Embeddings in Symbolic Dynamical Systems

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A Dynamical System

$$(X, f)$$

- Phase Space
- Function

$$X \xrightarrow{f} X$$

Example Dynamical System

$$[0,1] \xrightarrow{f} [0,1]$$

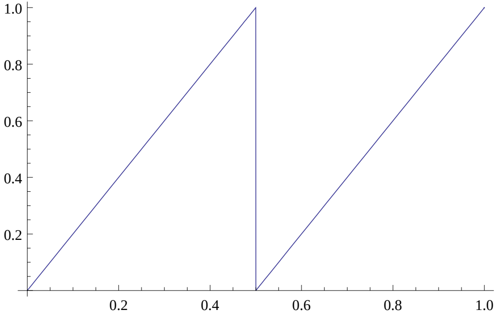

Figure: $f(x) = 2x \pmod 1$

Category of Dynamical Systems

Objects Dynamical systems

Maps Commuting Maps

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

Subsystem

Let (X, d, f) be a dynamical system. If $\Lambda \subseteq X$ is an invariant subset of f, i.e., $f(\Lambda) \subseteq \Lambda$, then the restriction of f on Λ, $f|_\Lambda : \Lambda \to \Lambda$, determines a dynamical system $(\Lambda, f|_\Lambda)$, which is called a subsystem of (X, f).

A Topological Dynamical System

$$(X, f)$$

- Topological Phase Space
- Continuous Function

$$X \xrightarrow{f} X$$

Example Topological Dynamical System

$$[-1,1] \xrightarrow{g} [-1,1]$$

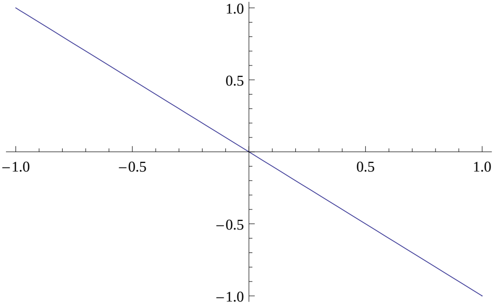

Figure: $g(x) = -x$

Category of Topological Dynamical Systems

Objects Topological Dynamical systems

Maps Continuous Commuting Maps

- Conjugacy
- Semi-Conjugacy
- Embedding
- Weak Embedding

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

Measurements of Chaos: Topological Entropy

- Topological entropy quantitatively describes complexity of a topological dynamical system.
- In a metric space, it describes the average exponential growth of the number of distinguishable orbit segments. [5]

Ordinary Symbolic Dynamical System

Let m be a positive integer and $\mathcal{M}$ the finite set consisting of positive integers which are less than $m+1$. Let $\mathcal{M}^{\mathbb{N}}$ be the set of all unilateral infinite sequences consisting of all elements of $\mathcal{M}$. Let n be a non-negative integer, r an element of $\mathcal{M}^{\mathbb{N}}$ and $(r)_n$ the n^{th} component of r. We define the following metric on $\mathcal{M}^{\mathbb{N}}$:

$$D(r,s) = \sum_{k=0}^{\infty} \frac{1-\delta((r)_k,(s)_k)}{m^k}, \quad r,s \in \mathcal{M}^{\mathbb{N}}, \tag{1}$$

where δ means Kronecker's delta. The mapping $\sigma : \mathcal{M}^{\mathbb{N}} \to \mathcal{M}^{\mathbb{N}}$ is called the shift transformation if $(\sigma r)_n = (r)_{n+1}$ holds for all $n \in \mathbb{N}$ and for all $r \in \mathcal{M}^{\mathbb{N}}$.

Product Symbolic Dynamical System

For any positive integer n, let $(\mathcal{M}_n^{\mathbb{N}}, D_n, S_n)$ be a symbolic dynamical system. Then, the product topological space constructed from $\{(\mathcal{M}_n^{\mathbb{N}}, D_n)\}_{n=1}^{\infty}$ which is denoted by $\Pi_{n=1}^{\infty}\mathcal{M}_n^{\mathbb{N}}$, is also a compact and totally disconnected topological space. $\Pi_{n=1}^{\infty}\mathcal{M}_n^{\mathbb{N}}$ can be metrized by the metric D which is defined as

$$D(\{r_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{D_n(r_n, s_n)}{2^n}, \quad \{r_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{M}_n^{\mathbb{N}}.$$

Let $\Pi_{n=1}^{\infty} S_n$ be the mapping on $\Pi_{n=1}^{\infty}\mathcal{M}_n^{\mathbb{N}}$ with values in $\Pi_{n=1}^{\infty}\mathcal{M}_n^{\mathbb{N}}$ defined as

$$\left(\prod_{n=1}^{\infty} S_n\right)(\{r_n\}_{n=1}^{\infty}) = \{S_n r_n\}_{n=1}^{\infty}.$$

Symbolic Embeddings

Let (X, d) be a compact and totally disconnected metric space, and T a continuous endomorphism.

