

# Cyclic Cayley Graphs and their Characteristic Polynomials

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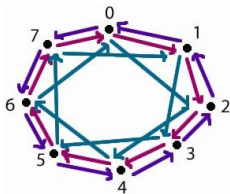
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# Cayley Graphs

Let  $G$  be a group. Let  $S$  be a set of elements in  $G$   
 $\text{Cay}(G\{S\})$  is the Cayley graph of  $G$  with the letters  $S$

## Definition

**Cayley Graph:**  $V(\text{Cay}(G\{S\})) = \{g \in G\}$  and  
 $\{g_1, g_2\} \in E(\text{Cay}(G\{S\}))$  iff  $g_1 \cdot s = g_2$  for some  $s$ .



$\text{Cay}(\mathbb{Z}_8, \{1, 2, 7\})$

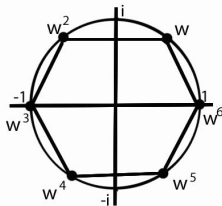
## Definition

**Characteristic Polynomial and Spectrum:** The characteristic polynomial and spectrum of a Cayley graph is the characteristic polynomial and spectrum of its adjacency matrix.

# Roots of Unity and Characteristic Polynomials

## Definition

**Root of Unity:**  $\omega_n$  is the  $n$ th root of unity. This means that  $\omega_n^n = 1$ . Roots of unity are evenly spaced on the unit circle in the complex plane.



## Definition

**Characteristic Polynomials:**  $\Phi_n(x)$  is the  $n$ th characteristic polynomial. It is a polynomial whose roots are  $\omega_n^k$ , where  $k$  is relatively prime to  $n$ .

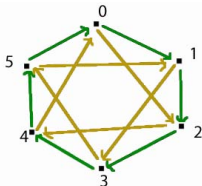
# Lazenby's Theorem

## Theorem

The eigen values of  $\text{Cay}(Z_n, \{S\})$  are

$$\lambda_x = \sum_{s \in S} \omega_n^{xs}$$

where  $x: 1, 2, \dots, n$



$\text{Cay}(Z_6, \{1, 2\})$

The spectrum of  $\text{Cay}(Z_6, \{1, 2\})$  is  
 $\omega + \omega^2, \omega^2 + \omega^4, \omega^3 + \omega^6,$   
 $\omega^4 + \omega^2, \omega^5 + \omega^4, \omega^6 + \omega^6$

# Characteristic Polynomial of $\text{Cay}(Z_n, \{s\})$

## Theorem

$$\chi(\text{Cay}(Z_n, \{s\})) = \prod_{k | \frac{n}{\text{GCD}(n,s)}} (\Phi_k(x))^{\text{GCD}(n,s)}$$

## Outline of Proof:

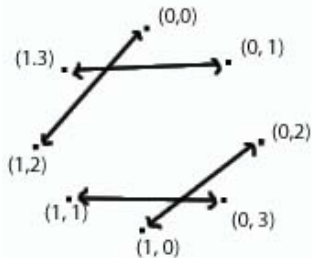
- From Lazenby's Theorem we have  $\lambda_x = \omega_n^{xs}$   $x: 1, 2, \dots, n$
- When  $s$  is relatively prime to  $n$  we get every root of unity
- The polynomial that gives every  $n$ th root of unity is  $\prod_{k|n} \Phi_k(x)$
- When  $\text{GCD}(s, n) = q$  we will get every  $\frac{q}{n}$  root of unity  $q$  times
- We can modify our polynomial to just give us these roots  $\prod_{k | \frac{n}{\text{GCD}(n,s)}} \Phi_k(x)$  and show the multiplicity  $\prod_{k | \frac{n}{\text{GCD}(n,s)}} (\Phi_k(x))^{\text{GCD}(n,s)}$

