Existence of topological entropy preserving subsystems weakly embeddable in symbolic dynamical systems

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Abstract

The topological entropy of a dynamical system is known to be equal to or greater than all of its subsystems. We show conditions in which subsystems have equal topological entropy. Furthermore, we characterize systems containing invariant subsets of equal entropy that are weakly embeddable in a symbolic dynamical system. It turns out that compact ϵ-expansive dynamical systems always contain such a subsystem.

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1. Introduction

The use of symbolic dynamics arose in 1898 when Jacques Hadamard used infinite symbol sequences to study geodesics on negatively curved surfaces. Symbolic systems have a discrete phase space and time evolution, and are used in many disciplines, including ergodic theory, information theory, and computer simulations of continuous systems [1]. Through the use of embedding maps, symbolic dynamical systems provide a rich framework to study general dynamical systems. In order to discern whether two dynamical systems are homeomorphic to each other, Adler-Konhiem-McAndrew introduced topological entropy which quantitatively describes the complexity of a system.

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In 2000, Akashi showed that ϵ-expansive dynamical systems with compact and totally disconnected phase space can be embedded in symbolic dynamical systems [2], and in 2006 Wang and Wei found necessary and sufficient conditions for such embeddings [3]. In the same year, Fedeli showed that a system being totally expansive is a necessary and sufficient condition for the system being weakly embedded into a symbolic dynamical system [4]. In 2002 Sakai gave a revised version of Akashi's upper bound on the topological entropy of an ϵ-expansive dynamical systems, and applies this to a Baire category theoretic classification of expansive dynamical systems [5]. By dropping the condition of being ϵ-expansive, in 2005 Akashi developed a product extension of symbolic dynamical systems into which symbolic embeddings can be made [6]. The first investigation into their properties appears in the 2009 paper by Cheng, Wang, and Wei [7].

When a continuous space is discretized, information is necessarily discarded. In this paper, the focus is to investigate the formation of a totally disconnected space from a connected one in a way that minimizes the information lost. The aim of this is to provide a method to obtain entropy preserving subsystems weakly embeddable in a symbolic dynamical system.

Section 2 presents a definition of topological entropy and other definitions used in the paper. In section 3 we show sufficient conditions in which a subsystem has topological entropy equal to the original system. In section 4 we show conditions for obtaining a totally disconnected subsystem such that topological entropy is preserved, and corollaries to this result.

2. Notation and Preliminaries

2.1. Basic Concepts and Definitions

A *topological dynamical system*, denoted as (X, T), consists of a topological space X and a continuous map $T : X \to X$. A set $Z \subseteq X$ is said to be *invariant on* T if $T(Z) \subseteq Z$. The restriction of T to Z, $T|_Z : Z \to Z$, determines a dynamical system $(Z, T|_Z)$, which is called a *subsystem* of (X, T).

Throughout this paper, $\mathbb{N}$ denotes the set of all non-negative integers. Let $\mathcal{M}$ be a finite set of symbols, and let $\mathcal{M}^{\mathbb{N}}$ be the set of all one sided infinite sequences consisting of elements of $\mathcal{M}$. Let $\sigma : \mathcal{M}^{\mathbb{N}} \to \mathcal{M}^{\mathbb{N}}$ be the shift map in which $\sigma(a_0 a_1 a_2 \ldots) = a_1 a_2 a_3 \ldots$. The dynamical system $(\mathcal{M}^{\mathbb{N}}, \sigma)$ is called a *symbolic dynamical system*.

For some indexing positive integer n, consider a shift space $\mathcal{M}_n^{\mathbb{N}}$ with shift map σ_n. The product space $\prod_{n=1}^{\infty} \mathcal{M}_n^{\mathbb{N}}$ together with a shift map $\prod_{n=1}^{\infty} \sigma_n$ de-

fined as $(\prod_{n=1}^{\infty} \sigma_n) \circ (\{r_n\}_{n=1}^{\infty}) = \{\sigma_n r_n\}_{n=1}^{\infty}$ forms a *product symbolic dynamical system.* For information on the properties of product symbolic dynamical systems see [7].

