Graph Fortresses and Cheeger Values

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1 Introduction

The Cheeger constant is a graph constant used to measure how connected a graph is.[2] Graphs with low Cheeger constants are easier to cut, while graphs with high Cheeger constants are more robust to losing edges. In a practical application, the Cheeger constant measures how robust a network is to connection failure; under this application, it is desirable to maximize the Cheeger constant.

In this paper, I introduce the concept of *fortresses*, subsets of the vertices of a finite graph that are at least as strongly connected to each other as they are to the rest of the graph. The concept of a fortress is derived from a natural cellular automaton that runs on the vertices of the graphs in a way similar to a battle between two forces on the vertices of the graph. In this light, a fortress is a set of vertices that, if all held by the same side, can never be taken by the other side. *Double fortresses*, or fortresses whose complements are also fortresses, often result from poor connectivity of the graph. For example, if the graph has a bottleneck, there is no way for the side with the majority of the vertices to use that power to take over the rest of the graph. This poor connectivity is related to a low Cheeger constant. In light of this, it is useful to investigate the relation between Cheeger constants and the presence of fortresses on graphs.

2 Preliminary results

Even without using a cellular automaton, it is possible to prove several useful results about fortresses. First, some definitions are in order.

2.1 Definitions

(Definitions from the first two paragraphs are due to [1].)

Unless otherwise noted, we take a graph G to be a finite, simple graph. We denote the vertices of G by $V(G)$ and the edges of G (which are unordered pairs

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of elements of $V(G)$) by $E(G)$. Two vertices u and v in $V(G)$ are adjacent if $\{u, v\} \in E(G)$; if this is true, we say that $u \sim_G v$, or if the graph discussed is obvious, $u \sim v$. The *degree* of a vertex $v \in V(G)$ is denoted $d_{v,G}$ and is defined as the number of vertices $u \in V(G)$ so that $u \sim v$. If it is clear what graph is being discussed, the notation $d_{v,G}$ can be simplified to d_v.

If $A \subseteq V(G)$, then the *induced subgraph* of A on G is denoted $\langle A \rangle$ and is defined as the graph where $V(\langle A \rangle) = A$ and $E(\langle A \rangle) = \{\{u, v\} \in E(G) : \{u, v\} \subseteq A\}$.

If S is a finite set, denote by $|S|$ the number of elements in S. Suppose that A and B are disjoint subsets of $V(G)$. Then the *edge boundary* between A and B is defined by $\partial(A, B) = |\{\{u, v\} \in E(G) : u \in A \text{ and } v \in B\}|$. If A is a nonempty proper subset of $V(G)$, then we define the *Cheeger value* of A as $h_G(A) = \frac{\partial(A, V(G) \setminus A))}{\min\{|A|, |V(G) \setminus A|\}}$. We note that $h_G(A) = h_G(V(G) \setminus A)$. Define the *Cheeger constant* of a graph G to be $h_G = \min_{\emptyset \subsetneq A \subsetneq V(G)} h_G(A)$. If A is a nonempty proper subset of $V(G)$, then A is a *minimal Cheeger set* if for all $\emptyset \subsetneq B \subsetneq A$, $h_G(B) > h_G(A)$. (Note that the definition of the Cheeger constant does not take into account the degrees of the vertices of A, in contrast to [2].)

Let A be a subset of $V(G)$. A is a *fortress* if for all $v \in A$, $\partial(\{v\}, A \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus A)$. A is a *double fortress* if both A and $V(G) \setminus A$ are fortresses. If A is a nonempty proper subset of $V(G)$, and A is a (double) fortress, then A is a *nontrivial* (double) fortress.

2.2 The Minimal Cheeger Set as Fortress

Proposition 1. *Let G be a finite graph. Then any minimal Cheeger set A with $h_G(A) < 1$, or $h_G(A) = 1$ and $d_v \geq 2$ for all $v \in A$, is a nontrivial fortress.*

Proof. Pick a minimal Cheeger set A, let $h_G(A) < 1$ (or equal to 1 provided that each vertex in A has degree at least 2), and pick $v \in A$. If $h_G(A) < 1$, then either $h_G(A) = 0$, which makes A a connected component and nontrivial fortress (which completes the proof), or $0 < h_G(A) < 1$, which means that the denominator is greater than the positive numerator, implying that $|A| \geq 2$. Also, if $h_G(A) = 1$ and each vertex has degree 2 or more, then the Cheeger value of a single vertex would be greater than 1, implying that A cannot be a single vertex. Therefore, A has at least two vertices, and $A \setminus \{v\}$ is nonempty. Since A is a minimal Cheeger set, $h_G(A \setminus \{v\}) > h_G(A)$. Now define $n = \min\{|A|, |V(G) \setminus A|\}$. Then,

$$\begin{aligned} \frac{\partial(A, V(G) \setminus A) + \partial(\{v\}, A \setminus \{v\}) - \partial(\{v\}, V(G) \setminus A)}{n-1} &\geq h_G(A \setminus \{v\}) \\ &> \frac{\partial(A, V(G) \setminus A)}{n} \\ &= h_G(A) \end{aligned}$$

Cross-multiplication by positive numbers does not change the sign of the inequality; this transformation gives us

$$n\partial(A, V(G) \setminus A) + n(\partial(\{v\}, A \setminus \{v\}) - \partial(\{v\}, V(G) \setminus A)) > n\partial(A, V(G) \setminus A) - \partial(A, V(G) \setminus A)$$

This implies $n(\partial(\{v\}, A \setminus \{v\}) - \partial(\{v\}, V(G) \setminus A)) > -\partial(A, V(G) \setminus A)$. Now, since the Cheeger value of A is no greater than 1, we know that $\frac{\partial(A,V(G)\setminus A)}{n} \leq 1$; multiplication by $-n$ reverses the sign of the inequality, giving us $-\partial(A, V(G) \setminus A) \geq -n$. Therefore, $n(\partial(\{v\}, A \setminus \{v\}) - \partial(\{v\}, V(G) \setminus A)) > -n$, and division by positive n gives us $\partial(\{v\}, A \setminus \{v\}) - \partial(\{v\}, V(G) \setminus A) > -1$. Since the left-hand side is an integer, we know that $\partial(\{v\}, A \setminus \{v\}) - \partial(\{v\}, V(G) \setminus A) \geq 0$, and therefore, $\partial(\{v\}, A \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus A)$, as desired. □

2.3 Extending Fortresses

Given a fortress A, it is possible to extend the fortress by union with other fortresses or adding appropriate vertices one at a time, as the following lemmas demonstrate.

