Spectra of Semidirect Products of Cyclic Groups

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Spectrum of a Graph

Definition

The **Characteristic Polynomial** of a graph X, denoted $\chi(X)$, is the characteristic polynomial of $A(X)$, the adjacency matrix of X.

Definition

The **Spectrum** of a graph X is the set of its **Eigenvalues**, which are the roots of $\chi(X)$.

Definition

Let $C(G, S)$ denote the Cayley graph of a group G using the set $S \subseteq G$ to generate the graph. (Precisely speaking, S is not a set, as it could contain duplicates.) The **Spectrum** of G given a set S is the spectrum of $C(G, S)$.

Graph Covering

Definition

A **Graph Morphism** is a map φ from a graph X to a graph Y such that each vertex of X maps to a vertex of Y, and a (directed) edge (v_1, v_2) in X maps to the edge $(\varphi(v_1), \varphi(v_2))$ in Y.

Definition

A **Graph Covering** is a surjective graph morphism $\varphi : X \longrightarrow Y$ which gives for each vertex v_1 in X a bijection between edges leaving v_1 and edges leaving $\varphi(v_1)$ and a bijection between edges entering v_1 and edges entering $\varphi(v_1)$. We say that the graph X **Covers** the graph Y if there exists a covering map from X to Y.

Graph Covering: Properties

Theorem

If a graph X covers a graph Y, then $\chi(Y)$ divides $\chi(X)$.

Proof involves showing that the adjacency matrix of X can be written in the form $\left[\begin{array}{c|c} B & C \\ \hline 0 & A(Y) \end{array}\right]$ (where $A(Y)$ is the adjacency matrix of Y).

Proposition

Given a group homomorphism φ from a group G onto a group K (or, equivalently, given $H \lhd G$ such that $G/H = K$), $C(G,S)$ covers $C(K,\varphi(S))$.

Semidirect Product

Definition

Given two groups G and H and a group homomorphism $\varphi : H \longrightarrow \mathrm{Aut}\,(G)$, the **Semidirect Product** of G and H with respect to φ, denoted $G \rtimes_\varphi H$ (or, simply, $G \rtimes H$) is a new group with set $G \times H$ and multiplication operation $(g_1, h_1)\,(g_2, h_2) = (g_1\varphi\,(h_1)\,g_2, h_1 h_2)$.

Proposition

Given cyclic groups $\mathbb{Z}/m$ and $\mathbb{Z}/n$, a semidirect product $\mathbb{Z}/m \rtimes \mathbb{Z}/n$ between them corresponds to a choice of integer k such that $k^n \equiv 1 \pmod{m}$. The semidirect product group is given by $\mathbb{Z}/m \rtimes \mathbb{Z}/n = \langle x, y \mid x^m = e, y^n = e, yxy^{-1} = x^k \rangle$.

Additional Introductory Information

- Semidirect products provide a simple way of building non-abelian groups.
- Little is known in general about the spectra of non-abelian groups.
- Techniques used here in analyzing the spectra:
  - Using graph coverings to find factors of the characteristic polynomials of the Cayley graphs.
  - Using block matrices to simplify computation.
  - Finding connections with roots of unity.

Dihedral Groups

For all cyclic groups, there is at least one automorphism of order 2. One of these automorphisms corresponds to a choice of $k = -1$, and is realized by inverting all of the elements of the group.

Definition

Given an integer n, the **Dihedral Group of Order 2n**, denoted D_{2n}, is given by $\langle x, y \mid x^n = e, y^2 = e, yxy^{-1} = x^{-1} \rangle = \mathbb{Z}/n \rtimes \mathbb{Z}/2$.

Theorem (Taylor Coon)

$\chi\left(C\left(D_{2n}, \{x, y\}\right)\right) = \lambda^n \chi\left(C\left(\mathbb{Z}/n, \{\pm 1\}\right)\right)$.

Taylor's proof uses the Figure Equation, which is a formula for the characteristic polynomial of a graph that does not involve the adjacency matrix. I found an alternative proof using the adjacency matrix.

