Extensions of the Heisenberg Group

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Abstract

The Heisenberg group is a Lie group with applications in quantum physics and Fourier analysis. Therefore, it is important to understand its extension groups and how to construct them. This paper discusses central extensions of the Heisenberg group and ways of constructing them using 2-cocycles. It then goes on to consider extension groups of the groups resulted from central extensions of the Heisenberg group, on which not much has been written. This should serve as a helpful result in the field of representation theory.

Contents

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1 The Heisenberg Group

Definition 1.1. *The Heisenberg Group, often denoted by H_1, is the group of 3×3 matrices consisting of elements of the form*

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

in $\mathbb{R}^3$ which we will denote (x, y, z). Thus, the identity element e is $(0, 0, 0)$.

Then we can define

$$(x, y, z) \cdot (x', y', z') = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & z+z'+xy' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix}$$

which is written $(x + x', y + y', z + z' + xy')$. [?]

The Heisenberg group is also a Lie group, which means it is not only a group but with the topology inherited from $\mathbb{R}^3$, it becomes a smooth manifold. It has many applications for various subjects including quantum mechanics and Fourier analysis.

The Heisenberg Group can also be thought of in higher dimensions. The Heisenberg group of $2n + 1$ dimensions in $\mathbb{R}^{2n+1}$ is usually denoted H_n and is the group of matrices under matrix multiplication consisting of elements of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & I_n & y \\ 0 & 0 & 1 \end{pmatrix},$$

with x a row vector of length n, y a column vector of length n, and I_n the identity matrix of n dimensions.

Example 1.2. *Let $x, y, z \in \mathbb{R}$. H_2, the 5-dimensional Heisenberg group is the group of matrices with elements of the form*

$$\begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we will denote this element (x_1, x_2, y_1, y_2, z). *Then the identity element of* H_2 *is* $(0,0,0,0,0)$ *and* $(x_1, x_2, y_1, y_2, z)^{-1} = (-x_1, -x_2, -y_1, -y_2, x_1x_2 + y_1y_2 - z)$. *As for group multiplication we have*

$$\begin{aligned}
&(x_1, x_2, y_1, y_2, z)(x_1', x_2', y_1', y_2', z') \\
&= \begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1' & x_2' & z' \\ 0 & 1 & 0 & y_1' \\ 0 & 0 & 1 & y_2' \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x_1 + x_1' & x_2 + x_2' & z + z' + x_1y_1 + x_2y_2 \\ 0 & 1 & 0 & y_1 + y_1' \\ 0 & 0 & 1 & y_2 + y_2' \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= (x_1 + x_1', x_2 + x_2', y_1 + y_1', y_2 + y_2', z + z' + x_1y_1 + x_2y_2).
\end{aligned}$$

2 Group Extensions

Definition 2.1. *An extension of a group* H *by a group* N *is a group* G *with a normal subgroup* M *such that* $M \cong N$ *and* $G/M \cong H$. *This information can be encoded into a short exact sequence of groups*

$$1 \to N \to G \to H \to 1,$$

where $\alpha : N \to G$ *is injective and* $\beta : G \to H$ *is surjective. Also, the image of* α *is the kernel of* β. *[?]*

In other words, if G is an extension of H by N then N is a normal subgroup of G and the quotient group G/N is isomorphic to group H. [?]
Definition 2.2. *An extension is called a central extension if the normal subgroup* N *of* G *lies in the center of* G. *[?]*
Theorem 2.3. *It is explained in [?] that a second extension* $1 \to N \to G' \to H \to 1$ *is equivalent to* $1 \to N \to G \to H \to 1$ *if there is a map* $G \to G'$ *that makes the following diagram commute. Note that this map needs to be an isomorphism.*

$$\begin{tikzcd}
 & & G \arrow[dd] \arrow[dr] & & \\
1 \arrow[r] & N \arrow[ur] \arrow[dr] & & H \arrow[r] & 1 \\
 & & G' \arrow[ur] & &
\end{tikzcd}$$

Example 2.4. *Consider a group N consisting of elements of the form*

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then N is a subgroup of the Heisenberg group H_1 since

