# Kazhdan Groups and Graphs

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# Outline

# Definition of Property (T)

## Definition ( [Kazhdan, 1967])

A topological group $G$ has *Kazhdan's Property (T)* if the trivial representation **1** is an isolated point in the space of unitary representations endowed with the Fell topology. Let $\kappa$ be the largest real number such that the ball of radius $\kappa$ about **1** contains only the trivial representation. Then $\kappa$ is called a *Kazhdan constant* for $G$.

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- This is an informal definition, and not the definition originally introduced by Kazhdan.
- Kazhdan constants can change depending on the group presentation.

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- Any finite group has Property (T)
- $SL_n(K)$ for $K = \mathbb{R}, \mathbb{C}, \mathbb{Z}$, $n \geq 3$.

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- $SL_2(\mathbb{Z})$ lacks (T), as it contains $F_2$ as a finite index subgroup.

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- Intuitively, expander graphs are graphs that have strong connectivity properties without having a large number of edges
- Margulis showed that the finite Schreier graphs of a group with Property (T) were in fact expander graphs

# Definition of a Kazhdan Graph Structure

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- Any finite graph is a KGS, as its automorphism group is finite and therefore Kazhdan
- The Cayley graph of a Kazhdan group is a KGS
  - If a group $G$ is generated by the finite symmetric set $S$, then $\text{Cay}(G, S)$ is the graph with vertex set $G$, and $(g_1, g_2)$ is an edge if $g_1 = sg_2$ for some $s \in S$.

# A Property of Kazhdan Graph Structures

**Proposition (Due to Furman)**

*Let $(X, \Gamma)$ be a Kazhdan graph structure, and let $G$ be a group acting on $X$ with finite index and finite isotropy subgroups. Then $G$ is a Kazhdan group, and $(X, G)$ is a Kazhdan graph structure.*

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From this proposition, we can obtain some examples of non-KGS:

- $T_k$, the $k$-regular infinite tree, cannot form a KGS. It is the Cayley graph of $*^k\mathbb{Z}_2$, the free product of $k$ copies of $\mathbb{Z}_2$, a group which lacks property (T).

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- $T_k$, the $k$-regular infinite tree, cannot form a KGS. It is the Cayley graph of $*^k\mathbb{Z}_2$, the free product of $k$ copies of $\mathbb{Z}_2$, a group which lacks property (T).
- The infinite grid in the plane cannot form a KGS, as it is the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$

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- The most immediate examples of infinite KGS are the Cayley graphs of infinite Kazhdan groups
- Many useful constructions involving Kazhdan groups often implicitly make use of the group's Cayley graph
- Are there simple constructions of infinite non-Cayley graphs that are still KGS?
- Do constructions on the Cayley graphs on Kazhdan groups generalize to similar constructions on non-Cayley KGS?

# A Theorem Concerning Cayley Graphs

**Theorem ( [Sabidussi, 1958])**

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**Corollary**

*A graph $X$ is a non-Cayley graph if it does not admit a free, transitive action by any group.*

# Construction of a Non-Cayley Kazhdan Graph Structure

## Theorem

*Let $\Gamma$ be a Kazhdan group generated by finite symmetric set $S$. Let $H \subset \Gamma$ be a finite, not-normal subgroup such that $H \cap S = \emptyset$ and there exists $s \in S$, $\gamma \in \Gamma$ such that $s \in \gamma H \gamma^{-1}$. Let $N_\Gamma(H) = \{\gamma \in \Gamma | \gamma H \gamma^{-1} = H\}$, and assume $N_\Gamma(H)$ has finite index in $\Gamma$. Let $X = Sch(\Gamma/H, S)$. Then*

1. *$X$ is not a Cayley graph*
2. *$(X, N_\Gamma(H))$ is a KGS*

Recall that if a group $G$ generated by $S$ acts on a set $V$, then $\mathrm{Sch}(V, S)$ is the graph with vertex set $V$, and $(v_1, v_2)$ is an edge if $v_1 = s(v_2)$ for some $s \in S$. Note that Schreier graphs can therefore have loops and multiple edges.

# Construction of a Non-Cayley Kazhdan Graph Structure II

**Part 1: $X$ is not Cayley.**

Observe that the vertex corresponding to $H$ has no loops in $X$ as $sH = H \implies s \in H$, but $S \cap H = \emptyset$ by assumption. Given that there exists $s \in \gamma H \gamma^{-1}$ for some $\gamma \in \Gamma$, we have $s \in \gamma H \gamma^{-1} \Leftrightarrow s\gamma \in \gamma H \Leftrightarrow s\gamma H = \gamma H$. Therefore, there is a loop at $\gamma H$ but no loop at $H$, and thus no automorphism can take $H$ to $\gamma H$. Thus no group can act transitively on $X$, and therefore $X$ is not Cayley. □

The conditions on $H \cap S$ and $\gamma H \gamma^{-1} \cap S$ are only used to prove that $X$ is non-Cayley. Without these assumptions, $X$ is still a KGS.

# Construction of a Non-Cayley Kazhdan Graph Structure III

**Part 2(a): X is a KGS.**

Define an action of $N_\Gamma(H)$ on $V(X)$ by $(\gamma_1 H)\gamma = \gamma_1\gamma H$ for $\gamma \in N_\Gamma(H)$. Note that this is well-defined, as $\gamma H = H\gamma$, and it is simple to check that this action preserves adjacencies. □

# Construction of a Non-Cayley Kazhdan Graph Structure IV

## Part 2(b): X is a KGS cont.

To see that the isotropy groups are finite, note that , for $\gamma \in N_\Gamma(H)$, we have $(\gamma_1 H)\gamma = \gamma_1\gamma H = \gamma_1 H \implies \gamma H = H \implies \gamma \in H$. As $H$ is finite, so are stabilizers. Finally, to see that the quotient $X/N(H)$ is finite, note that $N(H)$ has finite index in $\Gamma$, so $(\Gamma/H)/N(H)$ is by assumption finite. □

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*Remark* This Non-Cayley KGS can be used to construct expander graphs, as discussed in the next talk.

# Example

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Let $\Gamma$ be a Kazhdan group generated by $S$, and let $F$ be any finite group, generated by $T$. Then $\Gamma \times F$ is a Kazhdan group generated by $R = S \times T$. If $H \subset F$ is a not-normal subgroup such that $H \cap T = \emptyset$ and $N_F(H) \cap T \neq \emptyset$, and $N$ is a normal subgroup of finite index in $\Gamma$, then $N \times H$ satisfies the assumptions of the theorem.

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- For a trivial example, consider $F = S_4$, $T = \{(12), (13), (14)\}$, and $H = \{e, (23)\}$. Then $H \cap T = \emptyset$, and $(14) \in N_F(H) \cap T$.

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- For a trivial example, consider $F = S_4$, $T = \{(12), (13), (14)\}$, and $H = \{e, (23)\}$. Then $H \cap T = \emptyset$, and $(14) \in N_F(H) \cap T$.
- More generally, a subgroup appropriate for the theorem can often be found by conjugating a not normal subgroup of $H$ that intersects $S$ by a suitable element of $\Gamma$.

# Further Research

- Define a useful Kazhdan constant for a KGS

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- A graph-theoretical characterization of Kazhdan Graph Structures

# Acknowledgments

- Stratos Prassidis

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