C*-algebras of Graph Products

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Abstract

For certain graphs, we can associate a universal C*-algebra, which encodes the information of the graph algebraically. In this paper we examine the relationships between products of graphs and their associated C*-algebras. We present the underlying theory of associating a C*-algebra to a direct graph as well as to a higher rank graph. We then provide several isomorphisms relating C*-algebras of product graphs to products of C*-algebras. Among these, we include a discussion of the direct sum, crossed product, and tensor product of graph algebras.

1 Introduction

The notion of a C*-algebra associated to a graph has been a subject of great interest since 1980 when it was first introduced by Cuntz and Krieger. These algebras provide many examples for operator algebraists and tend to show up in many different fields, including non-abelian duality, non-commutative geometry, and in the study of C*-algebra structure. The algebraic properties of a graph algebra are related to the combinatorial properties of the underlying directed graph. Another idea of great interest in recent years is the notion of higher rank graphs which comes from category theory.

Product graphs occur naturally in discrete mathematics and are of combinatorial interest. They can often carry structure such as that of a metric space. Recent interest in product graphs has largely focused on recognition algorithms to identify a graph as a product of smaller graphs. Another point of interest is in the visualization of direct and semidirect products of groups by the product graphs of their Cayley graphs. In this paper we explore the link between these two ideas. That is, we provide a discussion of the structure of C*-algebras associated to product graphs.

In section 1 we define some basic graph terminology along with various graph products as well as higher rank graphs. We will find 2-graphs to be very useful in making the connection between some product graphs and their associated C*-algebras. We also provide a method for constructing a higher rank graph from two directed graphs due to Alex Kumjian and David Pask in [8].

In section 2 we present the methodology of associating a C*-algebra to a graph, together with some examples. It is known that there is a universal C*-algebra of a directed graph unique up to isomorphism [9]. Using a modified methodology taken from [8], [9], and [10] we can also form a C*-algebra for a higher rank graph. We show that the higher rank graph construction discussed in section 1 has the same uniqueness properties as the graph algebras for directed graphs, by satisfying a known criterion of local convexity. We also show that the graph algebra of a directed graph is isomorphic to the graph algebra of the 1-graph which is its path category, allowing us to discuss these two ideas equivalently.

In section 3 we present relationships between graph products and C*-algebra constructions. First we provide a brief discussion of the direct sum of graph algebras as the graph algebra of the union of graphs. We then present a result of Alex Kumjian and David Pask in [7] which characterizes the crossed product of a graph algebra as the graph algebra of a factor graph. Finally we use the notion of higher rank graphs to describe the tensor product of graph algebras, and we present the graph algebra of the tensor product of graphs as a sub-C*-algebra of this tensor product algebra.

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2 Graphs

2.1 Directed Graphs

A **graph** $G = (G^0, G^1, r, s)$ consists of a vertex set G^0, together with an edge set G^1, along with two mappings $r : G^1 \to G^0$ (range) such that $r((x, y)) = y$ and $s : G^1 \to G^0$ (source) such that $s((x, y)) = x$.

Let $u, v \in G^0$. We say that u is **adjacent** to v in G, denoted $u \sim v$, if there is an element $e \in G^1$ such that $s(e) = u$ and $r(e) = v$. A vertex v is called a **source** if $r^{-1}(v) = \emptyset$. It is common to illustrate these relationships by drawing the vertices of the graph as points, with the adjacencies (or edges) of the graph indicated by arrows pointing from the source vertex $s(e)$ to the range vertex $r(e)$. We then sometimes denote the edge e by the ordered pair $(r(e), s(e))$. Given a finite graph G (i.e. G^0 and G^1 are both finite), we define the

adjacency matrix A_G to be the $|G^0| \times |G^0|$ matrix defined by

$$A_G(v, w) = |\{e \in G^1 : r(e) = v, s(e) = w\}|.$$

We then say that the graph G is **row-finite** if each row of A_G has finite sum, that is each vertex is the range of finitely many edges.

A graph is said to be **undirected** if the adjacency relation on the vertex set G^0 is symmetric, that is $u \sim v \iff v \sim u$ for all $u, v \in G^0$. Hence, in the case of an undirected graph all arrows between points are double-headed arrows, and these are often suppressed so that adjacencies are illustrated with line segments. All other graphs are said to be directed graphs or **digraphs**. Note that the edges of directed graphs are frequently referred to as arcs to distinguish them from undirected edges. Since it is more common to work with undirected graphs, it is conventional to refer to undirected graphs just as graphs and to refer to all other graphs as directed graphs. However, for the purposes of this paper all graphs will be assumed to be directed.

A **graph homomorphism** from E to F is a map $\varphi : E^0 \to F^0$ that preserves the range and source maps r, s. Then, if φ is also bijective we say it is a **graph isomorphism**, and if further $E = F$ we call φ a **graph automorphism**. It is clear that the set of all automorphisms on E, denoted by $Aut(E)$, is a group under composition.

We say a group G acts on a graph E if there exists a homomorphism $\varphi : G \to Aut(E)$. Then the map $\cdot$ from $G \times E^0$ to E^0 defined by $g \cdot x = \varphi(g)(x)$ for all $g \in G, x \in E^0$ is called a **group action of** G **on** E.

A **path of length** n in a graph G is then a sequence of edges $\mu = \mu_1, \mu_2, \ldots, \mu_n$ such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \le i \le n-1$. We write $|\mu|$ for n, the length of the path. Vertices are then said to be paths of length 0. Given a graph G, the set of all paths of length n is denoted G^n, and we define G^* to be $\cup_{n \ge 0} G^n$. For $n > 1$ naturally extend the range and source maps r, s to elements of G^* with $r(\mu) := r(\mu_1)$ and $s(\mu) := s(\mu_{|\mu|})$. For $v \in E^0$, we define $r(v) = v = s(v)$. Let $\mu, \nu \in E^*$. Then we say μ is composable with ν if $r(\nu) = s(\mu)$, and we write their composition $\mu\nu = \mu_1\mu_2 \ldots \mu_{|\mu|}\nu_1\nu_2 \ldots \nu_{|\nu|}$.

$$\begin{tikzcd}
s(\nu) \arrow[r, "\nu"] \arrow[dr, "\mu\nu"'] & r(\nu) = s(\mu) \arrow[d, "\mu"] \\
& r(\mu)
\end{tikzcd}$$

2.2 Graph Products

Given two graphs $E = (E^0, E^1, r_E, s_E)$ and $F = (F^0, F^1, r_F, s_F)$, there are various ways of combining them to form a larger graph. This larger graph is then usually called their product graph. We now define several graph products that we will discuss. Note that each of these products is associative and commutative. As a visual aid, we will illustrate each of these product graphs for the following graphs E, F.

$$\begin{tikzcd} E: u_1 \arrow[d] & F: u_2 \arrow[r] \arrow[rr, bend left] & v_2 \arrow[r] & w_2 \\ v_1 \end{tikzcd}$$

Definition 2.1. *The* ***(disjoint) union*** *of E with F is the graph $E \cup F = (E^0 \cup F^0, E^1 \cup F^1, r_\cup, s_\cup)$ such that for all $e \in E^1$, we define $r_\cup(e) = r_E(e)$ and $s_\cup(e) = s_E(e)$, and for all $e \in F^1$ we define $r_\cup(e) = r_F(e)$ and $s_\cup(e) = s_F(e)$.*

$$\begin{tikzcd} E \cup F: & u_1 \arrow[d] & u_2 \arrow[r] \arrow[rr, bend left] & v_2 \arrow[r] & w_2 \\ & v_1 \end{tikzcd}$$

Note that here it is assumed that E^0 and F^0 are disjoint. Also note that the union of a graph G with a copy of itself n times is denoted nG.

