Hitting Times for Finite and Infinite Graphs

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Positive Recurrent Infinite Graphs

Recurrence vs. Transience

Recurrent: The probability of returning to the starting vertex goes to one as time goes to infinity.

Transient: There is a non-zero probability of never returning to the starting vertex.

In a strongly connected graph, independent of starting vertex.

Expected First Return Time

First Return Time (T_u^+): Given starting vertex u, the time a given random walk takes to return to u.

Expected First Return Time ($E_u(T_u^+)$): Over a large number of random walks starting at u, the average first return time.

Positive Recurrence vs. Null Recurrence

For any vertex u in a transient graph, $E_u(T_u^+) = \infty$.

In a recurrent graph, $E_u(T_u^+)$ can be finite or infinite.

Positive Recurrent: $E_u(T_u^+) < \infty$.

Null Recurrent: $E_u(T_u^+) = \infty$.

Independent of starting vertex.

Stationary Measures and Positive Recurrence

Measure (π): A non-negative, real-valued function on the vertices of a graph.

Transition operator (P): The generalization of the transition matrix to the infinite case.

P acts on measures in the following way:

$$P\pi(u) = \sum_{v \to u} \frac{\pi(v)}{outdeg(v)}$$

If a graph is recurrent, then there exists a measure π such that $P\pi = \pi$, unique up to scalar multiples.

The graph is positive recurrent if:

$$\sum_{u \in G} \pi(u) < \infty$$

The graph is null recurrent if:

$$\sum_{u \in G} \pi(u) = \infty$$

Graphs with indeg = outdeg

Theorem

Let G be a strongly connected, infinite graph with $indeg(u) = outdeg(u)$ for all $u \in G$.

G is not positive recurrent.

$\pi(u) = outdeg(u)$ is a stationary measure and is not summable.

No infinite undirected or Cayley graphs are positive recurrent.

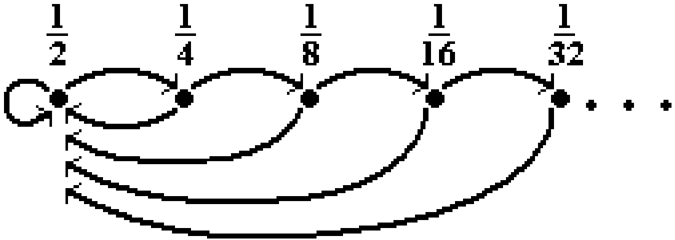
Stationary Distributions and Expected Return Times

Distribution: A measure π such that:

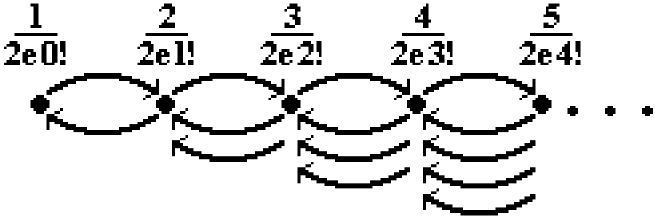
$$\sum_{u \in G} \pi(u) = 1$$

A graph is positive recurrent if and only there exists a distribution π such that $P\pi = \pi$. In that case, $E_u(T_u^+) = \frac{1}{\pi(u)}$.

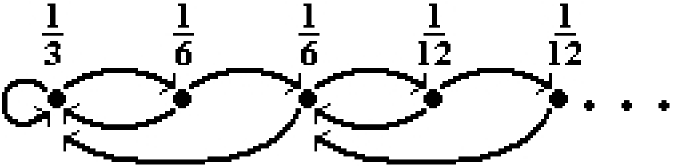
Some Examples of Positive Recurrent Graphs



A locally finite, positive recurrent graph:



A bounded degree, single-edged, positive recurrent graph:



References

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