1} \otimes (S^2 \circ \sigma \circ S^{-2})\big)\big(\Delta(\sigma'(a))\big).$$

Note that $\Delta\big(\sigma'(a)\big) = (\sigma' \otimes S^{-2})(\Delta a)$, by Proposition 5.2. As a result, we thus obtain the following:

$$\big(\mathrm{id} \otimes (\sigma \circ S^{-2})\big)(\Delta a) = \big(\mathrm{id} \otimes (S^2 \circ \sigma \circ S^{-2} \circ S^{-2})\big)(\Delta a).$$

Since Δ is full, it follows that $\sigma \circ S^{-2} = S^2 \circ \sigma \circ S^{-2} \circ S^{-2}$. Since S^2 and S^{-2} are automorphisms on $\mathcal{A}$, we thus have: $S^2 \circ \sigma = \sigma \circ S^2$.

(2). Proof for $\sigma' \circ S^2 = S^2 \circ \sigma'$ is similar.

(3). Let $a \in \mathcal{A}$ be arbitrary. We have:

$$\begin{aligned}\Delta\big(\sigma(\sigma'(a))\big) &= (S^2 \otimes \sigma)\big(\Delta(\sigma'(a))\big) \\ &= \big((S^2 \circ \sigma') \otimes (\sigma \circ S^{-2})\big)(\Delta a) = \big((\sigma' \circ S^2) \otimes (S^{-2} \circ \sigma)\big)(\Delta a) \\ &= (\sigma' \otimes S^{-2})\big(\Delta(\sigma(a))\big) = \Delta\big(\sigma'(\sigma(a))\big),\end{aligned}$$

by Proposition 5.2 and by (1), (2) above (the commutativity of σ and S^2, and of σ' and S^2). As Δ is injective, this shows that $\sigma \circ \sigma' = \sigma' \circ \sigma$. □

Remark. In the C^* or vN-algebraic weight theory, the commutativity of the modular automorphism groups for the weights φ and ψ implies the existence of a suitable Radon–Nikodym derivative. See discussion in §4.1. Meanwhile in the theory of locally compact groupoids (see [19], [18]), the existence of a Radon–Nikodym derivative between the left integral and the right integral is typically assumed as part of the definition, which is referred to as the *quasi-invariance* condition. For this reason, we termed our Assumption as the "quasi-invariance" earlier, as it gives rise to the commutativity of the modular automorphisms σ and σ' as shown in Proposition 5.11.

In Proposition 5.5, we gathered some results on the modular element δ under the modular automorphisms σ and σ'. With the quasi-invariance Assumption, we can prove another result:

Proposition 5.12. *Consider the modular automorphisms σ and σ', which can be naturally extended to the multiplier algebra level. Under the quasi-invariance Assumption, we have:*

$$\sigma^{-1}(a) = \delta[\sigma']^{-1}(a)\delta^{-1} \quad \text{and} \quad [\sigma']^{-1}(a) = \delta^{-1}\sigma^{-1}(a)\delta,$$

for any $a \in \mathcal{A}$.

Proof. As a consequence of the quasi-invariance Assumption, we can use Proposition 5.11 (3), the commutativity of σ and σ'. Since $\sigma' \circ \sigma = \sigma \circ \sigma'$, we have $\sigma^{-1} \circ \sigma' \circ \sigma = \sigma'$, and it is easy to see that $\sigma^{-1} \circ [\sigma']^{-1} \circ \sigma = [\sigma']^{-1}$. Or, $\sigma \circ [\sigma']^{-1} = [\sigma']^{-1} \circ \sigma$.

Applying this commutativity result to Proposition 5.5 (4), we obtain: $[\sigma']^{-1}\big(\sigma(a)\big) = \delta^{-1} a\delta$. Here let $a = \sigma^{-1}(x)$, for $x \in \mathcal{A}$. Then it becomes: $[\sigma']^{-1}(x) = \delta^{-1}\sigma^{-1}(x)\delta$, true for any $x \in \mathcal{A}$. Equivalently, we have: $\sigma^{-1}(x) = \delta[\sigma']^{-1}(x)\delta^{-1}$, $\forall x \in \mathcal{A}$. □

There is a good reason to believe that under the quasi-invariance Assumption, the modular element δ becomes self-adjoint (actually, positive). See discussion given in Section §4.1 (see Proposition 4.1). As such, from this point on, we will restrict ourselves to the situation when δ is self-adjoint. This will make some of the results in the previous subsection become simpler.

Proposition 5.13. *Assume that δ is self-adjoint. Then we have the following simpler results:*

1. $(\varphi \circ S)(a) = \varphi(a\delta)$ *and* $(\varphi \circ S^{-1})(a) = \varphi(\delta a)$, *for all* $a \in \mathcal{A}$.
2. $S(\delta) = \delta^{-1}$ *and* $S^2(\delta) = \delta$.
3. $(\varphi \otimes \mathrm{id})(\Delta a) = (\mathrm{id} \otimes \varphi)\big(F_1(1 \otimes a)\big)\delta = \delta(\mathrm{id} \otimes \varphi)\big((1 \otimes a)F_3\big)$, *for all* $a \in \mathcal{A}$.
4. $\Delta(\delta) = (\delta \otimes \delta)E = E(\delta \otimes \delta) = E(\delta \otimes \delta)E$.