Lemma 2. *Let A and B be fortresses on G. Then $A \cup B$ is a fortress on G.*

Proof. We want to show that for each vertex $v \in A \cup B$, that $\partial(\{v\}, (A \cup B) \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus (A \cup B))$. But

$$\begin{aligned} \partial(\{v\}, (A \cup B) \setminus \{v\}) &\geq \partial(\{v\}, A \setminus \{v\}) \\ &\geq \partial(\{v\}, V(G) \setminus A) \\ &\geq \partial(\{v\}, V(G) \setminus (A \cup B)) \end{aligned}$$

Therefore, $A \cup B$ is a fortress on G. □

Lemma 3. *Suppose A is a fortress, and u is a vertex in $V(G) \setminus A$. Then if $\partial(A, \{u\}) \geq \partial(V(G) \setminus (A \cup \{u\}), \{u\})$, then $A \cup \{u\}$ is also a fortress.*

Proof. We need to show that $\forall v \in (A \cup \{u\}), \partial(\{v\}, A \cup \{u\} \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus (A \cup \{u\}))$. By hypothesis, this is true for $v = u$. Suppose, that $v \in A$. Then

$$\begin{aligned} \partial(\{v\}, A \cup \{u\} \setminus \{v\}) &\geq \partial(\{v\}, A \setminus \{v\}) \\ &\geq \partial(\{v\}, V(G) \setminus A) \\ &\geq \partial(\{v\}, V(G) \setminus (A \cup \{u\})) \end{aligned}$$

Every $v \in A \cup \{u\}$ having satisfied the condition, $A \cup \{u\}$ is a fortress, as desired. □

3 The Cellular Automaton that Characterizes Fortresses

At this point, it is useful to use the cellular automaton (CA) approach to generate more results about fortresses. First, I will describe how the cellular automaton works. At time $t = 0$, each vertex is assigned either a $+$ or a $-$. Then, the state of the vertices at each successive time interval depends on the state at the previous tick. A vertex becomes $+$ if it has more $+$ neighbors than $-$

neighbors; similarly, a vertex becomes $-$ if the $-$ neighbors outnumber the $+$ neighbors. If the number of $+$ and $-$ neighbors are equal, then the vertex keeps the same sign it had the previous tick.

It will be helpful to describe the effects of the CA on a state by associating with the state at each time t a set A_t of the $+$ vertices at that time. Then the CA acts on A_0, the starting state, by changing the set of positive vertices to $A_1.A_2,\ldots$. Through this association, a formal definition of how the automaton operates is possible.

If $n \geq 1$, then A_n equals

$$\left\{v \in V(G) : \begin{cases} \partial(\{v\}, A_{n-1} \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus A_{n-1}) & : v \in A_{n-1} \\ \partial(\{v\}, A_{n-1}) > \partial(\{v\}, V(G) \setminus (A_{n-1} \cup \{v\})) & : v \notin A_{n-1} \end{cases}\right\}$$

Since A_n depends only on A_{n-1}, the following results are immediate.

Fact 4. *The following are true: (1) $(A_i)_j = A_{i+j}$; (2) for all $i, j, t \in \mathbb{Z}^+, A_i = A_j \Rightarrow A_{i+t} = A_{j+t}$; and (3) if $A_i = A_{i+t}$ for some $i, t \in \mathbb{Z}^+$, then $A_i = A_{i+nt}$ for all $n \in \mathbb{Z}^+$.*

3.1 Ultimate Behavior

We are interested in the limiting behavior of the sequence $A_0, A_1, A_2, \ldots$. Define $A_\infty = \{B \subseteq V(G) : \forall N \in \mathbb{Z}^+, \exists n > N : B = A_n\}$. Further, define the *center* of A, $\mathrm{ctr}(A) = \{v \in V(G) : \exists N \in \mathbb{Z}^+ : \forall n > N, v \in A_n\}$. The center of A and A_∞ are closely related, as the proposition below shows. First, a lemma is needed.

Lemma 5. *A_∞ is nonempty.*

Proof. Arbitrarily fix $S \subseteq V(G)$. If there is an element $b \in \mathcal{P}(V(G)) \setminus \{S\}$ so that $b \in A_\infty$, then A_∞ is clearly nonempty. Suppose that the opposite is the case. Then, for each $b \in \mathcal{P}(V(G)) \setminus \{S\}$, there is $N_b \in \mathbb{Z}^+$ such that $\forall n > N_b, b \neq A_n$. Since $\mathcal{P}(V(G)) \setminus \{S\}$ is finite, the expression $\max_{b \subseteq V(G)} N_b$ is well-defined; let N be that value. Now select $n > N$. We know that $A_n \in \mathcal{P}(V(G))$, but by construction of N, $A_n \notin \mathcal{P}(V(G)) \setminus \{S\}$. Therefore, $A_n = S$, so $S \in A_\infty$; therefore, A_∞ is nonempty. □

Proposition 6.

$$\mathrm{ctr}(A) = \bigcap_{B \in A_\infty} B$$

Proof. First, suppose $v \in \mathrm{ctr}(A)$. Then, there is $N \in \mathbb{Z}^+$ so that $v \in A_n \forall n > N$. Now, let B be an arbitrarily fixed member of A_∞. Then, for the positive integer N we just obtained, there is $n > N$ so that $B = A_n$, but since $n > N$, $v \in A_n$, and consequently, $v \in B$. But B was arbitrarily fixed; therefore, v is in every such B, and so $v \in \bigcap_{B \in A_\infty} B$.

