# Extensions of the Heisenberg Group

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# The Heisenberg Group

**Definition**

The Heisenberg Group, often denoted by $H_1$, is the group of $3 \times 3$ matrices consisting of elements of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

in $\mathbb{R}^3$ which we will denote $(x, y, z)$. Thus, the identity element $e$ is $(0, 0, 0)$.

Then we can define

$$(x, y, z) \cdot (x', y', z') = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x + x' & z + z' + xy' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{pmatrix}$$

which is written $(x + x', y + y', z + z' + xy')$.

The Heisenberg Group can also be thought of in higher dimensions. The Heisenberg group of $2n+1$ dimensions in $\mathbb{R}^{2n+1}$ is usually denoted $H_n$ and is the group of matrices under matrix multiplication consisting of elements of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & I_n & y \\ 0 & 0 & 1 \end{pmatrix},$$

with $x$ a row vector of length $n$, $y$ a column vector of length $n$, and $I_n$ the identity matrix of $n$ dimensions.

## Example

Let $n, m, x, y, z \in \mathbb{R}$. $H_2$, the 5-dimensional Heisenberg group is the group of matrices with elements of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we will denote this element $(x_1, x_2, y_1, y_2, z)$. Then the identity element of $H_2$ is $(0, 0, 0, 0, 0)$ and
$(x_1, x_2, y_1, y_2, z)^{-1} = (-x_1, -x_2, -y_1, -y_2, x_1 y_1 + x_2 y_2 - z)$.

## Example

As for group multiplication we have

$$
\begin{aligned}
&(x_1, x_2, y_1, y_2, z)(x_1', x_2', y_1', y_2', z') \\
&= \begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1' & x_2' & z' \\ 0 & 1 & 0 & y_1' \\ 0 & 0 & 1 & y_2' \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x_1 + x_1' & x_2 + x_2' & z + z' + x_1 x_2 + y_1 y_2 \\ 0 & 1 & 0 & y_1 + y_1' \\ 0 & 0 & 1 & y_2 + y_2' \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= (x_1 + x_1', x_2 + x_2', y_1 + y_1', y_2 + y_2', z + z' + x_1 y_1' + x_2 y_2').
\end{aligned}
$$

# Group Extensions

### Definition

An extension of a group $H$ by a group $N$ is a group $G$ with a normal subgroup $M$ such that $M \cong N$ and $G/M \cong H$. This information can be encoded into a short exact sequence of groups

$$1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1,$$

where $\alpha : N \rightarrow G$ is injective and $\beta : G \rightarrow H$ is surjective. Also, the image of $\alpha$ is the kernel of $\beta$.

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In other words, if $G$ is an extension of $H$ by $N$ then $N$ is a normal subgroup of $G$ and the quotient group $G/N$ is isomorphic to group $H$.

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## Example

Consider a group $N$ consisting of elements of the form

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Then $N$ is a subgroup of the Heisenberg group $H_1$ since

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}$$

which is clearly in $N$ and

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & z+z' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix}$$

which is also in $N$.

## Example

Furthermore, $N$ is a normal subgroup of $H_1$ because

$$= \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -a & ab - c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & z - ay \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in N.$$

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Since $H_1/N \cong \{(x, 0, 0) \in H_1\}$, we can construct an exact sequence of groups.

$$e \longrightarrow N \longrightarrow H_1 \longrightarrow \{(x, 0, 0)\} \longrightarrow e$$

## Example

Now consider a group $A$ in $\mathbb{R}^4$ with elements of the form $(x, y, z, d)$. The group multiplication is defined by
$(x, y, z, d) \cdot (x', y', z', d') = (e^{d'}x + x', e^{-d'}y + y', z + z' + e^{d'}(x \cdot y'), d + d')$.