- The dynamical system (X, d, T) can be embedded into a product symbolic dynamical system. [2]
- If T is expansive, then (X, d, T) may be embedded into an ordinary symbolic dynamical system. [1]
- If T is expansive, then (X, d, T) can be weakly embedded in an ordinary symbolic dynamical system, even if X is not compact. [4]
- Nothing is known about weak embeddings in product symbolic dynamical systems.

Disconnecting the System

For a given dynamical system, we want an invariant subsystem with a totally disconnected phase space.

- Removing open sets
- Removing single points

Tent Map

$$\mathbb{R} \xrightarrow{h} \mathbb{R}$$

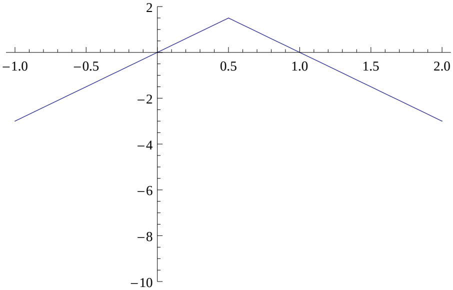

Figure: $h(x) = \begin{cases} 3x & \text{if } x < 1/2 \\ -3(x-1) & \text{if } x \geq 1/2 \end{cases}$

Tent Map

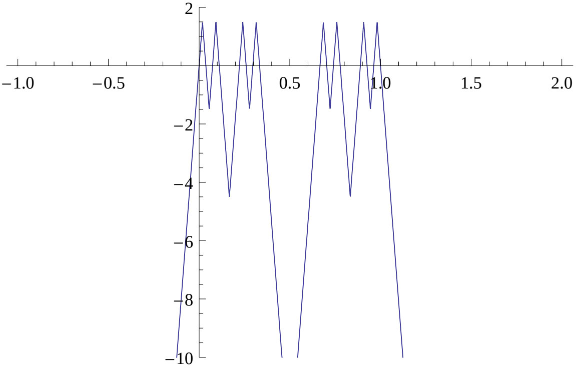

Figure: Four iterations of h(x)

Tent Map

$$[0,1] \xrightarrow{h} [0,1]$$

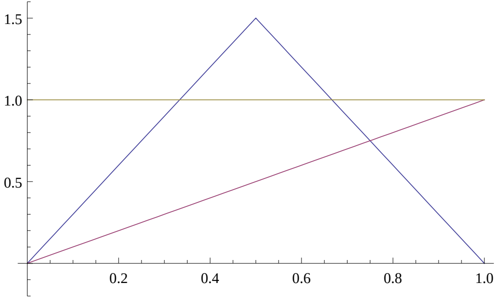

Figure: Restriction to the unit interval

Tent Map

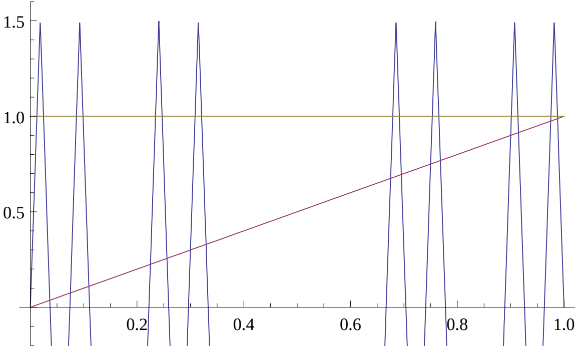

Figure: Four Iterations

Symbolically Embedding the System

For a given point $x \in \Gamma$, has the ternary expansion $x_0 x_1 x_2 x_3 \ldots$ where $x_i \in \{0, 1, 2\}$ for all $i \in \mathbb{Z}_+$. Define a map $\pi(x)$ for a given point $x \in \Gamma$ mapping to a point $y = y_0 y_1 y_2 y_3 \ldots \in \{0, 1\}^{\mathbb{N}}$, recursively, with respect to the ternary expansion of x, by

$$y_0 = \begin{cases} 0 \text{ if } x < 1/2 \\ 1 \text{ otherwise} \end{cases}$$

$$y_{i+1} = \begin{cases} 0 \text{ if } x_i = 0, \ x_{i+1} = 0 \\ 0 \text{ if } x_i = 2, \ x_{i+1} = 2 \\ 1 \text{ if } x_i = 2, \ x_{i+1} = 0 \\ 1 \text{ if } x_i = 0, \ x_{i+1} = 2 \end{cases}$$