# Direct Product of Cyclic Groups

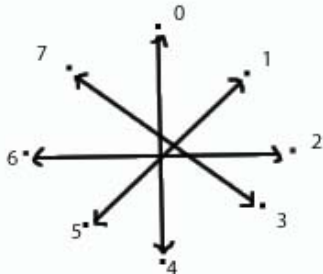
## Lemma

$$\text{Cay}(Z_n \oplus Z_m, \{(s, r)\}) \cong \text{Cay}(Z_{n \cdot m}, \{q\})$$

$$\text{Where } \nu = \frac{n}{\text{GCD}(n,s)} \quad \mu = \frac{m}{\text{GCD}(m,r)} \quad q = \frac{n \cdot m}{\text{LCM}(\nu, \mu)}$$



$\text{Cay}(Z_2 \oplus Z_4, \{(1, 2)\})$



$\text{Cay}(Z_8, \{4\})$

$$\nu=2 \quad \mu=2 \quad q=4$$

## Outline of Proof:

- Show the graphs have the same number of vertices
- Both groups have  $n \cdot m$  elements
- Show the graphs have the same number of disjoint cycles
- LHS: We treat the direct product as two separate cycles. One cycle is  $Z_n, \{s\}$  and the other is  $Z_m, \{r\}$ . Multiplying by  $(s,r)$  moves one vertex along each cycle. We can show that the number of moves required to complete a cycles is  $\nu$  in  $Z_n, \{s\}$  and  $\mu$  in  $Z_m$ . To reach the beginning of both cycles at the same time takes  $LCM(\nu, \mu)$  moves. So there are  $\frac{n \cdot m}{LCM(\nu, \mu)}$  disjoint cycles in with length  $LCM(\nu, \mu)$ .
- RHS:  $Cay(Z_{n \cdot m}, \{q\})$  has  $q$  cycles of length  $\frac{nm}{q}$  but  $q = \frac{n \cdot m}{LCM(\nu, \mu)}$ , so there are  $\frac{n \cdot m}{LCM(\nu, \mu)}$  cycles of length  $LCM(\nu, \mu)$ .

# How Cayley Graphs of $Z_n \oplus Z_m$ and $Z_{nm}$ Relate

## Theorem

$$\text{Cay}(Z_n \oplus Z_m, \{(s_1, r_1), (s_1, r_2), (s_2, r_1) \dots (s_i, r_j)\}) \cong \\ \text{Cay}(Z_{n*m}, \{q_{(s_1, r_1)} \dots q_{(s_i, r_j)}\})$$

$$\text{where } q_{(s_i, r_j)} = \frac{n*m}{\text{LCM}(\nu_{s_i}, \mu_{r_j})} \\ \nu_{s_i} = \frac{n}{\text{GCD}(n, s_i)} \text{ and } \mu_{r_j} = \frac{m}{\text{GCD}(m, r_j)}$$

The proof of this theorem is similar to the proof of the lemma. First show that the vertex set is the same, and then show that the graphs have the same type of cycles.

This theorem is important because isomorphic graphs have the same characteristic polynomial, and I already found the characteristic polynomial of  $\text{Cay}(Z_n, S)$ .



# Semidirect Products and the Dihedral Group

The semidirect product is a more complex way of multiplying a group. The dihedral group is a semidirect product of cyclic groups.

## Definition

$$Z_2 \ltimes Z_n = D_{2n} = \langle x, y : x^n = e, y^2 = e, yxy = x^{-1} \rangle$$

There are two basic sets of elements  $S_x = \{d \in D_n \mid d = x^a y^0\}$  and  $S_y = \{d \in D_n \mid d = x^a y^1\}$ .