Let (X, T) and (Y, S) be two topological dynamical systems. If there exists a subspace Z of Y and a continuous injective map $\varphi : X \rightarrow Z$ such that $S \circ \varphi = \varphi \circ T$, then (X, T) is said to be *weakly embeddable* into (Y, S). If in addition, φ is a homeomorphism then we say (X, T) is *embeddable* in (Y, S).

Let (X, T) be a dynamical system. If there exists a countable partition of $X = \bigcup_{i \in \mathbb{N}} P_i$ such that for each P_i the restriction $T|_{P_i} : P_i \rightarrow X$ is injective, then T is *countably injective.* A metric space X is *totally bounded* if for every $\epsilon > 0$ there is a finite subset A of X such that $X = \bigcup \{B(x, \epsilon) : x \in A\}$, where $B(x, \epsilon)$ is the open ball of center x and radius ϵ.

A topological space is *totally disconnected* if the connected components of X are one-point sets. We call a set X *strongly zero-dimensional* if for every finite open cover $U = \{U_i : 0 \leq i \leq m\}$ of X, there exists an open refinement $V = \{V_i : 0 \leq i \leq m\}$ such that $V_i \subset U_i$ for $i \in \{0, \ldots, m\}$ and $V_i \cap V_j = \emptyset$ whenever $i \neq j$.

Let (X, T) be a topological dynamical system and let d be a metric on X. Then T is said to be ϵ-*expansive* if there exists some $\epsilon > 0$ such that for every distinct pair of points $x, y \in X$, there is some $n \in \mathbb{N}$ for which $d(T^n(x), T^n(y)) \geq \epsilon$. A topological dynamical system (X, T) is *totally expansive* if there is a finite closed and open partition $\mathcal{B}$ of X such that for any pair x, y of distinct points of X there is some $n \in \mathbb{N}$ such that the points $T^n(x)$ and $T^n(y)$ belong to different members of $\mathcal{B}$.

Using the notion of totally expansive, Fedeli obtained necessary and sufficient conditions for topological dynamical systems to be weakly embeddable in symbolic dynamical systems. A corollary from his results, which we will use later in the paper, is stated in Proposition 1 [4].

Proposition 1. *Let (X, T) be an ϵ-expansive topological dynamical system whose phase space is a strongly zero-dimensional totally bounded metric space. Then (X, T) is totally expansive and can be weakly embedded into a symbolic dynamical system.*

The above proposition can be modified for weak embeddings into product symbolic dynamical systems.

Proposition 2. *Let (X,T) be a topological dynamical system. If X is a strongly zero-dimensional and totally bounded metric space, then (X,T) can be weakly embedded into a product symbolic dynamical system.*

If X is strongly zero dimensional and totally bounded, then for any $\epsilon > 0$ there exists a finite covering of X by clopen balls of radius ϵ. From here, the proof of Proposition 2 follows almost exactly the proof of Theorem 1 in Akashi's paper [6]. Furthermore, if (X,T) is a topological dynamical system on a compact space and φ is a weak embedding map from (X,T) to an ordinary or product symbolic dynamical system, then φ is a embedding map.