Now, suppose $v \in \bigcap_{B \in A_\infty} B$. We want to show that $v \in \mathrm{ctr}(A)$. Consider the class $C = \{B \subseteq V(G) : v \notin B\}$. Clearly, $C \cap A_\infty = \emptyset$. Since no element of C is in A_∞, each $c \in C$ must fail to meet the criterion for being a member of A_∞.

This means that for all $c \in C$, there exists $N_c \in \mathbb{Z}^+$ so that $\forall n > N_c, c \neq A_n$. Because C is a subset of finite set $\mathcal{P}(V(G))$, the expression $\max_{c \in C} N_c$ is well-defined; let N equal that maximum. Now consider any $n > N$. By construction of N, $A_n \notin C$, which implies that $v \in A_n$. Since this holds for all $n > N$, it follows that $v \in \mathrm{ctr}(A)$.

This completes the proof. $\square$

3.2 "Fairness" of the Automaton

By how the automaton is defined, it acts equally on + and − vertices. In other words, suppose the vertices at time $t = 0$ are all flipped, so that + vertices become − and − become +. Then, run the automaton for n steps and flip the vertices again. The result is the same as if the automaton was run for n steps on the original position. The following proposition proves this "fairness" property.

Proposition 7. *For all* $n \in (\mathbb{Z}^+ \cup \{0\})$, $(V(G) \setminus A_0)_n = V(G) \setminus (A_n)$

Proof. By induction. When $n = 0$, both sides are equal. Suppose than $(V(G) \setminus A_0)_k = V(G) \setminus (A_k) = S$; we need to show that the claim holds for $k = 1$. First, suppose $v \in S$. Then

$$\begin{aligned} v \in (V(G) \setminus A_0)_{k+1} &\Leftrightarrow \partial(\{v\}, S \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus S) \\ &\Leftrightarrow \neg[\partial(\{v\}, V(G) \setminus S) > \partial(\{v\}, S \setminus \{v\})] \\ &\Leftrightarrow v \in V(G) \setminus (A_{k+1}) \end{aligned}$$

Now suppose $v \notin S$. Then

$$\begin{aligned} v \in (V(G) \setminus A_0)_{k+1} &\Leftrightarrow \partial(\{v\}, S) > \partial(\{v\}, V(G) \setminus (S \cup \{v\})) \\ &\Leftrightarrow \neg[\partial(\{v\}, V(G) \setminus (S \cup \{v\})) \geq \partial(\{v\}, S)] \\ &\Leftrightarrow v \in V(G) \setminus (A_{k+1}) \end{aligned}$$

Therefore, the inclusion criteria in both sets are equivalent; so $(V(G) \setminus A_0)_{k+1} = V(G) \setminus A_{k+1}$, so the induction holds and the proof is complete. $\square$

As a result of Proposition 7, we can "pull out" complements. In light of this, we define $A_n^c = V(G) \setminus (A_n) = (V(G) \setminus A_0)_n$.

3.3 Monotonicity

Start with an initial position A_0 at time $t = 0$. Changing some of the − vertices to + vertices could create more + vertices at future ticks. It, of course, may happen that the extra + vertices provide no additional gain, but the presence of additional + vertices should never cause a vertex at a later tick to go from + to −. Thus, the automaton is *monotonic*, as the following proposition shows.

Proposition 8. *Suppose* $B_0 \subseteq A_0$. *Then, for all* $i \in (\mathbb{Z}^+ \cup \{0\})$, $B_i \subseteq A_i$.

Proof. By induction. By hypothesis, the base case is true. Suppose now that $B_k \subseteq A_k$; we must show that $B_{k+1} \subseteq A_{k+1}$. To do this, pick any $v \in B_{k+1}$. Then there are two cases. If $v \in B_k$, then $\partial(\{v\}, B_k \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus B_k)$. But it follows that

$$\begin{aligned} \partial(\{v\}, A_k \setminus \{v\}) &\geq \partial(\{v\}, B_k \setminus \{v\} \\ &\geq \partial(\{v\}, V(G) \setminus B_k) \\ &\geq \partial(\{v\}, V(G) \setminus A_k) \end{aligned}$$

and since $v \in A_k$, $v \in A_{k+1}$. Now suppose $v \notin B_k$. Then $\partial(\{v\}, B_k) > \partial(\{v\}, V(G) \setminus (B_k \cup \{v\}))$. But

$$\begin{aligned} \partial(\{v\}, A_k \setminus \{v\}) &\geq \partial(\{v\}, B_k) \\ &> \partial(\{v\}, V(G) \setminus (B_k \cup \{v\})) \\ &\geq \partial(\{v\}, V(G) \setminus (A_k \cup \{v\})) \end{aligned}$$

Regardless of whether $v \in A_k$ or $v \notin A_k$, v satisfies the criterion for being in A_{k+1}. Since this holds for all $v \in B_{k+1}$, the proof is complete. $\square$

Monotonicity allows us to demonstrate the following property of A_∞:

Proposition 9. *Suppose $X, Y \in A_\infty$ so that $Y \subseteq X$. Then $X = Y$. In other words, there are no pairs of elements in A_∞ in which one is properly contained in the other.*

Proof. Choose $N \in \mathbb{Z}^+$. Since $X \in A_\infty$, there is $n > N$ so that $A_n = X$. Since $Y \in A_\infty$, there is $p > 0$ so that $A_{n+p} = Y$. Again, since $X \in A_\infty$, there is $q > p$ so that $A_{n+q} = X$. Now we want to find the value of A_{n+pq}. By the periodicity formalized in Fact 4, we know that since $A_n = A_{n+q}$, $A_{n+pq} = A_n = X$. However, $A_{n+2p} = (A_{n+p})_p \subseteq (A_n)_p = A_{n+p} = Y$, and similarly for $n + 3p, n + 4p, \ldots, n + qp$. Therefore, $X = A_{n+pq} \subseteq Y$. Since $X \subseteq Y$ and $Y \subseteq X$, we know that $X = Y$. $\square$

