Graph Fortresses and Cheeger Values

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Outline

What Is a Fortress?

- A fortress is a set of vertices on a finite graph so that at least half of each fortress vertex's neighbors are in the fortress.
- Double fortress: both a set and its complement are fortresses.

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The Fortress as the Basis of a Cellular Automaton

- Cellular automaton on graphs:
  - Each vertex is $+$ or $-$.
  - Each tick, vertex changes sign to majority of neighbors.
  - ID A_0 as all $+$ verts. at $t = 0$, then succ. states of $+$ verts. are $A_1, A_2, A_3, \ldots$.
- Double fortresses = steady states
- Also interested in other endings as $t \to \infty$

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Cheeger Value and Constant

- For $\emptyset \subsetneq A \subsetneq V(G)$, define *Cheeger value* $h_G(A) = \frac{\partial(A, V(G) \setminus A)}{\min\{|A|, |V(G) \setminus A|\}}$.
  - Answers "how connected or robust is G?"
  - Contrast Chung's def'n.
- *Cheeger constant* is minimum Cheeger value.
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Extending to the Infinite

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- Possible to extend to infinite with *graph families*
- Nested subgraphs $G_1, G_2, G_3, \ldots$
- Applications: Expander & Cayley graphs of infinite groups

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Minimal Cheeger Sets as Fortresses

Theorem

If A is a minimal Cheeger set on finite G with $h_G(A) < 1$, then A is a fortress. Further, if $h_G(A) = 1$, then A is a fortress if each vertex in G has degree at least 2.

Proof uses both minimality and that the Cheeger value is no more than 1.

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Nontrivial Double Fortress Guarantee

Theorem

If G is a finite graph with $h_G < 1$, or with $h_G = 1$ and every vertex having degree 2 or more, then G has a nontrivial double fortress.

Much stronger version possible using our CA
All conditions needed; converse not true

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Some Basic CA Properties

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$(A^c)_n = (A_n)^c$. *(The CA acts equally on A and its complement.)*

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If $B_0 \subset A_0$*, then for all positive integers* i*,* $B_i \subset A_i$*.*

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Fortress Results from CA Interpretation

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A_0 is a fortress iff $A_0 \subset A_1 \subset A_2 \subset \cdots$. Moreover, if either condition holds, each A_i is also a fortress.

Corollary

A_0 is a double fortress iff $A_0 = A_1 = A_2 = \cdots$.

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Creating a Double Fortress from Disjoint Fortresses

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Suppose A and B are disjoint nontrivial fortresses on G. Then G has a nontrivial double fortress.

- Create double fortress by iterating A until it reaches steady state.
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Brief Word on Graph Families

- Recall a graph family is a sequence of nested subgraphs $G_1, G_2, G_3, \ldots$.
- A vertex v in a graph family member G_n is *finalized* if the neighborhood of v does not change in any later graph.
- A fortress F on G_n is *stable* if it is also a fortress on any later graph.
- Any fortress consisting of only finalized vertices is stable.

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Summary

- Fortresses are strongly related to Cheeger constants.
- Stable fortresses in graph families are important in analyzing Cayley graphs.
- Outlook
  - How do fortresses depend on or affect a graph's spectrum?
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For Further Reading I

F. R. K. Chung.
Spectral Graph Theory.
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