Definition 2.2. *The* ***box (cartesian) product*** *of E with F is the graph $E \square F = (E^0 \times F^0, (E^1 \times F^0) \cup (E^0 \times F^1), r_\square, s_\square)$, where $r_\square, s_\square$ are defined as follows: For all $e \in E^1, f \in F^1, u \in E^0, v \in F^0$,*

$$r_\square(e, v) = (r_E(e), v) \qquad r_\square(u, f) = (u, r_F(f))$$

$$s_\square(e, v) = (s_E(e), v) \qquad s_\square(u, f) = (u, s_F(f))$$

$$\begin{tikzcd} E \square F: & (u_1, u_2) \arrow[r] \arrow[rr, bend left] \arrow[d] & (u_1, v_2) \arrow[r] \arrow[d] & (u_1, w_2) \arrow[d] \\ & (v_1, u_2) \arrow[r] \arrow[rr, bend left] & (v_1, v_2) \arrow[r] & (v_1, w_2) \end{tikzcd}$$

Definition 2.3. *The* ***tensor (categorical) product*** *of E with F is the graph $E \times F = (E^0 \times F^0, E^1 \times F^1, r_\times, s_\times)$, such that for all $(e, f) \in E^1 \times F^1$ we define $r_\times(e, f) = (r_E(e), r_F(f))$ and $s_\times(e, f) = (s_E(e), s_F(f))$.*

$$\begin{tikzcd} E \times F: & (u_1, u_2) \arrow[dr] \arrow[drr] & (u_1, v_2) \arrow[dr] & (u_1, w_2) \\ & (v_1, u_2) & (v_1, v_2) & (v_1, w_2) \end{tikzcd}$$

Definition 2.4. *Let E and F be graphs such that $E^0 = F^0$. Then the* ***overlay product*** *of E with F is the graph $E \bowtie F = (E^0 = F^0, E^1 \cup F^1, r_{\bowtie}, s_{\bowtie})$, such that for all $e \in E^1 \cup F^1$ if $e \in E^1$ then $r_{\bowtie}(e) = r_E(e)$ and $s_{\bowtie}(e) = s_E(e)$, and if $e \in F^1$ then $r_{\bowtie}(e) = r_F(e)$ and $s_{\bowtie}(e) = s_F(e)$.*

We then define the ***strong product*** *of E with F to be $E \boxtimes F = (E \Box F) \bowtie (E \times F)$.*

$$E \boxtimes F: \quad \begin{tikzcd} (u_1,u_2) \arrow[r] \arrow[rr, bend left] \arrow[d] \arrow[dr] \arrow[drr] & (u_1,v_2) \arrow[r] \arrow[d] \arrow[dr] & (u_1,w_2) \arrow[d] \\ (v_1,u_2) \arrow[r] \arrow[rr, bend right] & (v_1,v_2) \arrow[r] & (v_1,w_2) \end{tikzcd}$$

Definition 2.5. *Let G be a countable group, and $E = (E^0, E^1, r_E, s_E)$ be a row-finite graph. Then, given a function $c : E^1 \to G$, we define the* ***skew product graph*** *$E(c) = (G \times E^0, G \times E^1, r, s)$, where for all $(g,e) \in E(c)^1$ we have*

$$r((g,e)) = (gc(e), r_E(e)),$$

$$s((g,e)) = (g, s_E(e)).$$

This graph $E(c)$ is sometimes referred to as a derived graph, and together with the labeling $c : E^1 \to G$ it is referred to as a voltage graph. Notice that the formation of the skew product graph is a very different operation from the others we have discussed, in that it is not a binary operation on two graphs, but a product of a group with a graph. Also notice that when c maps every edge of E to the identity of G the resulting skew product graph is isomorphic to the graph $|G|E$ (i.e. E unioned with itself $|G|$ times).

$$E: \quad \begin{tikzcd} x \arrow[r, "e"] \arrow[d, "h"'] & y \arrow[d, "f"] \\ w \arrow[r, "g"'] & z \end{tikzcd} \qquad G = \mathbb{Z}/3\mathbb{Z}$$

$$c: \quad e \mapsto 0, g \mapsto 0, f \mapsto 1, h \mapsto 2$$

$$
\begin{tikzcd}
E(c): & (0,x) \arrow[r] \arrow[ddd, bend right] & (0,y) \arrow[drr] & (1,x) \arrow[r, rightsquigarrow] \arrow[dll, rightsquigarrow] & (1,y) \arrow[dddll, rightsquigarrow] \\
& (0,w) \arrow[r] & (0,z) & (1,w) \arrow[r, rightsquigarrow] & (1,z) \\
& (2,x) \arrow[r, Rightarrow] \arrow[urr, Rightarrow] & (2,y) \arrow[u, Rightarrow] & & \\
& (2,w) \arrow[r, Rightarrow] & (2,z) & &
\end{tikzcd}
$$

Additionally, it is easily shown that for a cyclic group $G =< 1 >$ and an arbitrary graph E, the skew product graph $E(c)$, where $c : e \mapsto 1$ for all $e \in E^1$, is isomorphic to the tensor product graph $E_G \times E$, where E_G is the graph with vertices as elements of G, directionally adjacent to each other if and only if their difference in G is 1. For we see that the vertex sets of these two graphs are the same, since

$$E(c)^0 = G \times E^0 = E_G^0 \times E^0 = (E_G \times E)^0$$

so we need only show that the edge sets are the same and there is a mapping preserving the structure maps of $E(c)$ and $E_G \times E$. Consider an edge $((h, b), (g, a)) \in E(c)^1$ for $(h, b), (g, a) \in E(c)^0$ (i.e. an edge from (g, a) to (h, b)). By the definition of a skew product graph, this is equivalent to $a \sim b$ in E and $h = g + c((b, a)) = g + 1$ in G. This means that $(h, g) \in E_G^1$, and so the conditions for $((h, b), (g, a)) \in E(c)^1$ are equivalent to those for $((h, b), (g, a)) \in (E_G \times E)^1$. Thus, the edge sets of these two graphs and their respective structure maps are the same. Hence, $E(c) \cong E_G \times E$, for c as chosen above.