3.4 The CA Characterization of Fortresses

Suppose A_0 is a fortress. The definition of a fortress in part one was created so that an infinite sequence based on a fortress has nice properties, such as the following:

Proposition 10. *A_0 is a fortress if and only if $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$. Further, if either condition holds, each A_n is also a fortress.*

Proof. ($\Rightarrow$) By induction. The claim is trivially true if $n = 0$. So suppose that the claim holds for $n = k$; we want to show that the claim is true for $n = k + 1$. Choose $v \in A_n$. Then, $v \in A_{n+1}$ if and only if $\partial(\{v\}, A_n \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus A_n)$. But the latter condition is true because A_n is a fortress. In light of this, $A_n \subseteq A_{n+1}$. Now we must show that A_{n+1} is a fortress. By the inclusion relation, we can write $A_{n+1} = A_n \cup V$, where $V \cap A_n = \emptyset$. If V is empty, then $A_{n+1} = A_n$, so A_{n+1} is a fortress. Otherwise, index the elements of V as $\{v_1, v_2, \ldots, v_m\}$. Then, by being in A_{n+1} but not A_n, each $v \in V$ satisfies $\partial(\{v\}, A) > \partial(\{v\}, V(G) \setminus (A \cup \{v\}))$. Therefore, by Lemma 3, the sets $A_n \cup \{v_1\}$, $A_n \cup \{v_1, v_2\}$, and so on, up to $A_n \cup \{v_i\}_{i=1}^m = A_{n+1}$ are all fortresses.

($\Leftarrow$) It is sufficient to assume $A_0 \subseteq A_1$. Then, for each $v \in A_0$, $\partial(\{v\}, A_0 \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus A_0)$, which implies that A_0 is a fortress. The "further" claim then follows from the proof above. □

Corollary 11. *A_0 is a double fortress if and only if $A_0 = A_1 = A_2 = \cdots$.*

Proof. ($\Rightarrow$) This part follows immediately from Propositions 7 and 10. Since A_0 is a fortress, $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$. Since A_0^c is a fortress, $A_0^c \subseteq A_1^c \subseteq A_2^c \subseteq \cdots$, which is equivalent to $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$. Together, these two nestings imply $A_0 = A_1 = A_2 = \cdots$, as desired.

($\Leftarrow$) Suppose $A_0 = A_1 = \cdots$. It is sufficient to assume $A_0 = A_1$. This relation says that for all $v \in A_0$, $\partial(\{v\}, A_0 \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus A_0)$, and since $A_0^c = A_1^c$, for all $v \in A_0^c$, $\partial(\{v\}, A_0^c \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus A_0^c)$. Together, these two results imply that A is a double fortress as desired. □

These two results allow us to prove this powerful theorem.

Theorem 12. *If A_0 and B_0 are disjoint nontrivial fortresses on G, then G has a nontrivial double fortress.*

Proof. Consider the sequence $A_0, A_1, A_2, \ldots$. By Proposition 10, these sets are nested fortresses: $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$. As long as each fortress is a proper subset of the next fortress, the cardinality of each fortress forms a strictly increasing sequence. However, the maximum cardinality of any fortress in the sequence is $|V(G)|$. Therefore, these must be some $i \in \mathbb{Z}^+$ so that $A_i = A_{i+1}$. Define a new sequence $\{C_j\}_{j=0}^{\infty}$. Then, $C_0 = C_1$, and by the ($\Leftarrow$) part of the proof of Corollary 11, C_0 is a double fortress. It remains to show that C_0 is nontrivial. First, we note that $\emptyset \subsetneq A_0 \subseteq A_i = C_0$. Second, we note that $\emptyset \subsetneq B_0 \subseteq B_1 \subseteq \cdots \subseteq B_i \subseteq A_i^c$, where the last relation holds by Proposition 8. Together, these results demonstrate that C_0 is the desired nontrivial double fortress. □

The following is an immediate consequence.

Theorem 13. *Let G be a finite graph with $h_G < 1$ or $h_G = 1$ and $d_v \geq 2$ for all $v \in V(G)$. Then G has a nontrivial double fortress.*

Proof. Let A be one of the nonempty proper subsets of $V(G)$ so that $h_G(A) = h_G$. Then, we know that $h_G(V(G) \setminus A) = h_G$ also. Now, let B be any nonempty minimal subset of A for which $h_G(B) = h_G$, and let C be any nonempty minimal subset of $V(G) \setminus A$ where $h_G(C) = h_G$. Then B and C are minimal Cheeger sets with Cheeger constant less than 1 (or equal to 1 with each vertex having degree at least 2); therefore, by Proposition 1, B and C are nontrivial fortresses. Further, by construction, B and C are disjoint. Therefore, there is a nontrivial double fortress on G by Theorem 12. □

All the conditions of the above theorem are necessary, and the converse is not necessarily true. Here are three examples.

1. **A graph lacking nontrivial double fortresses with Cheeger constant 1 where there is a vertex of degree 1.** Consider the graph S_n (for "star") where $V(S_n) = \{0, 1, \ldots, n\}$ and $x \sim y \Leftrightarrow xy = 0 \,\text{and}\, x + y > 0$. This graph's Cheeger constant is 1. However, one can verify that every nontrivial fortress must contain the center point, $\{0\}$. Therefore, there is no nontrivial double fortress.

2. **A graph lacking nontrivial double fortresses where the graph's Cheeger constant exceeds 1 by an arbitrarily small amount.** Consider the sequence of graphs W_{2n} (for "wheel") where $V(W_{2n}) = \{-1, 1, \ldots, 2n\}$ and $x \sim y \Leftrightarrow xy < 0 \,\text{or}\, x - y \equiv \pm 1 \pmod{2n}$. The Cheeger constant of W_{2n} is $1 + \frac{2}{n}$. Each of the positive (outer) vertices has degree 3, and the -1 (center) vertex has degree $2n$. Yet every nontrivial fortress consists either of all the positive (outer) vertices or the center vertex with at least half of the outer vertices (provided that each outer vertex in the fortress has an outer vertex neighbor also in the fortress). It is clear that the complement of each nontrivial fortress is not a fortress, so there is no double fortress.