Dihedral Groups

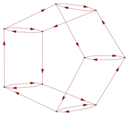

Figure: $C\left(D_{10}, \{x, y\}\right)$

- The λ^n factor in the characteristic polynomial comes from the fact that every row in the adjacency matrix for a dihedral group repeats.
- The other factor in the characteristic polynomial comes from a covering of $\mathbb{Z}/n$ given by the map $\varphi : D_{2n} \longrightarrow \mathbb{Z}/n$, where $\varphi\left(x^a y^b\right) = x^{a+b}$ for $0 \leq a < n$ and $0 \leq b < 2$.
- But, φ is NOT a group homomorphism. However, there is another covering given by the group homomorphism $\psi : D_{2n} \longrightarrow \mathbb{Z}/2$, where $\psi\left(x\right) = e$ and $\psi\left(y\right) = y$.

Semidirect Products of Cyclic Groups of Odd Prime Order

For odd primes p_1 and p_2, a nontrivial semidirect product $\mathbb{Z}/p_2 \rtimes \mathbb{Z}/p_1$ will exist if and only if p_1 divides $p_2 - 1$.

Definition

Given odd primes p_1 and p_2 with p_1 dividing $p_2 - 1$, their semidirect product, which will be denoted $H_{p_1 p_2}$, is given by $\left\langle x, y \mid x^{p_2} = e, y^{p_1} = e, yxy^{-1} = x^k \right\rangle = \mathbb{Z}/p_2 \rtimes \mathbb{Z}/p_1$ for some k such that $k^{p_1} \equiv 1 \pmod{p_2}$.

Proposition

For any two values of $k \neq 1$ with $k^{p_1} \equiv 1 \pmod{p_2}$, the resulting semidirect products are isomorphic.

(Proof involves showing that changing the value of k in this way is an automorphism of the group.)

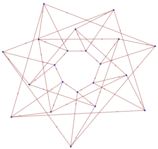

Figure: $C\left(H_{21}, \{x, y\}\right)$

- Based on their definitions, these groups appear similar in structure to the dihedral groups.
- Here, however, there appears to be no covering of a cyclic group that does not come from a group homomorphism.
- The expected covering of $\mathbb{Z}/p_1$ sending x to the identity and y to y is present, and this behavior is well-understood.
- Therefore, I was more concerned with investigating the rest of the characteristic polynomial of $C\left(H_{p_1 p_2}, \{x, y\}\right)$.

Semidirect Products of Cyclic Groups of Odd Prime Order

Main Theorem

Let ω be a primitive p_2^{th} root of unity, and W is a specific matrix with characteristic polynomial the p_2^{th} cyclotomic polynomial.
$\chi\left(C\left(H_{p_1 p_2}, \{x, y\}\right)\right) = \left((1-\lambda)^{p_1} + 1\right) s\left(\lambda\right)$, where

$$s\left(\lambda\right) = \prod_{n=1}^{p_2-1} \left(\prod_{j=0}^{p_1-1} \left(\lambda - \omega^{nk^j}\right) - 1 \right) = \det \left(\prod_{j=0}^{p_1-1} \left(\lambda I - W^{k^j}\right) - I \right)$$

Example

If $p_2 = 7$, $W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Corollary

$s\left(\lambda\right) = r\left(\lambda\right)^{p_1}$ *for some integer polynomial* $r\left(\lambda\right)$.