2.2. Definition of Topological Entropy

Let $f : X \to X$ be a continuous map on the space X with metric d. For $n \in \mathbb{N}$ define a metric $d_{n,f}(x,y) = \sup\{d(f^j(x), f^j(y))|0 \leq j < n : j \in \mathbb{N}\}$. A set $S \subset X$ is (n, ϵ)-separated for f provided $d_{n,f}(x,y) > \epsilon$ for every pair of distinct points $x, y \in S, x \neq y$. Note that on a totally bounded metric space, the (n, ϵ)-separated sets are finitely large. The number of different orbits of length n as measured by ϵ is defined by

$$r(n, \epsilon, f) = max\{card(S) \mid S \subset X \text{ is a } (n, \epsilon)\text{- separated set for } f\}.$$

To measure the growth rate of $r(n, \epsilon, f)$ as n increases, we define

$$h(\epsilon, f) = \limsup_{n \to \infty} \frac{1}{n} log(r(n, \epsilon, f)).$$

Note that $r(n, \epsilon, f) \geq 1$ for any pair (n, ϵ), so $0 \leq h(\epsilon, f) \leq \infty$. The topological entropy of f is defined as

$$h(f) = \lim_{\epsilon \to 0, \epsilon > 0} h(\epsilon, f).$$

Topological entropy was introduced by Adler-Konhiem-McAndrew as an invariant for continuous mappings on compact topological spaces [8]. We may also use the definition of entropy on metric spaces, which was introduced by Bowen [9] and independently by Dinaburg [10] which is equivalent with the Adler-Konhiem-McAndrew version. Topological entropy quantitatively describes complexity of a topological dynamical system. Positive topological entropy implies Li-Yorke chaos, and for a topological dynamical system (X, f), if X is compact, then $h(\epsilon, f) < \infty$ [11].

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3. Topological entropy

Theorem 1. *Let (X, T) be a topological dynamical system and let d be a metric on X. If there exists an set $Z \subseteq X$ such that Z is invariant for T and dense in X, then*

$$h(T) = h(T|_Z).$$

Proof For all $n \in \mathbb{N}$, we find $r(n, \epsilon, T)$ is bounded above by $r(0, \epsilon, T)^n$, thus $h(\epsilon, T) = \infty$ if and only if $r(0, \epsilon, T) = \infty$. Suppose for some $n \in \mathbb{N}$ and $\epsilon > 0$ that $r(n, \epsilon, T) = \infty$. This implies $h(T) = \infty$. We show that $h(T|_Z) = \infty$ by contradiction. If we assume $h(T|_Z) < \infty$ then $r(0, \epsilon, T|_Z) < \infty$ for all $\epsilon > 0$ i.e. Z is totally bounded. Since X contains a totally bounded dense subset, X itself is totally bounded which contradicts the original assumption that $r(n, \epsilon, T) = \infty$. This contradiction thus proves that if $r(n, \epsilon, T) = \infty$ then $h(T) = h(T|_Z) = \infty$.

Now suppose for all $n \in \mathbb{N}$ and $\epsilon > 0$ that $r(n, \epsilon, T) < \infty$ and further, there exists some $n \in \mathbb{N}$ and $\epsilon > 0$ such that $r(n, \epsilon, T) \neq r(n, \epsilon, T|_Z)$. This means there is an (n, ϵ)-separated set $S \subset X$ such that $card(S) > r(n, \epsilon, T|_Z)$.

Fix $\epsilon_0 > 0$. The continuity of T implies that for each $x_i \in S$ there exist a $\delta_i > 0$ for which all $y \in X$ such that $d(x_i, y) < \delta_i$ implies $d_{n,T}(x_i, y) < \epsilon_0$. As Z is dense in X, we can construct a set $S_0 \subset Z$ of cardinality equal to that of S such that for each point $x_i \in S$ there exists a point $y_i \in S_0$ such that $d_{n,T}(x_i, y_i) < \epsilon_0/2$. By the triangle inequality, for all $x_i, x_j \in S$ and $y_i, y_j \in S_0$ such that $i \neq j$, then $|d_{n,T}(x_i, x_j) - d_{n,T}(y_i, y_j)| \leq \epsilon_0$. Thus, for all $n \in \mathbb{N}, \epsilon > 0$, there exists an $\epsilon_0 < \epsilon$ such that $r(n, \epsilon, T) = r(n, \epsilon_0, T|_Z)$. This means that for all $\epsilon > 0$ there exists an $\epsilon_0 < \epsilon$ such that $h(\epsilon, T) = h(\epsilon_0, T|_Z)$. Since $h(T)$ is determined as $\epsilon \to 0$, we may conclude that $h(T) = h(T|_Z)$.