3. **A graph with arbitrarily large Cheeger constant that has a nontrivial double fortress.** Consider the sequence of nearly-complete graphs J_{2n} defined by $V(J_{2n}) = \{1, 2, \ldots, 2n\}$ and $x \sim y \Leftrightarrow |x - y| \neq n$. Each vertex has degree $2n-2$, and the Cheeger constant of J_{2n} is $\frac{n(n-1)}{n} = n-1$. Yet the sets of vertices $\{1, 2, \ldots, n\}$ and $\{n+1, n+2, \ldots, 2n\}$ are fortresses, and the two sets are complementary; therefore, there is a nontrivial double fortress on J_{2n}.

4 Graph Families—The Extension to Infinity

4.1 Background and Definitions

Many of the previously-proved results use the fact that G is a finite graph. However, it is often useful to consider infinite graphs and their Cheeger values and fortresses. A workaround is to use graph families.

A *graph family* is a sequence of nested graphs $G_1, G_2, G_3, \ldots$, where $V(G_1) \subseteq V(G_2) \subseteq V(G_3) \subseteq \cdots$ and $E(G_1) \subseteq E(G_2) \subseteq E(G_3) \subseteq \cdots$. (This is the same notation as running the CA; the context will make it clear what operation the notation indicates.) We say that a vertex $v \in V(G_n)$ is *finalized* if for all $N > n$, $\{w : v \sim_{G_N} w\} = \{z : v \sim_{G_n} z\}$. We also say that a nonempty fortress $F \subset V(G_n)$ on G_n is *stable* if F is a fortress on G_N for all $N > n$. The following fact is clear:

Fact 14. *Let F be a fortress on G_n, and suppose that for all $v \in F$, v is finalized. Then F is stable.*

Proof. This follows directly from the fact that since the vertices in the fortress are finalized, being a fortress on G_n is the exact same condition as being a fortress on any later graph in the sequence. □

Determining whether a sequence of graphs has a stable nontrivial fortress is not obvious. The following two examples demonstrate the extreme possibilities.

1. Suppose G_n is the toroidal square lattice with n vertices on each side (with coordinates ranging from 0 to $n-1$). We will take $n \geq 3$ to ensure the graph remains simple. Because each G_n is 4-regular, any cycle is a fortress. In particular, the square with coordinates $(0,0)$, $(1,0)$, $(1,1)$, $(1,0)$ is a fortress on every G_n. Additionally, every vertex (x,y) on G_n in which $1 \leq x \leq n-2$ and $1 \leq y \leq n-2$ is finalized.

2. Suppose T_n designates a rooted binary tree on n levels ($n \geq 2$), where the levels are numbered from 0 to $n-1$. I claim there is no stable fortress F because I will show that every fortress on T_n must contain a vertex on level $n-1$. Suppose it does not. Then, there being only a finite number of vertices in the fortress, let k be the highest-numbered level of any vertex in the fortress, and let v be a vertex on level k. Then $\partial(\{v\}, V(G_n) \backslash F) \geq 2$ because the two children of v are not in F. However, every vertex in T_n has at most degree 3. This means it is impossible that $\partial(\{v\}, F \setminus \{v\}) \geq \partial(\{v\}, V(G) \setminus F)$. So no such stable fortress exists.

4.2 Cheeger Constant of Graph Families

It is natural to take the Cheeger constant of a graph family to be the limit inferior of the Cheeger constant of each graph. However, this leads to problems. For example, the infinite analogue to the Cheeger constant for the infinite k-regular tree is $k-2$, yet the limit inferior of the Cheeger constant of the natural graph family that generates the tree (all vertices of level n or less) is zero (since one possible set contains all but one branch, resulting in one border edge that cuts the graph into larger and larger pieces). Therefore, we need a refined definition.

Define the *modified Cheeger constant of a graph family* $G_1, G_2, G_3, \ldots$ by $h_G^* = \lim_{n\to\infty} h_{G_n}^*$, where $h_{G_n}^* = \min\{h_{G_n}(A) : A \subseteq V(G_n), v \,\text{finalized}\, \forall v \in A\}$. Because the vertices in the Cheeger set are finalized, the number of border edges is constant, and denominator can only increase. This means that $h_{G_1}^* \geq h_{G_2}^* \geq h_{G_3}^* \geq \cdots$.

Proposition 15. *Let $G_1, G_2, G_3, \ldots$ be a graph family such that $h_G^* < 1$. Then there is a stable fortress $F \subseteq G_n$ for some n.*

Proof. Choose k so that $h_G^* < h_{G_k}^* < 1$ (possible because the limit is less than 1). Let A be the Cheeger set on finalized vertices of G_k so that $h_{G_k}(A) = h_{G_k}^*$. Select B to be a minimal nonempty subset of A with $h_{G_k}(B) = h_{G_k}(A)$. (This is possible because A has a finite number of vertices.) Since $B \subseteq A$, every vertex in B is finalized. By Proposition 1, B is a fortress, and since B comprises finalized vertices, it is a stable fortress by Fact 14. ☐

4.3 Collapsing a Graph Family

It is easier to show that a fortress is stable if each vertex is finalized quickly. The following set of definitions will allow us to define *collapsed graph families*, where each vertex in a graph is finalized in the next graph.

Define a graph family $G_1, G_2, G_3, \ldots$ to be *finite-degree* if for all $n \in \mathbb{Z}^+$ and $v \in V(G_n)$, there exists $N_v \in \mathbb{Z}^+, N_v > n$, so that v is finalized in G_{N_v}. We say that a finite-degree graph family $G_1, G_2, G_3, \ldots$ is *collapsed* if for all $n \in \mathbb{Z}^+$ and $v \in V(G_n)$, v is finalized in G_{n+1}.