2.3 Higher-rank Graphs

The formal definition of a k-graph, given in [1, p.89], relies on some simple concepts from category theory. This is the definition we give here.

Definition 2.6. *A* ***category*** C *consists of two classes* C^0 *of objects and* C^* *of morphisms, and two functions* $r, s : C^* \to C^0$ *called codomain and domain respectively, as well as a partially defined product (composition)* $(f, g) \mapsto f \circ g$ *from* $\{(f, g) \in C^* \times C^* : s(f) = r(g)\}$ *to* C^**, satisfying*

$$r(f \circ g) = r(f) \text{ and } s(f \circ g) = s(g)$$

$$(f \circ g) \circ h = f \circ (g \circ h) \text{ when } s(f) = r(g) \text{ and } s(g) = r(h),$$

and distinguished elements (identity morphisms) $\{i_v \in C^* : v \in C^0\}$ *satisfying*

$r(i_v) = v = s(i_v)$ and $i_v \circ f = f, g \circ i_v = g$ when $r(f) = v$ and $s(g) = v$.

*A **functor** $F : C \to D$ is a pair of maps $F^0 : C^0 \to D^0$ and $F^* : C^* \to D^*$ which respect the domain and codomain maps and composition, and which satisfy $F^*(i_v) = i_{F^0(v)}$.*

*Given categories C and D, the **product category** $C \times D$ is the category with*

$(C \times D)^0 = C^0 \times D^0$

$(C \times D)^* = C^* \times D^*$, with composition component-wise from the contributing categories and identities $(1_A, 1_B)$ where $A \in C^0, B \in D^0$.

We will immediately become less formal and write fg for $f \circ g$. We also write $f : s(f) \to r(f)$ to indicate a morphism f, together with its domain and codomain.

Example 2.7.

Examples of categories are:

Grp with objects groups and morphisms group homomorphisms

Ab with objects abelian groups and morphisms group homomorphisms

Ring with objects rings and morphisms ring homomorphisms

Top with objects topological spaces and morphisms homeomorphisms

Set with objects arbitrary sets and morphisms mappings between them

While these are the standard morphisms for these classes of objects, any other set of mappings satisfying the necessary properties would make them into categories. For example, the set of all rings with morphisms between them being group homomorphisms is a category distinct from Ring.

Example 2.8.

Given a graph G, there is an associated path category $P(G)$, with objects G^0 and morphisms G^*, the set of all finite paths in the graph G. When it is clear that we are speaking of the path category of G, we will often denote it G^* and identify its objects with the paths of length zero.

It is natural to visualize a category with its diagram, which is a graph, where we draw the objects as vertices and the morphisms as edges, generally suppressing the loops that arise as identity morphims.

Example 2.9.

Let C be the category with objects $C^0 = \{a, b\}$ and morphisms $C^* = \{i_a, e : a \to b, i_b\}$. Then the product category of C with itself is the category $C \times C$ with objects

$$(C \times C)^0 = \{(a,a), (a,b), (b,a), (b,b)\}, \text{ and morphisms}$$
$$(C \times C)^* = \{(i_a, i_a), (i_a, e), (i_a, i_b), (e, i_a), (e, e), (e, i_b), (i_b, i_a), (i_b, e), (i_b, i_b)\},$$

where the domain and codomain for each of these morphisms is determined by the definition of the product category. The digrams for C and $C \times C$ are then:

$$C: \quad a \xrightarrow{e} b \qquad\qquad C \times C: \quad \begin{tikzcd} (a,a) \arrow[r, "{(i_a,e)}"] \arrow[d, "{(e,i_a)}"'] \arrow[dr, "{(e,e)}"] & (a,b) \arrow[d, "{(e,i_b)}"] \\ (b,a) \arrow[r, "{(i_b,e)}"'] & (b,b) \end{tikzcd}$$

Definition 2.10. *A **monoid** $(M, \cdot)$ is a set M together with a binary operation $\cdot$, satisfying*

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity), for all $a, b, c \in M$,

there exists an identity $e \in M$ such that $e \cdot a = a \cdot e = a$, for all $a \in M$.

Note that a monoid may equivalently be defined as a one-object category. That is, the morphisms in the one-object category are the elements of the monoid, and the binary operation $\cdot$ is composition of morphisms in the category.

Example 2.11.

$(\mathbb{N}^k, +)$ is a monoid, where addition is component-wise addition of natural numbers, and the identity is the k-tuple with zeros in all entries. We find it useful to define e_i to be the k-tuple with a 1 in the ith entry and zeros elsewhere.

Much of our discussion will be dependent on the following definition of a higher-rank graph taken from Chapter 10 of [9].

Definition 2.12. *A **graph of rank** k is a countable category Λ, together with a functor $d : \Lambda \to \mathbb{N}^k$, called the degree map, with the following unique factorization property:*

for every morphism λ and every decomposition $d(\lambda) = m + n$ with $m, n \in \mathbb{N}^k$, there exist unique morphisms μ and ν such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

*We also call (Λ, d) a **k-graph** or just a **higher-rank graph**, and we usually abbreviate (Λ, d) to Λ. Λ^n is the set of morphisms, or paths, of degree n. And $\Lambda^m(v) := \{\lambda \in \Lambda^m : r(\lambda) = v\}$. (Λ, d) is **row-finite** if for each $v \in \Lambda^0$ and $m \in \mathbb{N}^k$, the set $\Lambda^m(v)$ is finite. (Λ, d) has **no sources** if $\Lambda^m(v) \neq \emptyset$ for all $v \in \Lambda^0, m \in \mathbb{N}^k$. A row-finite k-graph Λ is **locally convex** if the existence of $\lambda \in \Lambda^{e_i}$ and $\mu \in \Lambda^{e_j}$ with $i \neq j$ and $r(\lambda) = r(\mu)$ implies the existence of $\nu \in \Lambda^{e_j}$ with $r(\nu) = s(\lambda)$; this is automatic is Λ has no sources.*

Note that we are implicitly identifying the objects in the category with the corresponding identity morphisms. For the purposes of this paper, we will be considering only graphs of ranks 1 and 2.

Example 2.13.

The path category of a graph G is a 1-graph, with $d(\mu) = |\mu|$ for all $\mu \in G^*$.

Definition 2.14. *The **1-skeleton** of 2-graph Λ is a 2-colored graph with vertices being the objects of Λ and edges of one color being the paths in $\Lambda^{(0,1)}$ and of the color being the paths in $\Lambda^{(1,0)}$.*

In our 1-skeletons, we will represent one color with squiggly edges and the other with straight edges.

Proposition 2.15. *Let $(\Lambda_1, d_1), (\Lambda_2, d_2)$ be 1-graphs. Then, their product category $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is a 2-graph where $d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \to \mathbb{N}^2$ is given by $d_1 \times d_2(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2))$, for $\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2$.*

This is a special case of Proposition 1.8 of [8],and its proof is straightforward.

Example 2.16.