$\square$

Remark Since a compact subset of a Hausdorff space is closed, if X is a compact Hausdorff space, then the only dense compact subspace is X itself. As the definition of topological entropy given by Adler-Konhiem-McAndrew is only for compact Hausdorff spaces, investigating entropy of dense subsystems necessitates the Bowen-Dinaburg definition for nontrivial results.

4. Subsystems

Let Y be a subset of a topological space X. We say Y is *nowhere dense* in X if the interior of its closure is empty. A set Y is *meagre* in X if it is a

union of countably many nowhere dense subsets and a set Y is *nonmeagre* in X if it is not meagre in X. A subset Y of X is *comeagre* if its complement $X \backslash Y$ is meagre. A topological space X is a *Baire space* if every non-empty open set is nonmeagre in X. For more on Baire Spaces and their properties, we refer the reader to any standard text book on topology.

Let (X, T) be a topological dynamical system. For a set $M \subset X$ define $\mathcal{O}_M^- = \bigcup_{p \in M} \bigcup_{n \in \mathbb{N}} \{x \in X | T^n(x) = p\}$, and define $\mathcal{O}_M^+ = \bigcup_{p \in M} \{T^n(p) | n \in \mathbb{N}\}$. To characterize regions with empty interior mapped to by sets with non-empty interior, let $\Xi(X) = \bigcup T(int(U))$ be the union over all $U \subset X$ with $int(U) \neq \emptyset$ and $int(T(U)) = \emptyset$. If there is no question as to which dynamical system is being referred to, we will denote $\Xi(X)$ by Ξ.

Lemma 1. *Let (X, T) be a topological dynamical system such that X is a Baire Space. If a set $B \subseteq X$ has an empty interior, is closed, and is disjoint from $\mathcal{O}_\Xi^+$, then $int(\mathcal{O}_B^-) = \emptyset$.*

Proof Assume that $T^{-1}(B)$ is not nowhere dense. Since B is closed and T is continuous, $T^{-1}(B)$ is closed and thus $T^{-1}(B) = \overline{T^{-1}(B)}$. Under the assumption that $T^{-1}(B)$ is not nowhere dense, there must exist a nonempty open set $U \subset T^{-1}(B)$. It follows that $T(U) \subset B$, and since B has an empty interior, then $T(U) \subset \Xi$, implying that $\mathcal{O}_\Xi^+$ is not disjoint with B. This contradiction shows that a closed set in $X \backslash \mathcal{O}_\Xi^+$ with an empty interior has a nowhere dense preimage. As X is a Baire space, the set $\mathcal{O}_B^- = \bigcup_{n \in \mathbb{N}} \{M \subset X | T^n(M) \subseteq B\}$ has an empty interior. $\qquad \square$

Remark For a topological dynamical system (X, T) in which $\Xi = \emptyset$, if X is a Baire space, then the iterated inverse image of a countable collection of closed sets with empty interior has an empty interior.

Theorem 2. *Let (X, T) be a topological dynamical system such that X is both a Baire and Hausdorff space and there are at most countably many nontrivial connected components of X. There exists a set $Y \subset X$ such that $X \backslash Y$ is totally disconnected, dense in X, and invariant on T if the set $\mathcal{O}_\Xi^+$ is strongly zero-dimensional.*

Proof Let $X = D_0 \cup C_0$ where D_0 is the largest totally disconnected subset of X and for an indexing set I, the set $C_0 = \bigcup_{i \in I} C_{0,i}$ is the union of all nontrivial connected components of X. We wish to show that for every $C_{0,i}$ we can select a set $B_{0,i} \subset C_{0,i} \backslash \mathcal{O}_\Xi^+$ such that $C_{0,i} \backslash B_{0,i}$ is not connected.