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}$$

which is clearly in N and

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & z+z' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix}$$

which is also in N. Furthermore, N is a normal subgroup of H_1 because

$$\begin{aligned}
&\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & z-ay \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in N.
\end{aligned}$$

Since $H_1/N \cong \{(x,0,0) \in H_1\}$, we can construct an exact sequence of groups.

$$e \to N \to H_1 \to \{(x,0,0)\} \to e$$

Example 2.5. *Now consider a group A in $\mathbb{R}^4$ with elements of the form (x,y,z,d). The group multiplication is defined by $(x,y,z,d)\cdot(x',y',z',d') = (e^{d'}x + x', e^{-d'}y + y', z + z' + e^{d'}(x \cdot y'), d + d')$. This operation is associative since*

$$\begin{aligned}
&((x,y,z,d)(f,g,h,k))(m,n,p,q) \\
&= (e^k x + f, e^{-q}y + g, z + h + e^k(x \cdot g), d + k)(m,n,p,q) \\
&= (e^q(e^k x + f) + m, e^{-k}(e^{-k}y + g) + n, z + h + e^k(x \cdot g) + p + e^q((e^k x + f)(n)), d + k + q) \\
&= (e^{k+q}x + e^q f + m, e^{-k-q}y + ge^{-q} + n, z + h + e^k xg + p + e^{k+q}xn + e^q fn, d + k + q) \\
&= (e^{k+q}x + d^q f + m, e^{-k-q}y + e^{-q}g + n, z + h + p + e^q(fn) + e^{k+q}(x(e^{-q}g + n)), d + k + q) \\
&= (x,y,z,d)(e^q f + m, e^{-q}g + n, h + p + e^q(fn), k + q) \\
&= (x,y,z,d)((f,g,h,k)(m,n,p,q)).
\end{aligned}$$

Let $(x,y,z,0) \in H_1$. Then H_1 is a subgroup of D because $(x,y,z,0)^{-1} = (-x,-y,-z+xy,0) \in D$ and

$$(x,y,z,0)\cdot(x',y',z',0) = (x + x', y + y', z + z' + xy', 0) \in D.$$

Then it is simple to show that the Heisenberg group H_1 is a normal subgroup of D and that $D/H_1 = \{(0,0,0,d) \in D\}$. Let $(x,y,z,d) \in D$ and $(a,b,c,0) \in H_1$. Then we have

$$\begin{aligned}
&(x,y,z,d)(a,b,c,0)(x,y,z,d)^{-1} \\
&= (e^0 x + a, e^0 y + b, z + c + e^0(x \cdot b)), d + 0)(-e^{-d}x, -e^d y, x \cdot y - d, -d) \\
&= (x + a, y + b, z + c + (x \cdot b), d)(-e^{-d}x, -e^d y, x \cdot y - d, -d) \\
&= (e^{-d}(x + a) - e^{-d}x, e^d(y + b) - e^d y, z + c + (x \cdot b) + x \cdot y - z + e^{-d}((x + a)(-e^d y)), d - d) \\
&= (e^{-d}a, e^d b, c + x \cdot b + x \cdot y + e^{-d}((x + a)(-e^d y)), 0),
\end{aligned}$$

which is an element of H_1. Therefore, we have the following exact sequence

$$e \to H_1 \to D \to \{(0,0,0,d)\} \to e.$$

3 2-Cocycles

Definition 3.1. *Let N be an abelian group. Given a group H, a 2-cocycle of H having values in N is a mapping ω from $H \times H$ to N satisfying the following cocycle identity*

$$\omega(r, s)\omega(rs, t) = \omega(s, t)\omega(r, st)\forall r, s, t \in H.$$

Often, it is further assumed that $\omega(1, s) = \omega(s, 1) = 1$ for all $s \in H$.

For a more precise definition and discussion refer to [?].