If G is a finite-degree graph family, define the *collapse* of G to be the sequence of graphs $(\psi(G))_1, (\psi(G))_2, (\psi(G))_3, \ldots$ created by the following construction:

- $V((\psi(G))_1) = V(G_1)$
- $V((\psi(G))_n) = V(G_n) \cup V((\psi(G))_{n-1}) \cup \{w : \exists N \in \mathbb{Z}^+ : \exists v \in (G_N \cap (\psi(G))_{n-1}) : v \sim_{G_N} w\}$, $n > 1$
- $E((\psi(G))_n) = \{\{v, w\} : v, w \in (\psi(G))_n \text{ and } \exists N \in \mathbb{Z}^+ : v \sim_{G_N} w\}$

Note that since G is finite-degree, each graph of the collapse is finite.

Lemma 16. *Suppose G is a finite-degree graph family. Then $\psi(G)$ is a collapsed graph family.*

Proof. It is clear that $V((\psi(G))_1) \subseteq V((\psi(G))_2) \subseteq V((\psi(G))_3) \subseteq \cdots$. Because of this inclusion relation and the fact that $E((\psi(G))_n)$ solely depends on G and $V((\psi(G))_n)$, it follows that $E((\psi(G))_1) \subseteq E((\psi(G))_2) \subseteq E((\psi(G))_3) \subseteq \cdots$. So $\psi(G)$ is a graph family. Further, for each vertex $v \in V((\psi(G))_n)$, each vertex that neighbors v in one of the graphs of G also neighbors v in $(\psi(G))_{n+1}$, quickly finalizing v. Hence, $\psi(G)$ is a collapsed graph family. ☐

Proposition 17. *Suppose G is a finite-degree graph family; $\psi(G)$ is its collapse. Then for any positive integer n, there exists a positive integer N so that $(\psi(G))_n$ is a subgraph of G_N.*

Proof. For every $v \in V((\psi(G))_n)$, let N_v be the least positive integer so that $v \in V(G_{N_v})$. For every $e \in E((\psi(G))_n)$, let N_e be the least positive integer so that $e \in E(G_{N_v})$. By construction, those integers are well-defined. Now let

$$N = \max\{\max_{v \in V((\psi(G))_n)} N_v, \max_{e \in E((\psi(G))_n)} N_e\}$$

(well-defined since each set is finite). Since the vertices and edges of G are nested subsets, this means that all the vertices and edges of $(\psi(G))_n$ are in G_N, making it a subgraph. ☐

4.4 There Is No Infinitely-Expanding Fortress

Let F be a nontrivial fortress on finite graph G. Then define the *attack* of f in F as $\operatorname{atk}(F, f) = \partial(\{f\}, V(G) \setminus F)$. The attack of F is $\operatorname{atk}(F) = \partial(F, V(G) \setminus F)$. Note that if F is a nontrivial fortress, then $\operatorname{atk}(F) \geq 0$.

Lemma 18. *Let F be a nontrivial fortress on finite graph G, and let $\langle F \rangle$ be the induced subgraph of G. Then $\operatorname{atk}(F) \leq 2|E(\langle F \rangle)|$.*

Proof. Since F is a fortress, we have that for all $f \in F$, $\partial(\{f\}, V(G) \setminus F) \leq \partial(\{f\}, F \setminus \{f\})$. Summing over all $f \in F$ gives us the desired inequality. □

The main effect of Lemma 18 is to show that the attack is finite. The following proposition demonstrates, as the CA is applied to a fortress, its attack strictly decreases as long as the fortress gets bigger.

Proposition 19. *Let F_0 be a nontrivial fortress on finite graph G. Then $\operatorname{atk}(F_0) \geq \operatorname{atk}(F_1) \geq \operatorname{atk}(F_2) \geq \cdots$, with $\operatorname{atk}(F_i) = \operatorname{atk}(F_{i+1})$ if and only if $F_i = F_{i+1}$.*

Proof. Suppose $F_0 \subsetneq F_1$. Then $F_1 = F_0 \cup V$, where $V = \{v_1, v_2, \ldots, v_k\}$ and $V \cap F_0 = \emptyset$. By the rules of the CA, for each $1 \leq i \leq k$, $\partial(\{v_i\}, F_0) > \partial(\{v_i\}, V(G) \setminus (F_0 \cup \{v_i\}))$. We can use this relation to compare $\operatorname{atk}(F_0)$ and $\operatorname{atk}(F_0 \cup \{v_i\})$. Adding v_i lowers the attack by one for each of the $\partial(\{v_i\}, F_0)$ vertices in F_0 adjacent to v_i: a loss of $\partial(\{v_i\}, F_0)$. On the other hand, adding v_i increases the total attack by v_i's contribution, $\partial(\{v_i\}, V(G) \setminus (F_0 \cup \{v_i\}))$. Therefore $\operatorname{atk}(F_0 \cup \{v_i\}) - \operatorname{atk}(F_0) = \partial(\{v_i\}, V(G) \setminus (F_0 \cup \{v_i\})) - \partial(\{v_i\}, F_0)$; define the right-hand quantity (which is negative) to be L_i. Then $\operatorname{atk}(F_1) - \operatorname{atk}(F_0) = \sum_{i=1}^{k} L_i - 2|E(\langle V \rangle)|$, where $\langle V \rangle$ is the induced subgraph on G. Since the right-hand quantity of the previous equation is negative, this implies $\operatorname{atk}(F_0) > \operatorname{atk}(F_1)$. Therefore, $\operatorname{atk}(F_0) \geq \operatorname{atk}(F_1)$, with equality iff $F_0 = F_1$. Since F_1 is a fortress (Proposition 10), we can apply the same argument repeatedly to get the statement of the proposition. □