For a given set $C_{0,i}$, if there are at least two points in both $\mathcal{O}_{\Xi}^{+}$ and $C_{0,i}$ then take a finite open covering of $\mathcal{O}_{\Xi}^{+} \cap C_{0,i}$ denoted $U_{0,i} = \bigcup_{0 \leq k \leq m} U_{0,i}^{k}$ for $m \geq 2$. Since $\mathcal{O}_{\Xi}^{+}$ is strongly zero-dimensional, there exists an open refinement $V_{0,i} = \bigcup_{0 \leq k \leq m} V_{0,i}^{k}$ such that $V_{0,i}^{j} \cap V_{0,i}^{k} = \emptyset$ for all $j \neq k$. Since $C_{0,i}$ is connected, and $V_{0,i}$ is a finite disjoint union of open subsets of $C_{0,i}$, there exists a non-empty closed complement of $V_{0,i}$ in $C_{0,i}$. By removing the boundary of this set, denoted by $B_{0,i}$, we can write $C_{0,i} \backslash B_{0,i}$ as the union of at least two disjoint open sets.

If there are zero or one elements in $C_{0,i} \cap \mathcal{O}_{\Xi}^{+}$, then consider two distinct points in $C_{0,i}$. As X is Hausdorff, there exist disjoint neighborhoods about these points (if there is one element of $\mathcal{O}_{\Xi}^{+}$ in $C_{0,i}$ we form a neighborhood about that point). As $C_{0,i}$ is connected, there exists a non-empty closed complement of these neighborhoods in $C_{0,i}$ whose boundary we denote by $B_{0,i}$. Note for each $C_{0,i}$, the corresponding $B_{0,i}$ is closed, has an empty interior, and and is disjoint with $\mathcal{O}_{\Xi}^{+}$.

Inductively define $C_{j+1} = \bigcup_{i \in I} C_{j+1,i}$ where $C_{j+1,i}$ are the nontrivial connected components of $X \backslash Y_{j}$, selecting for every connected component $C_{j,i}$ a set $B_{j,i}$ as described above and $Y_{j} = \bigcup_{0 \leq k \leq j} (\bigcup_{i \in I} \mathcal{O}_{B_{k,i}}^{-})$. Define $Y = \bigcup_{j \in \mathbb{N}} (\bigcup_{i \in I} \mathcal{O}_{B_{j,i}}^{-})$. Note that by construction $X \backslash Y$ is totally disconnected.

We will now show that $(X \backslash Y, T|_{X \backslash Y})$ is an invariant subsystem of (X,T). Assume there exists a point $x \in X \backslash Y$ such that $T(x) \in Y$. This implies that there exists a $k \in \mathbb{N}$ such that $T(x) \in Y_{k} = \bigcup_{0 \leq j \leq k} (\bigcup_{i \in I} \mathcal{O}_{B_{j,i}}^{-})$. Thus, there must exist an $a : 0 \leq a \leq k$ and some $b \in I$ such that $T(x) \in \mathcal{O}_{B_{a,b}}^{-} = \bigcup_{\beta \in B_{a,b}} (\bigcup_{n \in \mathbb{N}} \{p \in X : T^{n}(p) = \beta\})$. Thus, there must also exist a $\beta \in B_{a,b}$ and an $n \in \mathbb{N}$ such that $T(x) \in \{p \in X : T^{n}(p) = \beta\}$. This implies that $T^{n} \circ T(x) = \beta$ which in turn implies that $x \in \bigcup_{n \in \mathbb{N}} \{p \in X : T^{n}(p) = \beta\}$ and further that $x \in Y$. From this contradiction it follows that $(X \backslash Y, T|_{X \backslash Y})$ is an invariant subsystem.