Proposition 3.2. *If we have a suitable 2-cocycle $\omega : H \times H \to N$, then we can use it to define a group $N \times H$ with group multiplication defined by*

$$(m, s) \times_\omega (n, t) = (mn\omega(s, t), st).$$

Proof. Let $(m, s), (n, t), (p, v) \in N \times H$ and assume ω satisfies the cocyle identity of Definition 3.1. Then we can verify associativity since

$$\begin{aligned}
&((m, s) \times_\omega (n, t)) \times_\omega (p, v) \\
&= (mn\omega(s, t), st) \times_\omega (p, v) \\
&= (pmn[\omega(s, t)\omega(st, v)], stv)
\end{aligned}$$

Using the fact that ω satisfies the cocycle indentity, we substitute $\omega(t, v)\omega(s, tu)$ for $\omega(s, t)\omega(st, y)$.

$$\begin{aligned}
&= (pmn[\omega(t, v)\omega(s, tv)], stv) \\
&= (m, s)(mp\omega(t, v), tv) \\
&= (m, s)((n, t)(p, v))
\end{aligned}$$

Notice that $(m, s) \times_\omega (1, 1) = (m1\omega(s, 1), s) = (m, s)$. Thus we have an identity element $e = (1, 1)$. We also have that $(m, s)^{-1} = (m^{-1}(\omega(s, s^{-1}))^{-1}, s^{-1})$. Hence, we have that a 2-cocycle satisfying the cocycle identity forms an associative multiplication on $N \times H$.

□

Definition 3.3. *A bilinear form is a map $B : V \times V \to F$ where V is a vector space and F a field with the following properties for all $u/v \in F$ and λ fixed in F.*

1. $B(u + u', v) = B(u, v) + B(u', v)$
2. $B(u, v + v') = B(u, v) + B(u, v')$

3. $B(\lambda u, v) = B(u, \lambda v) = \lambda B(u, v)$

Definition 3.4. *An alternating bilinear map is a bilinear map B such that $B(u, v) = -B(v, u)$ for all u, v in V.*

Definition 3.5. *For finite dimensions, a bilinear form is nondegenerate if and only if $B(u, v) = 0$ for all $v \in V$ implies that $u = 0$.*

2.1 of [**?**] states the following.

Proposition 3.6. *If α is a nondegenerate alternating $\mathbb{R}$-bilinear form $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ then the $(2n+1)$-dimensional Heisenberg group H_n^α fits into an exact sequence*

$$e \to \mathbb{R} \to H_n^\alpha \to \mathbb{R}^{2n} \to e$$

and is the set of pairs $(t, v) \in \mathbb{R} \times \mathbb{R}^{2n}$, with the group operation defined as

$$(t_1, v_1)(t_2, v_2) = (t_1 + t_2 + \alpha(v_1, v_2), v_1 + v_2).$$

Proof. Let $a, b, c \in \mathbb{R}^{2n}$. The map α is a bilinear form so it has the property that

$$\alpha(a + b, c) = \alpha(a, c) + \alpha(b, c) \text{ and } \alpha(a, b) + \alpha(a, c) = \alpha(a, b + c).$$

Adding the two equations together gives

$$\alpha(a + b, c) + \alpha(a, b) + \alpha(a, c) = \alpha(a, c) + \alpha(b, c) + \alpha(a, b + c).$$

Canceling $\alpha(a, c)$ on both sides, which is allowable since we are working in $\mathbb{R}$, gives

$$\alpha(a + b, c) + \alpha(a, b) = \alpha(b, c) + \alpha(a, b + c).$$

Thus, α satisfies the cocycle identity. Since α is alternating, $\alpha(a, 0) = \alpha(0, a) = 0$, which means it satisfies our cocycle identity. Therefore, by Proposition 2.2, α is a valid group operation on $\mathbb{R}^{2n} \times \mathbb{R}$.

□

Example 3.7. *Let $z \in \mathbb{R}$ and $v \in \mathbb{R}^2$ with $\alpha_1(v, v') = \alpha_1((x, y), (n, m)) = xm - yn$. Then α_1 is satisfies the criterion for bilinear forms since*

$$\begin{aligned}
&\alpha_1(((x, y) + (x', y')), (n, m)) \\
&= \alpha_1((x + x', y + y'), (n, m)) \\
&= (x + x')m - (y + y')n \\
&= xm - yn + x'm - ny' \\
&= \alpha_1((x, y), (n, m)) + \alpha_1((x', y'), (n, m))
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_1(((x,y)),(n,m)+(n',m'))\\
&=\alpha_1((x,y),(n+n',m+m'))\\
&=x(m+m')-y(n+n')\\
&=xm-yn+xm'-n'y\\
&=\alpha_1((x,y),(n,m))+\alpha_1((x,y),(n',m'))
\end{aligned}$$

and

$$\alpha_1((x,y),(0,0))=\alpha_1((0,0)(n,m))=0\cdot((x,y),(n,m)).$$

α_1 *is an alternating bilinear form because*

$$\begin{aligned}
&\alpha_1((x,y)(n,m))\\
&=xm-yn\\
&=-(ny-mx)\\
&=-\alpha_1((n,m)(x,y)).
\end{aligned}$$

