C*-algebras of graph products

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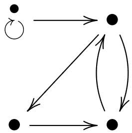
Definition of a (directed) Graph

A graph $E = (E^0, E^1, r, s)$ consists of two countable sets

- E^0: vertices
- E^1: edges
- and functions $r, s : E^1 \to E^0$ called *range and source*



Definition of a C*-algebra

Let F be a field. A vector space A over F with a binary operation $A \times A \to A$ (called multiplication) is an *associative algebra over F* if $\forall x, y \in A$ and $\forall a, b \in F$

- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $x \cdot (y + z) = x \cdot y + x \cdot z$
- $(ax) \cdot (by) = (ab)(x \cdot y)$
- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Definition of a C*-algebra

Let A be an associative algebra over $\mathbb{C}$.
A *norm* on A is a map $\|\cdot\| : A \to \mathbb{R}$ satisfying

- $\|v\| \geq 0$ for all $v \in A$ and $v = 0$ if and only if $v = 0$ in A
- $\|\lambda v\| = |\lambda| \|v\|$ for all $v \in A$ and all $\lambda \in \mathbb{C}$
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in A$.

An *involution* on a complex algebra A is a real-linear map $T \mapsto T^*$ such that for all $S, T \in A$ and $\lambda \in \mathbb{C}$, we have
$T^{**} = T$, $(ST)^* = T^* S^*$, $(\lambda T)^* = \bar{\lambda} T^*$.

Definition of a C*-algebra

A *C*-algebra* A is an associative algebra over $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$, such that A is complete with respect to the norm, and such that $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for every $a, b \in A$.

Examples of C*-algebras

In general a C*-algebra can be infinite dimensional. It is a theorem (Gelfand-Naimark) that every C*-algebra is isomorphic to a sub C*-algebra of $B(H)$, the algebra of all bounded linear operators on some Hilbert space H.

Our examples will be limited to finite dimensional case, and so our C*-algebras can be thought of as $M_n(\mathbb{C})$, with involution defined as taking the conjugate transpose, and the norm being the operator norm on matrices:

$$\|A\|_{op} = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\|.$$

Forming the C*-algebra of a graph

Let E be a row-finite directed graph. A *Cuntz-Krieger E-family* $\{S, P\}$ on a hilbert space H consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections on H, and a set $\{S_e : e \in E^1\}$ of partial isometries on H such that

- $S_e^* S_e = P_{s(e)} \forall e \in E^1$; and
- $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ wherever v is not a source.

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It is a theorem that for E there is a C*-algebra, $C^*(E)$ generated by a Cuntz-Krieger E-family which has a universal property. It is unique up to isomorphism and is called the *graph algebra* of E.

Note: Since we are dealing only with finite-dimensional cases, we will always have $H \cong \mathbb{C}^n$ for some n.

Example

$$E: \; a \xrightarrow{e} b$$

- Relations:
  - $S_e^* S_e = P_a$
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- Relations:
  - $S_e^* S_e = P_a$
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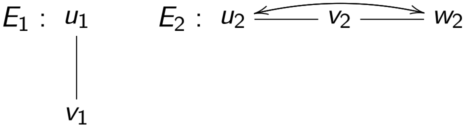
- Generators:
  $$P_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
- $C^*(E) = C^*(P_a, P_b, S_e) = M_2(\mathbb{C})$.

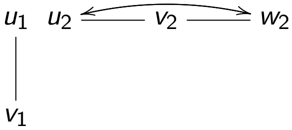
Products of graphs



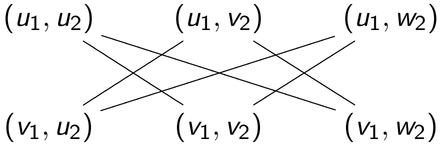
There are many ways of forming larger graphs given two smaller graphs. Examples are:

Products of graphs

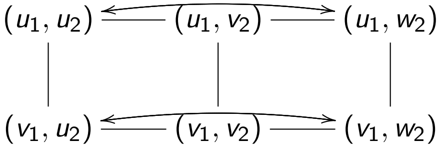
- union $E_1 \cup E_2$:



- tensor product
  $E_1 \times E_2$:



- box (cartesian) product
  $E_1 \square E_2$:



Products of graphs

The *box product* of E_1 with E_2 is the graph $E_1 \square E_2$ with vertex set $E_1^0 \times E_2^0$, where for all $(u_1, u_2), (v_1, v_2) \in E_1^0 \times E_2^0$ we define $(u_1, u_2) \sim (v_1, v_2)$ if either one of the following holds :

- $u_1 = v_1$ in G_1 and $u_2 \sim v_2$ in G_2,
- $u_2 = v_2$ in G_2 and $u_1 \sim v_1$ in G_1.

$$E : \quad a \xrightarrow{e} b, \qquad E \square E : \quad \begin{array}{ccc} (a, a) & \longrightarrow & (a, b) \\ \downarrow & & \downarrow \\ (b, a) & \longrightarrow & (b, b) \end{array}$$