Let G be the following graph:

$$a \xrightarrow{e} b\ .$$

Then, we see that the path category G^* is C from example 1.9,and that $G^* \times G^*$ is a two graph with the degree map $d = d_1 \times d_1$ as given in the previous proposition.

Its 1-skeleton is

$$\begin{tikzcd}
(a,a) \arrow[r, "{(i_a,e)}"] \arrow[d, squiggly, "{(e,i_a)}"'] & (a,b) \arrow[d, squiggly, "{(e,i_b)}"] \\
(b,a) \arrow[r, "{(i_b,e)}"'] & (b,b)
\end{tikzcd}$$

In the 2-graph $G^* \times G^*$, the path (e, e) is equal to both of the paths $(e, i_b)(i_a, e)$ and $(i_b, e)(e, i_a)$ by the unique factorization property of 2-graphs, so

$$d(e, e) = d((e, i_b)(i_a, e)) = d((i_b, e)(e, i_a)) = (1, 1).$$

3 Graph Algebras

Definition 3.1. *A **C*-algebra** A is a vector space over $\mathbb{C}$, together with a binary operation $\cdot$ called multiplication, a map $* : A \to A$ called involution, and a map $\|\cdot\| : A \to \mathbb{R}$ called the norm, such that for all $x, y, z \in A$, $\lambda \in \mathbb{C}$ we have*

$$(x+y)\cdot z = x\cdot z + y\cdot z \tag{1}$$
$$x\cdot(y+z) = x\cdot y + x\cdot z \tag{2}$$
$$(ax)\cdot(by) = (ab)(x\cdot y) \tag{3}$$
$$(x\cdot y)\cdot z = x\cdot(y\cdot z) \tag{4}$$
$$x^{**} = x \tag{5}$$
$$(yx)^* = x^*y^* \tag{6}$$
$$(\lambda x)^* = \overline{\lambda}x^* \tag{7}$$
$$\|x\| \geq 0 \quad \textit{with equality if and only if} \quad x = 0 \tag{8}$$
$$\|\lambda x\| = |\lambda|\|x\| \tag{9}$$
$$\|x+y\| \leq \|x\| + \|y\| \tag{10}$$
$$\|x\cdot y\| \leq \|x\|\|y\| \tag{11}$$
$$\|x^*\cdot x\| = \|x\|^2, \tag{12}$$

and such that $(A, \|\cdot\|)$ is a complete metric space.

*A **sub-C*-algebra** of A is a nonempty subset of A which is a C*-algebra with respect to the operations given on A.*

Given a C-algebra A and a subset S of A, the **sub-C*-algebra of** A **generated by** S is the smallest C*-algebra of A that contains S, and we denote it by $C^*(S)$. This can equivalently be described as follows: For each $n \in \mathbb{N}$, put*

$$W_n = \{x_1x_2\ldots x_n : x_j \in S \cup S^*\},$$

where $S^* = \{x^* : x \in S\}$, and put $W = \cup_{n=1}^{\infty} W_n$. The set W is the set of all words in $S \cup S^*$. Using that $W = W^*$ and that W is closed under multiplication, if follows that $C^*(S) = \overline{span(W)}$.

Remark 3.2.

Conditions (8), (11), and (12) above together imply that the involution in a C*-algebra is isometric since for all $a \in A$,

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\| \Rightarrow \|a\| \leq \|a^*\|.$$

In particular, $\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\| \Rightarrow \|a\| = \|a^*\|$.

Definition 3.3. *A ***-homomorphism** $\varphi : A \to B$ between C*-algebras A and B is a linear multiplicative map which satisfies $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.*

Example 3.4.

1. Let X be a compact Hausdorff space. $C(X)$, the set of all complex-valued continuous functions on X is a C*-algebra with operations defined as follows for all $f, g \in C(X)$, $x \in X$, $c \in \mathbb{C}$.

$$\begin{aligned}
&(f+g)(x) = f(x) + g(x) \\
&(cf)(x) = cf(x) \\
&f^*(x) = \overline{f(x)} \\
&\|f\| = \max_{x \in X} |f(x)|.
\end{aligned}$$

2. For each $n = 1, 2, 3, \ldots$, the complex vector space $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$, viewed as operators on $\mathbb{C}^n$, is a C*-algebra with multiplication defined as matrix multiplication, the involution defined as taking the conjugate transpose, and the norm being the operator norm on matrices, given by $\|A\|_{op} = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\|$. This example is essentially the same as the following example, in finite dimensional cases.

3. An especially important example is the following. $B(H)$, the algebra of all bounded operators on a Hilbert space H, is a C*-algebra with the usual adjoint operation. This follows from the well-known identity

$$\|A^*A\| = \sup_{\|x\|=\|y\|=1} |(A^*Ax, y)| = \sup_{\|x\|=\|y\|=1} |(Ax, Ay)| = \|A\|^2.$$

Remark 3.5.

The **Gelfand-Naimark theorem** says that for any C*-algebra A, there exist a Hilbert space H and an isometric *-homomorphism $\varphi : A \to B(H)$, so that every C*-algebra is isomorphic to a sub-C*-algebra of $B(H)$. We will usually be dealing with finite dimensional cases in this paper, so our C*-algebras will usually be subalgebras of $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

3.1 Cuntz-Krieger Families for Directed Graphs

Every row-finite directed graph has an associated C*-algebra. In cases where we do not assume row-finiteness, there still may be ways of forming a C*-algebra, but these cases are not addressed here. The following is a description of a method for finding the associated C*-algebra for a given graph. Some uniqueness issues are then discussed.

Because of the Gelfand-Naimark theorem, as we attempt to form the graph algebra of a particular graph, we may always have in mind some Hilbert space on which the elements of the algebra will be operators. A nice Hilbert space to work with is $\ell^2 = \left\{(x_0, x_1, \dots) : x_i \in \mathbb{C}, \sum |x_i|^2 < \infty\right\}$, if the graph is countable. As we shall see, it is often important that the Hilbert space be infinite dimensional, hence the choice of ℓ^2. In this paper, we will be working mostly with finite dimensional Hilbert space, but we will provide an example of an infinite dimensional case for completeness.

Definition 3.6. *Let H be a Hilbert space and E be a row-finite graph. A* ***Cuntz-Krieger E-family*** $\{S, P\}$ ***on H*** *is a set $P = \{P_v : v \in E^0\}$ of mutually orthogonal projections and $S = \{S_e : e \in E^1\}$ of partial isometries on H such that*

1. $S_e^* S_e = P_{s(e)}$ *for all* $e \in E^1$
2. $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ *whenever this is not an empty sum.*

From the theory of partial isometries, we have $S_e = S_e S_e^* S_e$, and so by the first Cuntz-Krieger relation $S_e = S_e P_{s(e)}$. The second relation then says that $S_e H \subset P_{r(e)} H$. Thus S_e is an isometry of $P_{s(e)} H$ onto a closed subspace of $P_{r(e)} H$. Expressed algebraically, we get the important relation

$$S_e = P_{r(e)} S_e = S_e P_{s(e)}.$$

In general there may be several Cuntz-Krieger E-families for a given graph E, generating non-isomorphic C*-algebras. But the following proposition given in [9] states that there is a specific Cuntz-Kreiger E-family which generates a C*-algebra that has a universal property. This C*-algebra is then called the **graph algebra of** E and is denoted $C^*(E)$.