Proof. (1). See Lemma 5.3, and let $\delta^* = \delta$.

(2). See Proposition 5.4, and let $\delta^* = \delta$.

(3). See Proposition 5.6, and let $\delta^* = \delta$.

(4). See Proposition 5.9, and let $\delta^* = \delta$. □

In the below is one more consequence of the quasi-invariance Assumption and the self-adjointness of δ:

Proposition 5.14. *Given the quasi-invariance Assumption and assuming that δ is self-adjoint, we have:*

$$(\sigma^{-1} \otimes \sigma')(\Delta x) = \Delta\big(S^{-2}(x)\big).$$

for any $x \in \mathcal{A}$.

Proof. Note that for any $a \in \mathcal{A}$, we have $\sigma'(a) = \delta\sigma(a)\delta^{-1}$, by Proposition 5.5 (3), and $\sigma^{-1}(a) = \delta[\sigma']^{-1}(a)\delta^{-1}$, by Proposition 5.12. We thus have, for any $x \in \mathcal{A}$,

$$(\sigma^{-1} \otimes \sigma')\big(\Delta(x)\big) = (\delta \otimes \delta)\big([\sigma']^{-1} \otimes \sigma\big)(\Delta x)(\delta^{-1} \otimes \delta^{-1}).$$

Apply here the result $\Delta\big([\sigma']^{-1}(x)\big) = \big([\sigma']^{-1} \otimes S^2)(\Delta x)$, from Corollary of Proposition 5.2. Then the expression becomes:

$$\begin{aligned} &= (\delta \otimes \delta)\big[(\mathrm{id} \otimes (\sigma \circ S^{-2}))(\Delta([\sigma']^{-1}(x)))\big](\delta^{-1} \otimes \delta^{-1}) \\ &= (\delta \otimes \delta)\big[(\mathrm{id} \otimes (S^{-2} \circ \sigma))(\Delta([\sigma']^{-1}(x)))\big](\delta^{-1} \otimes \delta^{-1}), \end{aligned}$$

as σ and S^{-2} commute (see Proposition 5.11).

Use here the result $\Delta\big(\sigma(a)\big) = (S^2 \otimes \sigma)(\Delta a)$, $a \in \mathcal{A}$, from Proposition 5.2, which can be also written as $(S^{-2} \otimes \mathrm{id})\Delta\big(\sigma(a)\big) = (\mathrm{id} \otimes \sigma)(\Delta a)$. Then the above expression becomes:

$$\begin{aligned} &= (\delta \otimes \delta)\big[(S^{-2} \otimes S^{-2})(\Delta(\sigma([\sigma']^{-1}(x))))\big](\delta^{-1} \otimes \delta^{-1}) \\ &= (\delta \otimes \delta)\big[(S^{-2} \otimes S^{-2})(\Delta(\delta^{-1}x\delta))\big](\delta^{-1} \otimes \delta^{-1}), \end{aligned}$$

where we used the result $\sigma\big([\sigma']^{-1}(x)\big) = \delta^{-1}x\delta$, from Proposition 5.5 (4).

Note that by Proposition 5.13, we have:

$$\Delta(\delta^{-1}x\delta) = \Delta(\delta^{-1})(\Delta x)\Delta(\delta) = (\delta^{-1} \otimes \delta^{-1})E(\Delta x)E(\delta \otimes \delta) = (\delta^{-1} \otimes \delta^{-1})(\Delta x)(\delta \otimes \delta).$$

Combining all these observations, we thus have:

$$\begin{aligned} (\sigma^{-1} \otimes \sigma')\big(\Delta(x)\big) &= \cdots = (\delta \otimes \delta)\big[(S^{-2} \otimes S^{-2})(\Delta(\delta^{-1}x\delta))\big](\delta^{-1} \otimes \delta^{-1}) \\ &= (\delta \otimes \delta)\big[(S^{-2} \otimes S^{-2})((\delta^{-1} \otimes \delta^{-1})(\Delta x)(\delta \otimes \delta))\big](\delta^{-1} \otimes \delta^{-1}) \\ &= (\delta \otimes \delta)(\delta^{-1} \otimes \delta^{-1})[(S^{-2} \otimes S^{-2})(\Delta x)](\delta \otimes \delta)(\delta^{-1} \otimes \delta^{-1}) \\ &= \Delta\big(S^{-2}(x)\big). \end{aligned}$$

Here, we used the result that $S(\delta) = \delta^{-1}$ and $S^2(\delta) = \delta$, from Proposition 5.13, and the property of the antipode that $\Delta\big(S(a)\big) = (S \otimes S)\big(\Delta^{\mathrm{cop}}(a)\big)$, $\forall a \in \mathcal{A}$, from Proposition 1.8, applied twice. □

[Additional remark]: As discussed in §4.1 and as indicated above, it seems to be the case that the quasi-invariance Assumption leads to the self-adjointness of δ. The reasoning requires going up to the von Neumann algebra level and back down, so not purely algebraic. It may be possible to find a more direct proof, but as the focus of the current paper is on the construction of a C^*-algebraic object, we did not make an attempt to develop such a proof. Moreover, in the purely algebraic setting, as there is no reason to have to require the quasi-invariance condition, the need for such a proof is probably not too significant.

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