We will now show that $X \backslash Y$ is dense in X. Given $Y = \bigcup_{j \in \mathbb{N}} (\bigcup_{i \in I} \mathcal{O}_{B_{j,i}}^{-})$, for all $j \in \mathbb{N}$ and $i \in I$, the set $B_{j,i}$ is closed, contains no elements of $\mathcal{O}_{\Xi}^{+}$, and $int(B_{j,i}) = \emptyset$, so by Lemma 1 the set $\mathcal{O}_{B_{j,i}}^{-}$ has an empty interior. By De Morgan's laws $X \backslash Y = \bigcap_{j \in \mathbb{N}} \left(\bigcap_{i \in I} X \backslash \mathcal{O}_{B_{j,i}}^{-} \right)$, and as the countable intersection of sets comeagre in X are comeagre, $X \backslash Y$ is comeagre and thereby dense in X. $\qquad \square$

Proposition 3. *Let (X, T) be a topological dynamical system. If X is a Baire space and T is countably injective, then $\Xi = \emptyset$.*

Proof Let $Z \subset X$ be such that $int(Z) = \emptyset$. Since T is countably injective, then the set $Y = T^{-1}(Z)$ can be represented by $\bigcup_{i \in \mathbb{N}} Y_i$ such that $Y_i \cap Y_j = \emptyset$ for all $i \neq j$ and for each $Y_i \subset Y$ there exists a set $Z_i \subset Z$ such that the restriction mapping $T|_{Y_i} : Y_i \to Z_i$ is a homeomorphism. Since each Z_i has an empty interior, then so does each Y_i, and as Y is the countable union of nowhere dense sets and X is a Baire space, then $Y = T^{-1}(Z)$ has an empty interior. We just showed that every set in X with empty interior has a preimage with an empty interior and it follows that $\Xi = \emptyset$. $\qquad\square$

Corollary 1. *If (X, T) is a topological dynamical system such that X is a compact metric space and T is ϵ- expansive, then there exists a set $Z \subset X$ such that Z is invariant on T, $h(T) = h(T|_Z)$ and $(Z, T|_Z)$ is weakly embeddable in a symbolic dynamical system.*

Proof For an $\epsilon > 0$, the system (X, T) is ϵ-expansive. Assume that $\Xi \neq \emptyset$. By Proposition 3 T is not countably injective, thus there exists a point $x \in X$ such that $card(T^{-1}(x)) = \infty$. Since X is compact there exists a finite covering of X by open balls of radius less than $\epsilon/2$. By the pigeon hole principle, there exists an open ball, B, such that for two points $p_0, p_1 \in B$ then $T(p_0) = x = T(p_1)$. Since $d(p_0, p_1) < \epsilon$ and $\mathcal{O}^+_{T(p_0)} = \mathcal{O}^+_{T(p_1)}$ then T is not ϵ-expansive. This contradiction proves that $\Xi = \emptyset$.

Since X is a compact metric space, it is Hausdorff, Baire, is totally bounded and has finite connected components. By Theorem 2 there exists a totally disconnected and dense subsystem, $(Z, T|_Z)$. By Theorem 1 the subsystem is of equal topological entropy. Since a totally disconnected metric space is strongly zero-dimensional, by Proposition 1 $(Z, T|_Z)$ is weakly embeddable in a symbolic dynamical system. $\qquad\square$

Example Consider the topological dynamical system (X, f) in which X is the closed interval $[-1, 2]$ of the real line and

$$f(x) = -x \left(1 - sin\frac{\pi}{2x} \right) \tag{1}$$

This function is countably injective, so by Proposition 3 the set $\Xi = \emptyset$. By Theorem 2, there exists a set $Y \subset X$ such that $X \backslash Y$ is totally disconnected, dense in X, and invariant on T. By Theorem 1, $h(f) = h(f|_{X \backslash Y})$. By Proposition 2, this subsystem can be weakly embedded into a product symbolic dynamical system.

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