Let $(x, y, z, 0) \in H_1$. Then $H_1$ is a subgroup of $A$ because $(x, y, z, 0)^{-1} = (-x, -y, xy - z, 0) \in A$ and

$$(x, y, z, 0) \cdot (x', y', z', 0) = (x + x', y + y', z + z' + xy', 0) \in A.$$

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Then it is simple to show that the Heisenberg group $H_1$ is a normal subgroup of $A$ and that $A/G = \{(0, 0, 0, d) \in A\}$. Therefore, we have the following exact sequence

$$e \to H_1 \to A \to \{(0, 0, 0, d)\} \to e.$$

# Cocycles

## Definition

Let $N$ be an abelian group. Given a group $H$, a 2-cocycle of $H$ having values in $N$ is a mapping $\omega$ from $H \times H$ to $N$ satisfying the following cocycle identity

$$\omega(r, s)\omega(rs, t) = \omega(s, t)\omega(r, st) \forall r, s, t \in H.$$

Often, it is further assumed that $\omega(1, s) = \omega(s, 1) = 1$ for all $s \in H$.

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## Proposition

*If we have a suitable 2-cocycle $\omega : H \times H \to N$, then we can use it to define a group $N \times H$ with group multiplication defined by*

$$(m,s) \times_{\omega} (n,t) = (mn\omega(s,t), st).$$

## Definition

A bilinear form is a map $B : V \times V \to F$ where $V$ is a vector space and $F$ a field with the following properties for all $u, v \in F$ and $\lambda$ fixed in $F$.

1. $B(u + u', v) = B(u, v) + B(u', v)$
2. $B(u, v + v') = B(u, v) + B(u, v')$
3. $B(\lambda u, v) = B(u, \lambda v) = \lambda B(u, v)$

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### Definition

For finite dimensions, a bilinear form is nondegenerate if and only if $B(u, v) = 0$ for all $v \in V$ implies that $u = 0$.

## Proposition

*If* $\alpha$ *is a nondegenerate alternating* $\mathbb{R}$*-bilinear form* $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ *then the* $(2n+1)$*-dimensional Heisenberg group* $H_n^\alpha$ *fits into an exact sequence*

$$e \to \mathbb{R} \to H_n^\alpha \to \mathbb{R}^{2n} \to e$$

*and is the set of pairs* $(t, v) \in \mathbb{R} \times \mathbb{R}^{2n}$*, with the group operation defined as*

$$(t_1, v_1)(t_2, v_2) = (t_1 + t_2 + \alpha(v_1, v_2), v_1 + v_2).$$

## Example

Let $z \in \mathbb{R}$ and $v \in \mathbb{R}^2$ with $\alpha_1(v, v') = \alpha_1((x, y), (n, m)) = xm - yn$. $\alpha_1$ is an alternating nondegenerate bilinear form.

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Therefore, by the last proposition, we have a group multiplication on $\mathbb{R}^3$ with identity element $(0, 0, 0)$ and $(x, y, z)^{-1} = (-x, -y, -z)$ with our group operation defined by

$$\begin{aligned}
&(z_1, v_1)(z_2, v_2) \\
&= (z_1, (x_1, y_1))(z_2, (x_2, y_2)) \\
&= (z_1 + z_2 + \alpha_2((x_1, y_1), (x_2, y_2)), (x_1, y_1) + (x_2, y_2)) \\
&= (z_1 + z_2 + x_1 y_2 - x_2 y_1, (x_1 + x_2, y_1 + y_2)).
\end{aligned}$$

In fact, this group is isomorphic to $H_1$.