And finally if $\alpha_1((x,y)(n,m))=0$ *for all (n,m) then* $(x,y)=0$*. Hence,* α_1 *is also a nondegenerate bilinear map.*

Therefore, by Proposition 3.6, we have a group multiplication on $\mathbb{R}^3$ *with identity element* $(0,0,0)$ *and* $(x,y,z)^{-1}=(-x,-y,-z)$ *with our group operation defined by*

$$\begin{aligned}
&(z_1,v_1)(z_2,v_2)\\
&=(z_1,(x_1,y_1))(z_2,(x_2,y_2))\\
&=(z_1+z_2+\alpha_2((x_1,y_1),(x_2,y_2)),(x_1,y_1)+(x_2,y_2))\\
&=(z_1+z_2+x_1y_2-x_2y_1,(x_1+x_2,y_1+y_2)),
\end{aligned}$$

which is associative since

$$\begin{aligned}
&((z_1,(x_1,y_1))(z_2,(x_2,y_2)))(z_3,(x_3,y_3))\\
&=(z_1+z_2+x_1y_2-x_2y_1,x_1+x_2,y_1+y_2)(z_3,(x_3,y_3))\\
&=(z_1+z_2+z+3+x_1y_2-y_1x_2+x_1y_3+x_2y_3-y_1x_3-y_2x_3,x_1+x_2+x_3,y_1+y_2+y_3)\\
&=(z_1,(x_1,y_1))(z_2+z_3+x_2y_3-x_3y_2,x_2+x_3,y_2+y_3)\\
&=(z_1,(x_1,y_1))((z_2,(x_2,y_2))(z_3,(x_3,y_3))).
\end{aligned}$$

In fact, this group, call it G, is isomorphic to H_1. Let $(z,x,y),(z',x',y') \in G$. Define a map $\phi : G \to H_1$ with $\phi((z,x,y)) = (z+xy, \sqrt{2}x, \sqrt{2}y)$. Then ϕ is a homomorphism because

$$\begin{aligned}
&\phi((z,x,y)(z',x',y')) \\
&= \phi(z+z'+xy'-yx', x+x', y+y') \\
&= (z+z'+xy'-yx'+xy+x'y+xy'+x'y', \sqrt{2}(x+x'), \sqrt{2}(y+y')) \\
&= (z+xy+z'+x'y'+2xy', \sqrt{2}(x+x'), \sqrt{2}(y+y')) \\
&= (z+xy, \sqrt{2}x, \sqrt{2}y)(z'+x'y', \sqrt{2}x', \sqrt{2}y') \\
&= \phi((z,x,y))\phi((z',x',y')).
\end{aligned}$$

Furthermore, ϕ is one-to-one because if we set $\phi(z,x,y) = \phi(c,a,b)$ then we have $(z+xy, \sqrt{2}x, \sqrt{2}y) = (c+ab, \sqrt{2}a, \sqrt{2}b)$, which implies that $(z,x,y) = (c,a,b)$. ϕ is also onto since for every $(z+xy, \sqrt{2}x, \sqrt{2}y) \in H_1$ there is an $(z,x,y) \in G$ such that $\phi(z,x,y) = (z+xy, \sqrt{2}x, \sqrt{2}y)$. Therefore, ϕ is an isomorphism as we wished to show.

Example 3.8. *Let $z \in \mathbb{R}$ and $v \in \mathbb{R}^2$ with $\alpha_2(v_1, v_2) = \alpha_2((a,b),(a',b')) = a \cdot b'$. Then we have a group call it A with group multiplication defined by*

$$\begin{aligned}
&(c, v_1)(c', v_2) \\
&= (c, (a,b))(c', (a',b')) \\
&= (c+c'+\alpha_2((a,b),(a',b')), (a,b)+(a',b')) \\
&= (c+c'+a \cdot b', (a+a', b+b')).
\end{aligned}$$

Then we have identity element $(0,0,0)$ and $(x,y,z)^{-1} = (-x,-y,xy-z)$.