Point Path

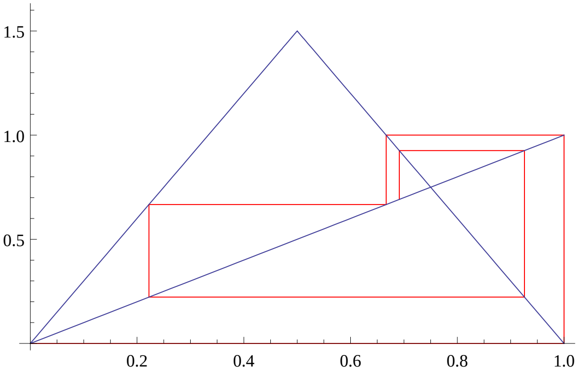

Figure: Iteration Path of the point $\frac{56}{81} = .2002000\ldots$

Wobble Function

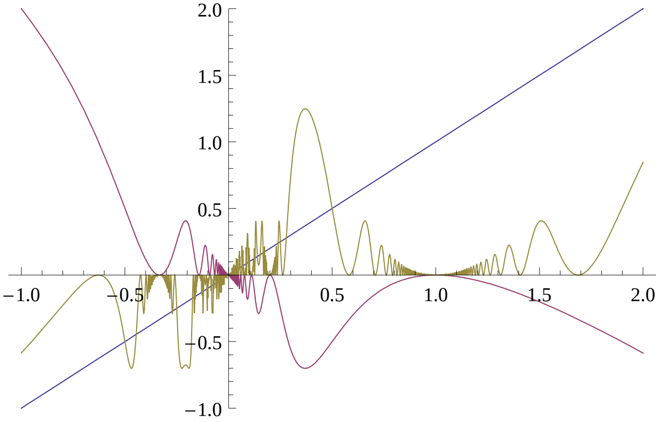

Figure: $w(x) = -x(1 - sin\left(\frac{\pi}{2x}\right))$

Results

Prop 1 Let (X, d) be a metric space and $T : X \to X$ a continuous mapping. If there exists an at most countable partition $X = \bigcup_{i \in \mathbb{N}} P_i$ such that each restriction of the map $T|_{P_i} : P_i \to X$ is injective, then there exists a countable set $Y \subset X$ such that $X \backslash Y$ is dense in X and $(X \backslash Y, T|_{X \backslash Y})$ is a totally disconnected subsystem of (X, T).

Prop 2 Let (X, d) be a metric space, $T : X \to X$ a continuous mapping and $h(X, T)$ denote the topological entropy of the dynamical system (X, T). If there exists a set $Y \subset X$ such that $X \backslash Y$ is dense in X and $(X \backslash Y, T|_{X \backslash Y})$ is a subsystem of (X, T), then

$$h(X, T) = h(X \backslash Y, T|_{X \backslash Y})$$

Results

Conj 1 Let (X, T) be a dynamical system whose phase space is a totally disconnected metric space. Then (X, T) is weakly embeddable into a product symbolic dynamical system.

Conj 2 Let (X,d,T) be a dynamical system with a totally disconnected domain and $(\Pi_{n=1}^{\infty} \mathcal{M}_n^{\mathbb{N}}, D, \Pi_{n=1}^{\infty} S_n)$ is a product symbolic dynamical system into which (X, d, T) can be weakly embedded. Then, the following equality holds:

$$h(X, d, T) = h\left(\prod_{n=1}^{\infty} \mathcal{M}_n^{\mathbb{N}}, D, \prod_{n=1}^{\infty} S_n\right)$$

Bibliography I

S. Akashi.
Embedding of expansive dynamical systems into symbolic dynamical systems.
Reports on Mathematical Physics, 46(1/2):11–14, 2000.

S. Akashi.
Embedding of nonlinear dynamical systems with compact and totally disconnected domains into the product symbolic dynamical systems.
Nonlinear Analysis, 63:1817–1821, 2005.

D. Cheng, Y. Wang, and G. Wei.
The product symbolic dynamical systems.
Nonlinear Analysis, 2009.

Bibliography II

A. Fedeli.
Embeddings into symbolic dynamical systems.
Reports on Mathematical Physics, 58(3):351–355, 2006.

C. Robinson.
Dynamical Systems : Stability, Symbolic Dynamics, and Chaos.
CRC Press, Boca Raton, Florida, 1998.