An initial nontrivial fortress F_0's attack, which is finite, serves as a cap on the number of vertices that can included in the double fortress that is the only member of F_∞. If G contains more than $|F| + \operatorname{atk}(F)$ vertices, then the lone element of F_∞ cannot be all the vertices of G. Therefore, if the number of vertices in a graph family is unbounded, we can always select a graph G_n that is large enough so that the lone element in F_∞ is not the entire graph. We state this formally below:

Proposition 20. *Let G be a graph family with $\lim_{n\to\infty} |V(G_n)| = +\infty$. Let $F \subset V(G_n)$ be a stable nontrivial fortress. Then there exists $N > n$ so that on G_N, $F_\infty = \{A\}$, where A is a nontrivial double fortress.*

Proof. First, I claim that each vertex $f \in F$ is eventually finalized. Suppose not; then we can choose $N \in \mathbb{Z}^+$ so that $d_{f,G_N} > 2d_{f,G_n}$. In other words, there are more new vertices than old vertices, and since the new vertices are definitely not in the fortress, f is not in the fortress, creating a contradiction. Therefore, every vertex $f \in F$ must eventually become finalized. For each $f \in F$, define K_f to be the smallest integer so that f is finalized in G_{K_f}. Then take $K = \max(\{K_f : f \in F\} \cup \{n+1\})$. Since F is stable, F is also a fortress in G_K (because $K > n$). Also, since each vertex in F is finalized, $\text{atk}(F)$ is constant in G_k for all $k \geq K$. We will let a denote that constant value. Now define $M = \max(\{K\} \cup \{i \in \mathbb{Z}^+ : |F| + a \geq |V(G_{i-1})|\})$. Then $|V(G_M)| > |F| + a$. By the explanation above, $F_\infty = \{A\}$ for some $A \subseteq V(G_M)$ (cf. Proposition 10), where A does not comprise the entire graph; therefore, there exists $j \in \mathbb{Z}^+$ so that $A = F_j = F_{j+1}$ and $F_j \subsetneq V(G_M)$. By Corollary 11, this implies that A is a double fortress, and moreover, a nontrivial double fortress, as desired. This completes the proof. □

We define a finite-degree graph family to be *k-regular* if there exists $n \in \mathbb{Z}^+$ where for finalized $v \in V(G_n)$, $d_{v,G_n} = k$.

The following theorem is the first one that guarantees the non-existence of a stable nontrivial fortress on a graph family.

Theorem 21. *Let $G_1, G_2, G_3, \ldots$ be a collapsed k-regular graph family with $\lim_{n\to\infty} |V(G_n)| = \infty$. Then, if $h^*_G > \frac{k}{2}$, then G has no stable nontrivial fortress.*

Proof. Suppose, for a contradiction, that there is a stable nontrivial fortress $A \subseteq G_n$. Then, define N to be the least positive integer so that $|V(G_N)| > 2|V(G_n)|$. Since A is a stable fortress, A is a fortress on G_N. Now $\partial(A, V(G_N) \setminus A) + 2|E(\langle A \rangle)| = k|A|$, where $\langle A \rangle$ is the induced subgraph on G_N. This is equivalent to $\text{atk}(A) + 2|E(\langle A \rangle)| = k|A|$. However, since $h^*_{G_N} > \frac{k}{2}$, this implies that $\frac{\text{atk}(A)}{|A|} \geq h^*_{G_N}(A) \geq h^*_{G_N} > \frac{k}{2}$. Therefore, $\text{atk}(A) > \frac{k|A|}{2}$. Thus, $2|E(\langle A \rangle)| = k|A| - \text{atk}(A) < \frac{k|A|}{2} < \text{atk}(A)$. But this directly contradicts Lemma 18, which tells us that $2|E(\langle A \rangle)| \geq \text{atk}(A)$. Therefore, our assumption that A was a stable nontrivial fortress is false; therefore, there is no stable nontrivial fortress on G. □

Corollary 22. *Let $G_1, G_2, G_3, \ldots$ be a finite-degree k-regular graph family with $\lim_{n\to\infty} |V(G_n)| = \infty$. Then, if $h^*_G > \frac{k}{2}$, then G has no stable nontrivial fortress.*

Proof. Again, suppose that there is a stable nontrivial fortress $A \subseteq V(G_n)$. Define Λ to be the least positive integer so that every vertex in A is finalized. Then G_Λ is a subgraph of $(\psi(G))_\Lambda$. Also, by Proposition 20, there exists a Λ^* so that $(\psi(G))_\Lambda$ is a subgraph of G_{Λ^*}. Now, since $\psi(G)$ is a collapsed graph family with $h^*_{\psi(G)} \geq h^*_G > \frac{k}{2}$, there is no stable fortress on $(\psi(G))_\Lambda$, and therefore no stable fortress on G_{Λ^*}, which contradicts the fact that A was assumed to be a stable fortress. So there is no such stable nontrivial fortress. □

5 The Relation to Modularity

5.1 Definitions

Modularity is a concept in network theory that describes how well a graph can be broken into individual strongly-connected modules. M. E. J. Newman [3] quantifies the modularity Q as follows:

$$Q = \frac{1}{4|E(G)|} \sum_{i,j \in V(G)} \left(A_{ij} - \frac{d_i d_j}{2|E(G)|} \right) s_i s_j$$

where A is the adjacency matrix and s_i is $+1$ or -1 depending on the sign of vertex i.

The motivation is to indicate partitions of graphs in which there are more intramodular edges and fewer intermodular edges than would be expected. A positive value indicates a clear break between modules as indicated by the partition, while a negative value indicates a bad choice of partition. Further, the trivial partition where all the vertices are in one partition has zero modularity, which allows zero to be the baseline.

5.2 Fortresses Have Nonnegative Modularity

From the definition, fortresses are strongly connected subsets of vertices. It should come as no surprise that double fortresses are a good choice for partitioning vertices into modules. The following theorem illustrates this property.