Proposition 3.7. *For any row-finite directed graph E, there is a C*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s, p\}$ such that for every Cuntz-Krieger E-family $\{T, Q\}$ in a C*-algebra B, there is a homomorphism $\pi_{T,Q}$ of $C^*(E)$ into B satisfying $\pi_{T,Q}(s_e) = T_e$ for every $e \in E^1$ and $\pi_{T,Q}(p_v) = Q_v$ for every $v \in E^0$.*

In [9], this C*-algebra is then shown to be unique up to isomorphism. That is, any other C*-algebra generated by a Cuntz-Krieger E-family which has this universal property is isomorphic to $C^*(E)$.

The following is a proposition which is proved in [9], which is useful in interpreting the link between a graph and its C*-algebra.

Proposition 3.8. *Suppose that E is a row-finite graph and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. Let $\mu, \nu \in E^*$. Then*

a) if $|\mu| = |\nu|$ and $\mu \neq \nu$, then $(S_\mu S_\mu^*)(S_\nu S_\nu^*) = 0$

b) $S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \textit{if} \quad \mu = \nu\mu' \quad \textit{for some} \quad \mu' \in E^* \\ S_{\nu'} & \textit{if} \quad \nu = \mu\nu' \quad \textit{for some} \quad \nu' \in E^* \\ 0 & \textit{otherwise} \end{cases}$

c) if $S_\mu S_\nu \neq 0$, then $\mu\nu$ is a path in E and $S_\mu S_\nu = S_{\mu\nu}$

d) if $S_\mu S_\nu^* \neq 0$, then $s(\mu) = s(\nu)$.

What this means is that the C*-algebra has encoded in it the structure of the graph. A path μ in the graph corresponds to the element S_μ in the C*-algebra. If the composition of two paths μ and ν in the graph is possible, then this composition $\mu\nu$ is represented in the C*-algebra by the multiplication of S_μ and S_ν, which is $S_{\mu\nu}$. On the other hand, if you try to "compose" two paths that are not composable in the graph, then the result is zero. However, not every non-zero element in the C*-algebra represents a path in the graph.

Example 3.9.

1. Let E be the following graph:

$$a \xrightarrow{e} b$$

   We define a Cuntz-Krieger family $\{P_v, P_w, S_e\}$ for this graph on $\mathbb{C}^2$ by

$$P_v(x, y) = (x, 0), \quad P_w(x, y) = (0, y), \quad \text{and} \quad S_e(x, y) = (0, x).$$

   Then $C^*(E)$ is $C^*(P_v, P_w, S_e) = C^*(S_e)$. Representing these operators as matrices, we find that $C^*(E) \cong M_2(\mathbb{C})$.

2. Consider the graph E as follows:

$$\overset{v}{\circlearrowleft}_e \xleftarrow[f]{} w$$

   Here it is implied by the Cuntz-Krieger relations that $P_v H$ must be infinite-dimensional as follows. $P_v = S_f S_f^* + S_e S_e^* \Rightarrow dim(P_v H) = dim(S_f H) + dim(S_e H)$. But S_e and S_f are isometries of $P_v H$ and $P_w H$ onto $S_e H$ and $S_f H$ respectively. Hence $dim(P_v H) = dim(P_w H) + dim(P_v H)$. So if both orthogonal projections are non-zero (this is referred to as non-degeneracy), then $P_v H$ is infinite-dimensional. Note, that the simultaneous existence of the arrow f leading to v, and the loop based at v, caused this phenomenon.

   With this in mind, we define $P_v(x_0, x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ and therefore (to make $H = \ell^2$ into a direct sum of the ranges of the orthogonal projections), define

$P_w(x_0, x_1, x_2, \dots) = (x_0, 0, 0, \dots)$. Since $S_f = S_f S_f^* S_f = S_f P_w$, to define S_f, we must simply decide where $(x_0, 0, 0, \dots)$ should go, remembering to keep the map isometric, and also that S_f^2 should be zero since ff is not a path in the graph. The easy choice is $S_f(x_0, x_1, x_2, \dots) = (0, x_0, 0, 0, \dots)$. Similarly, since $S_e = S_e P_v$, we need to decide where S_e should send $(0, x_1, x_2, \dots)$. Since $P_v H = S_e H \oplus S_f H$, we are forced to define $S_e(x_0, x_1, x_2, \dots) = (0, 0, x_1, x_2, \dots)$.

It is worth noting that the C*-algebra we get in this case is isomorphic to an important C*-algebra, called the Toeplitz C*-algebra, $C^*(U)$, where $U(x_0, x_1, \dots) = (0, x_0, x_1, \dots) = (S_e + S_f)(x_0, x_1, \dots)$. Indeed, the generators are expressible in terms of $S_e + S_f$.

$$\begin{aligned}
&P_v + P_w = (S_e + S_f)^*(S_e + S_f) \\
&P_v = (S_e + S_f)(S_e + S_f)^* \\
&S_e = (S_e + S_f)P_v \\
&S_f = (S_e + S_f)P_w.
\end{aligned}$$

3. We can always find a Cuntz-Krieger E-family by following a simple procedure. Consider the graph G:

$$a \xrightarrow{e} b \xleftarrow{f} c$$

Since there are three vertices we require at least a 3-dimensional Hilbert space on which there will be mutually orthogonal projections P_a, P_b, P_c. Noting that b is the range of two different edges, the second Cuntz-Krieger relation at b says that $P_b H = S_e H \oplus S_f H$. Thus, we require at least a 4-dimensional Hilbert space. We demonstrate by example that this is enough.

Take the Hilbert space to be $\mathbb{C}^4$. P_a and P_c then need to be mutually orthogonal projections onto 1-dimensional subspaces of $\mathbb{C}^4$. Take these to be $P_a(w, x, y, z) = (w, 0, 0, 0)$ and $P_c(w, x, y, z) = (0, x, 0, 0)$. We take P_b to be the projection onto the 2-dimensional subspace of $\mathbb{C}^4$ that is orthogonal to both $P_a H$ and $P_c H$. That is, we take $P_b(w, x, y, z) = (0, 0, y, z)$, and so $P_b H = span(0, 0, 1, 0) \oplus span(0, 0, 0, 1)$.

We then take S_e to be an isometric linear transformation from $P_a H$ onto $span(0, 0, 1, 0)$, and S_f to be an isometric linear transformation from $P_c H$ onto $span(0, 0, 0, 1)$, so that $P_b H = S_e H \oplus S_f H$ as required by the second Cuntz-Krieger relation. Explicitly we take $S_e(w, x, y, z) = (0, 0, w, 0)$ and $S_f(w, x, y, z) = (0, 0, 0, x)$.