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**Proof.** Let $(z,x,y),(z',x',y') \in G$. Define a map $\phi : G \to H_1$ with $\phi((z,x,y)) = (z+xy, \sqrt{2}x, \sqrt{2}y)$. Then $\phi$ is a homomorphism because

$$\begin{aligned}
&\phi((z,x,y)(z',x',y')) \\
&= \phi(z+z'+xy'-yx', x+x', y+y') \\
&= (z+z'+xy'-yx'+xy+x'y+xy'+x'y', \sqrt{2}(x+x'), \sqrt{2}(y+y')) \\
&= (z+xy+z'+x'y'+2xy', \sqrt{2}(x+x'), \sqrt{2}(y+y')) \\
&= (z+xy, \sqrt{2}x, \sqrt{2}y)(z'+x'y', \sqrt{2}x', \sqrt{2}y') \\
&= \phi((z,x,y))\phi((z',x',y')).
\end{aligned}$$

Furthermore, $\phi$ is one-to-one because if we set $\phi(z,x,y) = \phi(c,a,b)$ then we have $(z+xy, \sqrt{2}x, \sqrt{2}y) = (c+ab, \sqrt{2}a, \sqrt{2}b)$, which implies that $(z,x,y) = (c,a,b)$. $\phi$ is also onto since for every $(z,x,y) \in H_1$ there is an $(z-\frac{xy}{2}, \frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{2}}y) \in G$ such that $\phi(z-\frac{xy}{2}, \frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{2}}y) = (z,x,y)$. Therefore, $\phi$ is an isomorphism as we wished to show. ■

## Example

Let $z \in \mathbb{R}$ and $v \in \mathbb{R}^2$ with $\alpha_2(v_1, v_2) = \alpha_2((a,b),(a',b')) = a \cdot b'$. Then we have a group call it $D$ with identity element $(0,0,0)$ and $(x,y,z)^{-1} = (-x,-y,xy-z)$ with group multiplication defined by

$$\begin{aligned}
&(c, v_1)(c', v_2) \\
&= (c,(a,b))(c',(a',b')) \\
&= (c + c' + \alpha_2((a,b),(a',b')),(a,b) + (a',b')) \\
&= (c + c' + a \cdot b',(a + a', b + b')).
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&= (c + c' + a \cdot b', (a + a', b + b')).
\end{aligned}$$

This group operation is the same as the matrix multiplication of matrices in the Heisenberg group, so $D$ and the Heisenberg group are isomorphic.

Theorem 6.7 of A.M. DuPre's paper "Real Heisenberg Group Extension Isomophism Classes" states the following.

**Theorem**

*Every 2-cocycle on the Heisenberg Group $H_1$ can be written in the form*

$$\lambda_1(x^2y' + 2xz') + \lambda_2(y^2x' - x(y')^2 + 4yx'y' - 2xyy' - 6yz')$$

*for fixed $\lambda_1, \lambda_2 \in \mathbb{R}$.*

## Example

Let $N$ an abelian group. Using the equation in Theorem 2.9, set $\lambda_1 = 2$ and $\lambda_2 = 3$. Then define the map $\beta : H_1 \times H_1 \to N$ with

$$\beta((x,y,z)(x',y',z')) =$$
$$2(x^2y' + 2xz') + 3(y^2x' - x(y')^2 + 4yx'y' - 2xyy' - 6yz').$$

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Since it can be shown that $\beta$ is a 2-cocycle satisfying the cocycle identity, Proposision 2.2 gives that there is a group operation on $H_1 \times N$. Let $x, y, z, d \in H_1 \times N$. Then $H_1 \times N$ has group operation defined by

$$\begin{aligned} &(x,y,z,d)(x',y',z',d') \\ &= (x+x', y+y', z+z'+xy', d+d'+\beta((x,y,z)(x',y',z'))) \\ &= (x+x', y+y', z+z'+xy', \\ &d+d'+2(x^2y'+2xz')+3(y^2x'-x(y')^2+4yx'y'-2xyy'-6yz')), \end{aligned}$$

with identity element $(0,0,0,0)$ and

$$(x,y,z,d)^{-1} = (-x, -y, -z+xy, -d-2(x^2y-2xz+9yz-3xy^2)).$$

The following theorem is arguably DuPre's most important result.

**Theorem**

*Any two non-trivial, one-dimensional central extension of the Heisenberg $H_1$ group yield isomorphic groups.*