$$x^a \cdot x^b = x^{a+b}$$

$$x^a \cdot x^b y = x^{a+b} y$$

$$x^a y \cdot x^b y = x^{a-b}$$

$$x^a y \cdot x^b = x^{a-b} y$$

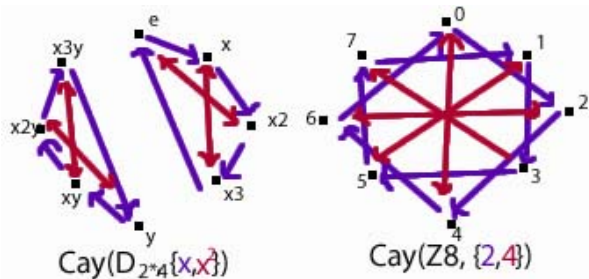
## Theorem

$$\text{Cay}(D_{2n}\{x^a y^\alpha, x^b y^\beta\}) \cong \begin{cases} \text{Cay}(Z_{2n}, \{2a, 2b\}) & \alpha, \beta = 0 \\ \text{Cay}(Z_{2n}\{\frac{+2n}{|x^a y, x^b y|}, \frac{-2n}{|x^a y, x^b y|}\}) & \alpha, \beta = 1 \\ \sum_1^{\text{GCD}(a,n)} \text{Cay}(D_{\frac{2n}{\text{GCD}(a,n)}}\{x, y\}) & \alpha = 0 \\ & \beta = 1 \end{cases}$$

# Case One

$$\text{Cay}(D_{2n}\{x^a, x^b\}) \cong \text{Cay}(Z_{2n}, \{2a, 2b\})$$

$$x^a \cdot x^b = x^{a+b} \quad x^a \cdot x^b y = x^{a+b} y$$



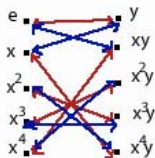
## Outline of Proof:

- multiplying by  $x^a$  splits the dihedral group into two sections
- each section acts just like the cyclic group

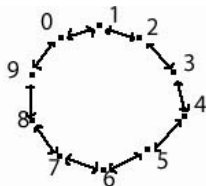
## Case Two

$$\text{Cay}(D_{2n}\{x^a y^\alpha, x^b y^\beta\}) \cong \text{Cay}(Z_{2n}\{\frac{+2n}{|x^a y, x^b y|}, \frac{-2n}{|x^a y, x^b y|}\})$$

$$x^a y \cdot x^b y = x^{a-b} \quad x^a y \cdot x^b = x^{a-b} y$$



$\text{Cay}(D_{10}, \{y, xy\})$



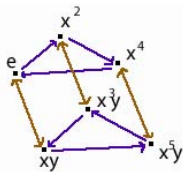
$\text{Cay}(Z_{10}, \{1, -1\})$

### Outline of Proof:

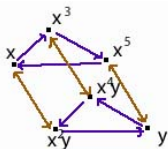
- multiplying by  $s_y$  goes between the two sets  $S_y$  and  $S_x$
- this makes a bipartite graph, which just creates cycles of the order of  $x^a y^\alpha, x^b y^\beta$

# Case Three

$$\text{Cay}(D_{2n}\{x^a, x^b y^\beta\}) \cong \sum_1^{\text{GCD}(a,n)} \text{Cay}(D_{\frac{2n}{\text{GCD}(a,n)}}\{x, y\})$$



$\text{Cay}(D_{12}, \{x^2, xy\})$



$\text{Cay}(D_6, \{x, y\})$

This is important because the characteristic polynomial of  $\text{Cay}(D_{2n}\{x, y\})$  is known.

# Semidirect Products of Prime Cyclic Groups

You can also take the semidirect product of some prime cyclic groups to create new groups. I am currently working on finding the characteristic polynomial of these groups.

## Preliminary Results

$$\chi(\text{Cay}(H_{p_1 p_2}\{x^a y^\alpha, x^b y^\beta\})) \cong \begin{cases} \sum_1^{p_1} \text{Cay}(Z_{p_2}\{a, \dots, b\}) & \alpha, \beta = 0 \\ \chi(\text{Cay}(Z_{p_1}\{\alpha, \beta\}))f(\lambda) & \alpha, \beta = 1 \\ ? & \alpha = 0 \\ & \beta = 1 \end{cases}$$