

Zeta Functions of Graph Covers

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Outline of Presentation

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1. Riemann Zeta Function

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2. Ihara Zeta Function

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3. Extend Zeta Function to Infinite Graphs

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6. $PSL_2(\mathbb{Z})$

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Riemann Zeta Function

- Riemann Zeta Function(Basic Definition)

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- Riemann Hypothesis

If $\zeta(s) = 0$ where $0 < \operatorname{Re}(s) < 1$, then $\operatorname{Re}(s) = 1/2$. This is to say that all non-trivial zeros of ζ are on the critical line.

Ihara Zeta Function

The Ihara-Zeta Function for a finite, connected graph G with no degree 1 vertices is defined as

$$Z_G(u) = \prod_{p \in \mathcal{P}} \frac{1}{1 - u^{\ell(p)}},$$

for $|u|$ small, and where the product is taken over all prime cycles p of G and $\ell(p)$ is the length of cycle p .

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This clearly resembles the Riemann Zeta Function,

$$\zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}}.$$

Ihara Zeta Function

Bass' Formula for the Ihara Zeta Function

Bass continued developing the definition of the Ihara Zeta Function by redefining it in terms of the regular adjacency matrix for a finite graph

$$Z_G(u)^{-1} = (1 - u^2)^{(\epsilon - v)} \det(I_n - uA + u^2Q),$$

where A is the adjacency matrix of G , ϵ is the number of edges, and v is the number of vertices. Q is a diagonal matrix such that the j th diagonal entry is equal to the degree of the j th vertex minus one.

Extension to Infinite Graphs

The Ihara Zeta Function $\zeta_X(t)$ for a finite graph X satisfies the relation,

$$\ln \zeta_X(t) = \sum_{r=1}^{\infty} \frac{c_r}{r} t^r,$$

where c_r is the number of closed, oriented loops of length r in the graph X .

Extension to Infinite Graphs

Definition (Grigorchuk and Zuk)

Let $X = \lim_{n \rightarrow \infty} X_n$ where X_n is a sequence of k -regular graphs such that the limit of $\tilde{c}_r = c_r(X_n)/|X_n|$ exists when $n \rightarrow \infty$. The zeta function $\zeta_X(t)$ of the graph X , with respect to the sequence $\{X_n\}$, is defined by

$$\ln \zeta_X(t) = \lim_{n \rightarrow \infty} \frac{1}{|X_n|} \ln \zeta_{X_n}(t) = \sum_{r=1}^{\infty} \tilde{c}_r t^r / r.$$

This series has a nontrivial interval of convergence of at least $1/k$ around 0.

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1. if $u_i \sim u_j$, then $p(u_i) \sim p(u_j)$, and
2. $p|_{N(u)} : N(u) \rightarrow N(p(u))$ forms a bijection where $N(u) = \{u_i | u \sim u_i\}$.

Covering Graphs

Example

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2. For any $g \in G, s \in S$, the vertices g and gs are joined by a line. Therefore the edge set $E(G)$ consists of pairs of the form (g, gs) .

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If \exists a surjective homomorphism $G \longrightarrow H$, then the map

$$p : \text{Cay}(G, S) \longrightarrow \text{Cay}(H, p(S))$$

is a covering map.

Graph Covers

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We will let the graph covers form naturally and retrieve the voltage assignment.

Method for Determining the Ihara Zeta Function

From Chae and Lee,

Theorem

Let $\Omega = Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_l}$ and let ϕ be a Ω -voltage assignment of G . Then, the adjacency matrix of a regular covering graph $G \times^\phi \Omega$ is

$$\sum_{(k_1, \dots, k_l)} A(\vec{G}_{\phi, (\rho_1^{k_1}, \dots, \rho_l^{k_l})}) \otimes P(\rho_1^{k_1}, \dots, \rho_l^{k_l}),$$

where $P(\rho_1^{k_1}, \dots, \rho_l^{k_l})$ is the permutation matrix associated with $(\rho_1^{k_1}, \dots, \rho_l^{k_l})$.

$$PSL_2(\mathbb{Z})$$

Motivation of Project

The modular group often seen as $PSL_2(\mathbb{Z})$ or $SL_2(\mathbb{Z})$ is a fundamental object of study in number theory and has many connections to other areas of mathematics.

We hope that finding the zeta function of the modular group will give us more insight on this important group.

To accomplish this, we will set up sequences of quotient groups, which will be associated with a sequence of Cayley graphs.

$PSL_2(\mathbb{Z})$

Set-up

Let $\Gamma = PSL_2(\mathbb{Z})$ and

$\Gamma_n = \ker(PSL_2(\mathbb{Z}) \longrightarrow PSL_2(\mathbb{Z}/2^n))$, for each $n \geq 1$.

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It is known that $PSL_2(\mathbb{Z})/\Gamma_n \simeq PSL_2(\mathbb{Z}/2^n)$. Therefore, we can form quotient groups Γ/Γ_n , from which we will construct Cayley graphs.

$PSL_2(\mathbb{Z})$

Consider the Cayley Graph of the quotient group denoted by $Cay(\Gamma/\Gamma_n, S)$, where

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

is the generating set of the Cayley graph.

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Each vertex of the graph will represent an element in the subgroups. The graphs will be 3-regular because there are three generators for each subgroup. Note that the voltage assignment for these coverings will always come from

$$\Omega = \Gamma_n/\Gamma_{n+1} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Current Status

Currently trying to generalize the sequence of graph covers by generalizing the subgraphs \vec{G} .

$$A(\vec{G}_{\phi, (\rho_1^{k_1}, \dots, \rho_l^{k_l})})$$