Notice that the given operators satisfy only the relations implied by the Cuntz-Krieger relations, and so $C^*(G) = C^*(\{P_a, P_b, P_c, S_e, S_f\})$. One easily checks that this is isomorphic to $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$.

3.2 Cuntz-Krieger Families for Higher-rank Graphs

There is a slightly modified set of Cuntz-Krieger relations for dealing with higher rank graphs. For a full discussion of the motivation for using these new relations, see [10]. In order to define these relations we must introduce the following notation.

Definition 3.10. *Let $m = (m_1, m_2, \ldots, m_k)$ and $n = (n_1, n_2, \ldots, n_k)$ be elements of $\mathbb{N}^k$. Then by $m \leq n$ we mean $m_i \leq n_i$ for $i = 1, 2, \ldots k$.*

Let (Λ, d) be a k-graph. For $q \in \mathbb{N}^k$ and $v \in \Lambda^0$ we define

$$\Lambda^{\leq q} := \{\lambda \in \Lambda : d(\lambda) \leq q \text{ and } \Lambda^{e_i}(s(\lambda)) = \emptyset \text{ when } d(\lambda) + e_i \leq q\},$$

and

$$\Lambda^{\leq q}(v) := \{\lambda \in \Lambda^{\leq q} : r(\lambda) = v\}.$$

In other words, $\Lambda^{\leq q}$ consists of the paths of degree q and the paths λ with $d(\lambda) \leq q$ which cannot be non-trivially extended to paths $\lambda\mu$ with $d(\lambda\mu) \leq q$.

Definition 3.11. *Let Λ be a row-finite k-graph. A* ***Cuntz-Krieger Λ-family in a C*-algebra*** *B consists of a family of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ satisfying the* ***Cuntz-Krieger relations****:*

1. *$\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections*
2. *$s_{\lambda\mu} = s_\lambda s_\mu$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$*
3. *$s_\lambda^* s_\lambda = s_{s(\lambda)}$*
4. *$s_v = \sum_{\lambda \in \Lambda^{\leq m}(v)} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.*

Note that in relation 4, the sum is not taken over all paths entering v as before. Instead, we get an equation involving s_v for each degree $m \in \mathbb{N}^k$. In [10] the uniqueness theorems involving graph algebras using these Cuntz-Krieger relations are proved for locally convex k-graphs.

Proposition 3.12. *Let E be a row-finite graph and E^* be its path category. Then $C^*(E) \cong C^*(E^*)$.*

Proof. Let $\varphi : C^*(E) \to C^*(E^*)$ be the *-homomorphism given by

$$\begin{aligned} P_v &\mapsto s_v \quad \text{for all } v \in E^0, \\ S_e &\mapsto s_e \quad \text{for all } e \in E^1. \end{aligned}$$

By proposition 2.8(c) φ is a *-isomorphism. □

When $k = 1$ the Cuntz-Krieger relations (1), (3), and (4) for E^* are the same as the Cuntz-Krieger relations for E

Proposition 3.13. *Let Λ_1, Λ_2 be 1-graphs. Then the 2-graph $\Lambda_1 \times \Lambda_2$ is locally convex.*

Proof. Let $e \in \Lambda_1^1$, $f \in \Lambda_2^1$, $a \in \Lambda_1^0$, and $b \in \Lambda_2^0$, and let $\lambda = (e, i_b) \in \Lambda_1 \times \Lambda_2^{(1,0)}$ and $\mu = (i_a, f) \in \Lambda_1 \times \Lambda_2^{(0,1)}$ with $r(e) = a$ and $r(f) = b$. Then there exist $\nu_1 = (i_{s(e)}, f) \in \Lambda_1 \times \Lambda_2^{(0,1)}$ and $\nu_2 = (e, i_{s(f)}) \in \Lambda_1 \times \Lambda_2^{(1,0)}$. Then,

$$\begin{aligned} r(\nu_1) &= (s(e), r(f)) = (s(e), b) = s(\lambda), \quad \text{and} \\ r(\nu_2) &= (r(e), s(f)) = (a, s(f)) = s(\mu). \end{aligned}$$

□

Remark 3.14.

The 2-graphs that we will be considering in this paper are those that are product categories of 1-graphs, and so will be locally convex by the previous proposition. Hence, the graph algebras of our 2-graphs will be unique under the definition given above, and by proposition 3.11 of [10] the fourth Cuntz-Krieger relation listed above can be replaced with the simpler relation

$$4'. \quad s_v = \sum_{\lambda \in \Lambda^{e_i}(v)} s_\lambda s_\lambda^* \quad \text{for } v \in \Lambda^0 \text{ and } i = 1, 2 \text{ with } \Lambda^{e_i}(v) \neq \emptyset.$$

Example 3.15.

Consider the following colored graph E.

$$\begin{tikzcd} w \arrow[r, "e_1"] \arrow[d, squiggly, "f_1"'] & x \arrow[d, squiggly, "f_2"] \\ y \arrow[r, "e_2"'] & z \end{tikzcd}$$

There is a unique 2-graph Λ for which this is the 1-skeleton. We define a set of partial isometries on $\mathbb{C}^4$ as follows.

$$\begin{array}{ll} S_w(a,b,c,d) = (a,0,0,0) & S_x(a,b,c,d) = (0,b,0,0) \\ S_y(a,b,c,d) = (0,0,c,0) & S_z(a,b,c,d) = (0,0,0,d) \\ S_{e_1}(a,b,c,d) = (0,a,0,0) & S_{f_1}(a,b,c,d) = (0,0,a,0) \\ S_{f_2}(a,b,c,d) = (0,0,0,b) & S_{e_2}(a,b,c,d) = (0,0,0,c). \end{array}$$

Check that these operators form a Cuntz-Krieger Λ-family in $B(\mathbb{C}^4)$. Remember that in this 2-graph, $e_2 f_1 = f_2 e_1$. The fourth Cuntz-Krieger relation at z says $S_z = S_{e_2} S_{e_2}^* = S_{f_2} S_{f_2}^* = S_{e_2 f_1} S_{e_2 f_1}^* = S_{f_2 e_1} S_{f_2 e_1}^*$, which is easily verified. It is clear that the C*-algebra generated by this Λ-family is isomorphic to $M_4(\mathbb{C})$.

4 C*-algebras of Graph Products

With the background from sections 1 and 2, we now provide a set of results discussing the structure of the C*-algebras of product graphs. In each of the following subsections, we will focus on a specific algebra structure.