Semidirect Products of Cyclic Groups of Odd Prime Order

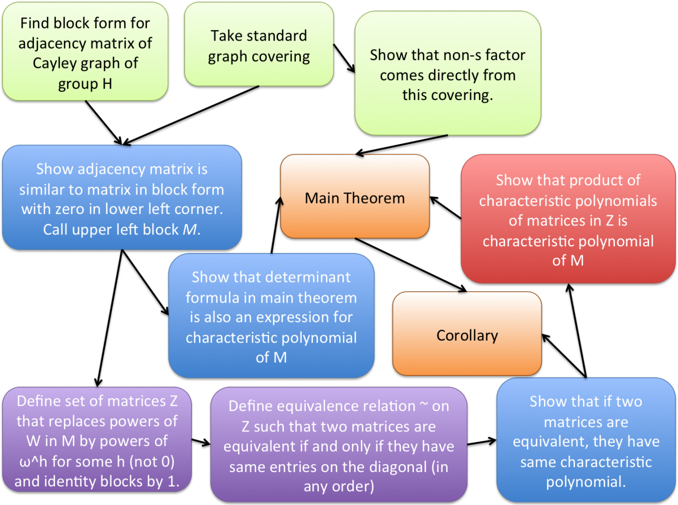

Figure: Proof Flowchart

Semidirect Products of Cyclic Groups of Odd Prime Order

Example

Let A be the adjacency matrix. If $p_1 = 3$ and $p_2 = 7$, then (for $k = 2$) $A = \left[\begin{array}{c|c|c} C & I & 0 \\ \hline 0 & C^2 & I \\ \hline I & 0 & C^4 \end{array}\right]$ and $M = \left[\begin{array}{c|c|c} W & I & 0 \\ \hline 0 & W^2 & I \\ \hline I & 0 & W^4 \end{array}\right]$, where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$\chi\left(C\left(H_{39}, \{x, y\}\right)\right) = \left((1-\lambda)^3 + 1\right)\left(\lambda^{12} + \lambda^{11} + \lambda^{10} - 3\lambda^9 - 2\lambda^8 + 5\lambda^6 + 7\lambda^5 - 3\lambda^4 - 4\lambda^3 + 8\lambda + 16\right)^3$$

Generalized Dihedral Groups

Definition

Given integers m, n, the **Generalized Dihedral Group of Order 2mn with cycle 2m**, denoted $\Delta_{m,n}$, is given by $\left\langle x, y \mid x^n = e, y^{2m} = e, yxy^{-1} = x^{-1} \right\rangle = \mathbb{Z}/n \rtimes \mathbb{Z}/2m$.

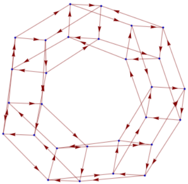

Figure: $C\left(\Delta_{2,7}, \{x, y\}\right)$

Generalized Dihedral Groups

Preliminary Results:

- There is a covering from $\Delta_{m,n}$ to $\Delta_{\frac{m}{s},\frac{n}{t}}$ for every s dividing m and t dividing n.
- For each positive integer i, there is a polynomial $p_i(\lambda)$ such that
$$\chi\left(C\left(\Delta_{m,n},\{x,y\}\right)\right)=\prod_{i\mid n}p_i(\lambda).$$
- For each positive integer i, there is a polynomial $q_i(\lambda)$ such that
$$\chi\left(C\left(\Delta_{m,n},\{x,y\}\right)\right)=\prod_{i\mid m}q_i(\lambda).$$
- $\deg(p_n)=2m\phi(n)$ and $\deg(q_m)=2n\phi(m)$, where ϕ is Euler's totient function.

Generalized Semidirect Products of Odd Prime Cyclic Groups

Theorem

For all positive integers b, there exists a positive integer a such that $(p_2 a + k)^{p_1} \equiv 1 \pmod{bp_2}$, *where* $k^{p_1} \equiv 1 \pmod{p_2}$.

Definition

Given odd primes p_1 and p_2 with p_1 dividing $p_2 - 1$ and given positive integers m and n, their generalized semidirect product, which will be denoted η_{m,n,p_1,p_2}, is given by $\left\langle x, y \mid x^{mp_2} = e, y^{np_1} = e, yxy^{-1} = x^{\kappa} \right\rangle = \mathbb{Z}/mp_2 \rtimes \mathbb{Z}/np_1$, where $\kappa = p_2 a + k$ is given by the previous theorem.

Generalized Semidirect Products of Odd Prime Cyclic Groups

Preliminary Results:

- There is a covering from η_{m,n,p_1,p_2} to $\eta_{\frac{m}{s},\frac{n}{t},p_1,p_2}$ for every s dividing m and t dividing n.
- For each pair of positive integers i, j, there is a polynomial $q_{i,j}(\lambda)$ such that

$$\chi\left(C\left(\eta_{m,n,p_1,p_2},\{x,y\}\right)\right)=\prod_{i|n}\prod_{j|m} q_{i,j}(\lambda).$$

- $\deg(q_{m,n}) = p_1 p_2 \phi(m)\phi(n)$.