C*-algebra of the box product

$$E: \quad a \xrightarrow{e} b\,, \qquad E \square E: \quad \begin{array}{ccc} w & \xrightarrow{e} & x \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ y & \xrightarrow[f]{} & z \end{array}$$

- Relations:
  - $S_e^* S_e = P_w = S_g^* S_g$
  - $S_h^* S_h = P_x = S_e S_e^*$
  - $S_f^* S_f = P_y = S_g S_g^*$
  - $P_z = S_f S_f^* + S_h S_h^*$

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  - $S_f^* S_f = P_y = S_g S_g^*$
  - $P_z = S_f S_f^* + S_h S_h^*$
- We see that this requires a five-dimensional hilbert space, and with calculations we find the graph algebra to be isomorphic to $M_5(\mathbb{C})$.
- We have seen that $C^*(E) \cong M_2(\mathbb{C})$, and so we see that $C^*(E) \otimes C^*(E) = M_4(\mathbb{C}) \ncong C^*(E\Box E)$.

2-graphs

A *category* C consists of two classes C^0 of objects, and C^* of morphisms, and two functions $r, s : C^* \to C^0$, domain and codomain, as well as a partially defined product (composition) $(f, g) \mapsto f \circ g$ from $\{(f, g) \in C^* \times C^* : s(f) = r(g)\}$ to C^*, composition, and distinguished elements (identity morphisms) $\{i_v \in C^* : v \in C^0\}$ satisfying

- $r(fg) = r(f)$ and $s(fg) = s(g)$
- $(fg)h = f(gh)$ when $s(f) = r(g)$ and $s(g) = r(h)$
- $r(i_v) = v = s(i_v)$ and $i_v f = f, gi_v = g$ when $r(f) = v$ and $s(g) = v$.

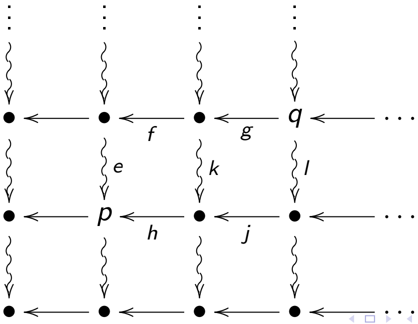
A *functor* $F : C \to D$ is a pair of maps $F^0 : C^0 \to D^0$ and $F^* : C^* \to D^*$ which respect the domain and codomain maps and composition, and which satisfy $F^*(i_v) = i_{F^0(v)}$.

2-graphs

A *rank 2 graph (or 2-graph)* (Λ, d) is a countable category Λ, together with a functor $d : \Lambda \to \mathbb{N}^2$, called the degree map, with the following unique factorization property:

> for every morphism λ and every decomposition $d(\lambda) = m + n$ with $m, n \in \mathbb{N}^2$, there exist unique morphisms μ and ν such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

For every 2-graph Λ, there is an associated 1-skeleton E_Λ, which is a colored directed graph with $E^0 = \Lambda^0$ and $E^1 = \Lambda^{(0,1)} \cup \Lambda^{(1,0)}$.



2-graphs

Given categories C and D, the *product category* $C \times D$ is the category with

- $(C \times D)^0 = C^0 \times D^0$
- $(C \times D)^* = C^* \times D^*$, with composition component-wise from the contributing categories and identities $(1_A, 1_B)$ where $A \in C^0, B \in D^0$.

Forming a 2-graph from two graphs

Viewing two graphs, E and F, as categories with objects as vertices and morphisms as paths, we can form their product category, which we then view as a 2-graph
$E \times F$, with the degree map being ordered pairs of path lengths.

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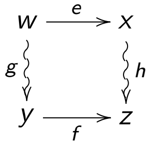
Example:

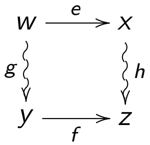
E : $a \xrightarrow{e} b$ 1-skeleton for $E \times E$:



Notice that ignoring color, $E \Box E$ is isomorphic to the 1-skeleton of $E \times E$.

C*-algebras of 2-graphs

$E: \quad a \xrightarrow{e} b$ 1-skeleton for $E \times E$:



There are slightly modified Cuntz-Krieger relations for 2-graphs, which account for the degrees of the morphisms of the graph. Using these relations, we still generate a universal graph algebra unique up to isomorphism. Under these relations $C^*(E \times E) \cong M_4(\mathbb{C}) \cong C^*(E) \otimes C^*(E)$.

In general, Alex Kumjian and David Pask have shown that the C*-algebra of the 2-graph constructed from two graphs, E, F is isomorphic to the tensor product of their corresponding C*-algebras. That is,

$$C^*(E \times F) \cong C^*(E) \otimes C^*(F).$$

Work in progress

We are in the process of investigating several other products. We have already investigated

- direct sum of algebras
- crossed product of an algebra with a group

We are still considering:

- tensor product of graphs
- an “overlay” product, that would lead to the strong product
- etc.

References

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