4.1 Direct Sum of Graph Algebras

Proposition 4.1. *Let E, F be row-finite directed graphs. Then*

$$C^*(E) \oplus C^*(F) \cong C^*(E \cup F).$$

Proof. $C^*(E)$ is generated by some Cuntz-Krieger E-family $\{S_1, P_1\}$ and $C^*(F)$ is generated by some Cuntz-Krieger F-family $\{S_2, P_2\}$ by proposition 2.7. So

$$C^*(E) = C^*(S_1, P_1) \subset C^*(S_1, P_1, S_2, P_2), \text{ and}$$
$$C^*(F) = C^*(S_2, P_2) \subset C^*(S_1, P_1, S_2, P_2).$$

By proposition 2.8,if $S \in C^*(E)$, $T \in C^*(F)$, then $ST = 0$. Hence by proposition A.7 in [9],

$$\begin{aligned} C^*(E) \oplus C^*(F) &\cong span\{C^*(E) \cup C^*(F)\} \\ &= span\{C^*(S_1, P_1) \cup C^*(S_2, P_2)\} \\ &= C^*(S_1, P_1, S_2, P_2) \\ &= C^*(E \cup F). \end{aligned}$$

This last equality is seen by noting that a universal Cuntz-Krieger $E \cup F$-family will be the union of universal Cuntz-Krieger E and F-families. □

4.2 Crossed Product of a Graph Algebra

The following is the discussion of discrete crossed products found in [1].Once this has been defined, we will summarize the result of [7]relating the crossed product of a graph algebra to the graph algebra of a factor graph.

A **C*-dynamical system** $(\mathfrak{A}, G, \alpha)$ consists of a C*-algebra $\mathfrak{A}$ together with a homomorphism α of a locally compact group G into Aut($\mathfrak{A}$). We will denote by α_s the automorphism $\alpha(s)$ for s in G.

Consider the algebra $\mathfrak{A}G$ of all finite sums $f = \sum_{t \in G} A_t t$ with coefficients in $\mathfrak{A}$ and multiplication determined by the formal rule $tAt^{-1} = \alpha_t(A)$. Then if $g = \sum_{u \in G} B_u u$ we have

$$\begin{aligned} fg &= \sum_{t \in G} \sum_{u \in G} A_t t B_u u \\ &= \sum_{t \in G} \sum_{u \in G} A_t (t B_u t^{-1}) t u \\ &= \sum_{t \in G} \sum_{u \in G} A_t \alpha_t(B_u) t u \\ &= \sum_{s \in G} (\sum_{t \in G} A_t \alpha_t(B_{t^{-1}s})) s. \end{aligned}$$

The adjoint is determined by $s^* = s^{-1}$, so that

$$(As)^* = s^* A^* = s^{-1} A^* s s^{-1} = \alpha_s^{-1}(A^*) s^{-1}.$$

Hence

$$f^* = \sum_{t \in G} \alpha_t(A^*_{t^{-1}}) t.$$

Define a norm by

$$||f|| = sup_\sigma ||\sigma(f)||$$

where σ runs over all *-representations of $\mathfrak{A}G$. By the discussion in [1], this supremum is always bounded by $||f||_1 = \sum_{t \in G} ||A_t||$. Taking the completion of $\mathfrak{A}G$ with respect to this norm gives a C*-algebra $\mathfrak{A} \times_\alpha G$ called the **crossed product of $\mathfrak{A}$ with G.**

With the notion of a group action on a graph, we expect the crossed product of this graph's C*-algebra to be related to the algebra of a graph derived from this action. This turns out to be true as shown by Kumjian and Pask in [7] by means of a well-known result in group theory, which involves the skew product notion.

Proposition 4.2. *If E is a locally finite directed graph and $\lambda : G \to Aut(E)$ a free action then*

$$C^*(E) \times_\lambda G \cong C^*(E/G) \otimes K(\ell^2(G)),$$

where $K(\ell^2(G))$ is the algebra of compact operators on $\ell^2(G)$ and E/G is the graph whose vertices are the orbits of E^0 under the action of G, two vertices (orbits) A, B being adjacent iff there are vertices $a \in A$, $b \in B$ such that a and b are adjacent.

4.3 Tensor Product of Graph Algebras

The following statement is part of Corollary 3.5 in [8].

Lemma 4.3. *Let (Λ_i, d_i) be k_i-graphs for $i = 1, 2$, then $C^*(\Lambda_1 \times \Lambda_2) \cong C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ via the map $s_{(\lambda_1,\lambda_2)} \mapsto s_{\lambda_1} \otimes s_{\lambda_2}$ for $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$.*

Here, $\Lambda_1 \times \Lambda_2$ is the $k_1 + k_2$-graph that is the product category of Λ_1 and Λ_2. In the case when Λ_1, Λ_2 are 1-graphs, we get a 2-graph. Since the path category of any directed graph becomes a 1-graph, this statement says that when we form the 2-graph from two directed graphs, the resulting graph algebra is isomorphic to the tensor product of the graph algebras of the directed graphs.

Example 4.4.

The 2-graph in example 2.14 is the product category of the 1-graph in example 2.9(1) with itself, and we see that their associated C*-algebras satisfy the above proposition, that is $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \cong M_4(\mathbb{C})$.

Lemma 4.5. *Let, Λ_1, Λ_2 be 1-graphs. Then the resulting 2-graph, $\Lambda_1 \times \Lambda_2$, has colorless 1-skeleton isomorphic to the box product $\Lambda_1 \Box \Lambda_2$.*

Proof. Consider the 1-skeleton of the 2-graph $\Lambda_1 \times \Lambda_2$, and call it E. We then have

$$E^0 = (\Lambda_1 \times \Lambda_2)^0 = \Lambda_1^0 \times \Lambda_2^0 = (\Lambda_1 \Box \Lambda_2)^0,$$

and so the vertex sets of our two 1-graphs are the same. Two vertices $(u_1, u_2), (v_1, v_2)$ are then adjacent in E, ignoring coloring, if there is a path $\lambda \in (\Lambda_1 \times \Lambda_2)^*$ such that $s(\lambda) = (u_1, u_2)$, $r(\lambda) = (v_1, v_2)$ and either $d(\lambda) = (0, 1)$ or $d(\lambda) = (1, 0)$. This is equivalent to the requirement that either $u_1 = v_1$ in Λ_1^0 and $(v_2, u_2) \in \Lambda_2^1$, or $(v_1, u_1) \in \Lambda_1^1$ and $u_2 = v_2$ in Λ_2^0, which are the exact requirements for adjacency in the box product $\Lambda_1 \Box \Lambda_2$. Hence, $E^1 = (\Lambda_1 \Box \Lambda_2)^1$. Thus, we conclude that the colorless 1-skeleton of the 2-graph constructed from Λ_1 and Λ_2 is, in fact, the box product of these same two graphs. $\Box$

Remark 4.6.

It is simple to extend the proof of the previous lemma to an argument for arbitrary k_1- and k_2-graphs, Λ_1 and Λ_2. We see that the box product of the 1-skeletons for Λ_1 and Λ_2 is the 1-skeleton for the $(k_1 + k_2)$-graph $\Lambda_1 \times \Lambda_2$.

The previous two lemmas lead naturally to the following theorem.