Theorem 23. *Let G be a finite graph, and $F \subseteq V(G)$ a double fortress. Then the modularity of the G based on F is nonnegative, with the modularity being zero only if F is trivial or if each vertex has the same number of neighbors in its own partition as in the other partition.*

Proof. First, for any $v \in A \subseteq V(G)$, define the *advantage* of v in A by $\operatorname{adv}(v) = \partial(\{v\}, A \setminus \{v\}) - \partial(\{v\}, V(G) \setminus A)$. Note that if A is a fortress, then $\operatorname{adv}(v) \geq 0$. Further, if $B \subseteq A \subseteq V(G)$, define the advantage of B in A by $\operatorname{adv}(B) = \sum_{v \in B} \operatorname{adv}(v)$.

Now, let us rewrite Q. First, split the summation to produce

$$Q = \frac{1}{4|E(G)|} \left(\sum_{i,j \in V(G)} A_{ij} s_i s_j - \sum_{i,j \in V(G)} \frac{d_i d_j s_i s_j}{2|E(G)|} \right)$$

Fix an arbitrary $i \in V(G)$ for a moment. Suppose $i \in F$ (alternatively, $i \in F^c$). Then the first term corresponds to how many more neighbors of i are in F than F^c (alternatively, in F^c than F). But this is just $\operatorname{adv}(i)$. Summing over all $i \in F$ and $i \in F^c$ gives

$$Q = \frac{1}{4|E(G)|} \left(\operatorname{adv}(F) + \operatorname{adv}(F^c) - \sum_{i,j \in V(G)} \frac{d_i d_j s_i s_j}{2|E(G)|} \right)$$

Now define $X = \sum_{v \in F} d_v$ and $Y = \sum_{v \in F^c} d_v$, where it is clear that $X + Y = 2|E(G)|$. Rearranging the terms produced by the last summation results in

$$Q = \frac{1}{4|E(G)|}\left(\operatorname{adv}(F) + \operatorname{adv}(F^c) - \frac{(X-Y)^2}{2|E(G)|}\right)$$

Let $Z = \partial(F, F^c)$. We will compute Z in two ways. First, for each vertex $v \in F$, $\partial(\{v\}, F \setminus \{v\}) + \partial(\{v\}, F^c) = d_v$. Since $\partial(\{v\}, F \setminus \{v\}) - \partial(\{v\}, F^c) = \operatorname{adv}(v)$, we substitute to get $\operatorname{adv}(v) + 2\partial(\{v\}, F^c) = d_v$. Solving for $\partial(\{v\}, F^c)$ gives $\partial(\{v\}, F^c) = \frac{1}{2}(d_v - \operatorname{adv}(v))$. Summing over all $v \in F$ gives $Z = \frac{1}{2}(X - \operatorname{adv}(F))$. Similarly, using a similar argument for $v \in F^c$ tells us that $\partial(\{v\}, F) = \frac{1}{2}(d_v - \operatorname{adv}(v))$, from which it follows that $Z = \frac{1}{2}(Y - \operatorname{adv}(F^c))$. Therefore, $\frac{1}{2}(X - \operatorname{adv}(F)) = \frac{1}{2}(Y - \operatorname{adv}(F^c))$, and $X - Y = \operatorname{adv}(F) - \operatorname{adv}(F^c)$.

Now we will produce the desired inequality. Since F and F^c are fortresses, $\operatorname{adv}(F)$ and $\operatorname{adv}(F^c)$ are nonnegative. Therefore, $\operatorname{adv}(F) + \operatorname{adv}(F^c) \geq |\operatorname{adv}(F) - \operatorname{adv}(F^c)|$, with equality holding if and only if either advantage is zero. Also, $2|E(G)| = X + Y \geq |X - Y| = |\operatorname{adv}(F) - \operatorname{adv}(F^c)|$, with equality holding if and only if X or Y is zero. Since all the numbers involved are nonnegative, we can multiply the two inequalities together to get

$$2|E(G)|(\operatorname{adv}(F) + \operatorname{adv}(F^c)) \geq (\operatorname{adv}(F) - \operatorname{adv}(F^c))^2$$

with equality holding if and only if $\operatorname{adv}(F) + \operatorname{adv}(F^c) = 0$ or F is trivial. Dividing by positive $2|E(G)|$ gives us that

$$\operatorname{adv}(F) + \operatorname{adv}(F^c) \geq \frac{(\operatorname{adv}(F) - \operatorname{adv}(F^c))^2}{2|E(G)|}$$

which implies that

$$\operatorname{adv}(F) + \operatorname{adv}(F^c) - \frac{(\operatorname{adv}(F) - \operatorname{adv}(F^c))^2}{2|E(G)|} \geq 0$$

Since $X - Y = \operatorname{adv}(F) - \operatorname{adv}(F^c)$,

$$\operatorname{adv}(F) + \operatorname{adv}(F^c) - \frac{(X-Y)^2}{2|E(G)|} \geq 0$$

Multiply by $\frac{1}{4|E(G)|}$ to produce the desired result,

$$Q = \frac{1}{4|E(G)|}\left(\operatorname{adv}(F) + \operatorname{adv}(F^c) - \frac{(X-Y)^2}{2|E(G)|}\right) \geq 0$$

with equality holding if and only if F is trivial or $\operatorname{adv}(F) + \operatorname{adv}(F^c) = 0$. $\square$

6 Summary

Cheeger constants and graph fortresses are strongly related. When the Cheeger constant is small enough, the existence of a nontrivial double fortress on a graph is guaranteed. On the other hand, the presence of a nontrivial double fortress does not necessarily say anything about the Cheeger constant.

On infinite graphs approximated by graph families, the stable fortress is the analogue of the double fortress. When $h^*_G < 1$, a stable fortress is guaranteed to exist. When G is a k-regular graph family and $h^*_G > \frac{k}{2}$, there is no stable fortress. Otherwise, it is an open question whether G has a stable fortress.

Finally, fortresses are bastions of nonnegative modularity. Since fortresses are a common result of running the CA, the CA will usually produce partitions of the vertices of a graph that have high modularity. In light of this, it may be fruitful to incorporate the CA into module-finding algorithms.

References

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