Theorem 4.7. *The tensor product of the C*-algebras associated with two 1-graphs Λ_1, Λ_2 is isomorphic to the C*-algebra of the 2-graph $\Lambda_1 \times \Lambda_2$, whose 1-skeleton is the box product of these 1-graphs.*

4.4 Subalgebras of the Tensor Product of Graph Algebras

Let E, F be directed graphs, and let $E^* \times F^*$ be the 2-graph formed from the product category of the path categories of E and F. Consider the C*-algebra associated with this 2-graph, $C^*(E^* \times F^*)$. We show that the C*-algebra associated with the tensor product graph $E \times F$ is a sub-C*-algebra of $C^*(E^* \times F^*) \cong C^*(E^*) \otimes C^*(F^*)$, and we specify the generators for this algebra.

We find the following definition from [8]to be useful in this process.

Definition 4.8. *Let $f : \mathbb{N}^j \to \mathbb{N}^k$ be a monoid homomorphism, then if (Λ, d_k) is a k-graph we may form the j-graph $(f^*(\Lambda), d_j)$ as follows: $f^*(\Lambda) = \{(\lambda, n) \in (\Lambda, \mathbb{N}^j) : d_k(\lambda) = f(n)\}$, with*

$$d_j((\lambda, n)) = n,$$

$$s((\lambda, n)) = s(\lambda), \text{ and}$$

$$r((\lambda, n)) = r(\lambda).$$

Note that in the above definition we can identify the objects of $f^*(\Lambda)$ with those of Λ.

Define the monoid homomorphism $f : \mathbb{N} \to \mathbb{N}^2$ by $f(m) = (m, m)$. From the 2-graph $(E^* \times F^*, d_2)$ together with f, we have a 1-graph $(f^*(E^* \times F^*), d_1)$, where

$$f^*(E^* \times F^*) = \{(\lambda, n) \in (E^* \times F^*, \mathbb{N}) : d_2(\lambda) = (n, n)\},$$

with the objects of $f^*(E^* \times F^*)$ identified with those of $E^* \times F^*$, and with

$$\begin{aligned} d_1((\lambda, n)) &= n, \\ s_1((\lambda, n)) &= s_2(\lambda), \text{and} \\ r_1((\lambda, n)) &= r_2(\lambda). \end{aligned}$$

We notice that the set of all paths in $f^*(E^* \times F^*)$ consists of all paths $\lambda = (e, f)$ in $E^* \times F^*$ such that $e \in E^*, f \in F^*$ are of equal length. But this describes exactly the set of all paths in the tensor product graph $E \times F$, and so with the observation that $(E \times F)^0 = (E^* \times F^*)^0 = (f^*(E^* \times F^*))^0$, we have $f^*(E^* \times F^*) = (E \times F)^*$. We have thus shown that $f^*(E^* \times F^*) \cong E \times F$.

By proposition 1.11 of [8], there is a *-homomorphism $\pi_f : C^*((E \times F)^*) \to C^*(E^* \times F^*)$ associated with f such that $s_{(\lambda,n)} \mapsto s_\lambda$, where $\{s_{(\lambda,n)}\}$ is a Cuntz-Krieger $(E \times F)^*$-family generating $C^*((E \times F)^*)$ and $\{s_\lambda\}$ is a Cuntz-Krieger $E^* \times F^*$-family generating $C^*(E^* \times F^*)$. Thus, taking $\{s_{Im(\pi_f)}\}$ to be the subset of $\{s_\lambda\}$ that forms the image of $\{s_{(\lambda,n)}\}$ under π_f , we have

$$C^*((E \times F)^*) \cong C^*(\{s_{Im(\pi_f)}\}).$$

It is clear that this is a sub-C*-algebra of $C^*(\{s_\lambda\}) = C^*(E^* \times F^*) \cong C^*(E) \otimes C^*(F)$. Hence, we conclude that $C^*(E \times F) \cong C^*((E \times F)^*)$ is a sub-C*-algebra of $C^*(E) \otimes C^*(F) \cong C^*(E^* \times F^*)$, with generating set $\{s_{Im(\pi_f)}\}$, that is, the set of all paths $(\lambda_1, \lambda_2) \in E^* \times F^*$ such that $d_1(\lambda_1) = d_1(\lambda_2)$.

We have now shown the following proposition.

Proposition 4.9. *Let E, F be row-finite graphs, and E^*, F^* their path categories. Then there is an isomorphic copy of $C^*(E \times F)$ in $C^*(E^* \times F^*) \cong C^*(E) \otimes C^*(F)$.*

5 Concluding Remarks

It is natural to consider what other graph products yield C*-algebras that might be subalgebras of the C*-algebra associated with the 2-graph constructed as the product category of two 1-graphs.

It is at first tempting to look for a monoid morphism f that would give $f^*(E^* \times F^*) = (E \Box F)^*$ and result in $C^*((E \Box F)^*)$ as a subalgebra of $C^*(E^* \times F^*)$, as given by the image of π_f. However, we show with a counterexample that there is no such f.

Consider the following graph and its box product graph:

$$E: \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd} \qquad E \Box E: \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

The following proposition from [9] allows us to easily compute their associated C*-algebras.

Proposition 5.1. *Suppose E is a finite directed graph with no cycles, and $w_1, w_2, \ldots, w_n$ are the sources in E. Then for every Cuntz-Krieger E-family $\{S, P\}$ in which each P_v is non-zero we have*

$$C^*(S, P) \cong \bigoplus_{i=1}^{n} M_{|s^{-1}(w_i)|}(\mathbb{C}),$$

where $s^{-1}(w_i) = \{\mu \in E^* : s(\mu) = w_i\}$.

We see that $C^*(E) \cong M_2(\mathbb{C})$ and $C^*(E \Box E) \cong M_5(\mathbb{C})$. Thus, $C^*(E^* \times E^*) \cong C^*(E) \otimes C^*(E) \cong M_4(\mathbb{C})$ clearly does not contain $M_5(\mathbb{C}) \cong C^*(E \Box E) \cong C^*((E \Box E)^*)$ as a sub-C*-algebra. The difficulty that we have run into here is the additional projective space required by the two incident edges on the lower-right vertex of $E \Box E$, giving a graph algebra acting on a five-dimensional Hilbert space, instead of the four-dimensional Hilbert space acted on by the graph algebra for $E^* \times E^*$.

By the same counterexample, we cannot expect the strong product to be a sub-C*-algebra of the tensor product, since the box product graph is a subgraph of the tensor product graph.

We also see that the C*-algebra of the union of two graphs cannot in general be represented as a sub-C*-algebra of the tensor product of their graph algebras, by a way of a simple counterexample. Consider the trivial graph E with one vertex and no edges. Then by proposition 4.10 $C^*(E) \cong \mathbb{C}$ and $C^*(E \cup E) \cong \mathbb{C} \oplus \mathbb{C} \cong \mathbb{C}^2$. This is clearly not a sub-C*-algebra of $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$.

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