This group operation is the same as the matrix multiplication of matrices in the Heisenberg group so A and the Heisenberg group are isomorphic.

Theorem 6.7 of [?] states the following.

Theorem 3.9. *Every 2-cocycle on the Heisenberg Group can be written in the form*

$$\lambda_1(x^2y' + 2xz') + \lambda_2(y^2x' - x(y')^2 + 4yx'y' - 2xyy' - 6yz')$$

for fixed $\lambda_1, \lambda_2 \in \mathbb{R}$. [?]

We do not give a proof of this theorem but will investigate a special case.

Example 3.10. *Let H a group and N an abelian group. Using the equation in Theorem 2.9, set $\lambda_1 = 2$ and $\lambda_2 = 3$. Then define the map $\beta : H \times H \to N$ with $\beta((x,y,z)(x',y',z')) = 2(x^2y' + 2xz') + 3(y^2x' - x(y')^2 + 4yx'y' - 2xyy' - 6yz')$.*

$$
\begin{aligned}
&\beta((a,b,c),(g,h,k)) + \beta((a+g,b+h,c+k),(x,y,z))\\
&= 2(a^2h+2ak)+3(b^2g-ah^2+4bgh-2abh-6bk)+2((a+g)^2y+2(a+g)z)+3((b+h)^2x\\
&\quad - (a+g)y^2 + 4(b+h)xy - 2(a+g)(b+h)y - 6(b+h)z)\\
&= 2a^2h+4ak+3b^2g-3ah^2+12bgh-6abh-18bk+2(a+g)^2y+4(a+g)z+3(b+h)^2x\\
&\quad - 3(a+g)y^2 + 12(b+h)xy - 6(a+g)(b+h)y - 18(b+h)z\\
&= 2a^2h + 4ak + 3b^2g - 3ah^2 + 12bgh - 6abh - 18bk + 2(a^2 + 2ag + g)y + 4(a+g)z\\
&\quad + 3(b^2 + 2bh + h^2)x - 3(a+g)y^2 + 12(b+h)xy - 6(a+g)(b+h)y - 18(b+h)z\\
&= 2g^2y+4gz+3h^2x-3gy^2+12hxy-6ghy-18hz+2a^2h+2a^2y+4ak+4az+3b^2g+3b^2x\\
&\quad - 3ah^2 - 6ahy - 3ay^2 + 12bgh + 12bhx + 12bgy + 12bxy - 6abh - 6aby - 18bk - 18bz\\
&= 2g^2y+4gz+3h^2x-3gy^2+12hxy-6ghy-18hz+2a^2(h+y)+4a(k+z)+3b^2(g+x)\\
&\quad - 3a(h+y)^2 + 12b(g+x)(h+y) - 6ab(h+y) - 18b(k+z)\\
&= 2(g^2y+2gz)+3(h^2x-gy^2+4hxy-2ghy-6hz)+2(a^2(h+y)+2a(k+z)+3(b^2(g+x)\\
&\quad - a(h+y)^2 + 4b(g+x)(h+y) - 2ab(h+y) - 6b(k+z)\\
&= \beta((g,h,k),(x,y,z)) + \beta((a,b,c),(g+x,h+y,k+z))
\end{aligned}
$$

Since it can be shown that β is a 2-cocycle satisfying the cocycle identity, Proposision 2.2 gives that there is a group operation on $H \times N$. Let $x, y, z, d \in H_1 \times N$. Then $H_1 \times N$ has a group operation defined by

$$
\begin{aligned}
&(x,y,z,d)(x',y',z',d')\\
&= (x+x', y+y', z+z'+xy', d+d'+\beta((x,y,z)(x',y',z')))\\
&= (x+x', y+y', z+z'+xy',\\
&d + d' + 2(x^2y' + 2xz') + 3(y^2x' - x(y')^2 + 4yx'y' - 2xyy' - 6yz')),
\end{aligned}
$$

with identity element $(0,0,0,0)$ and $(x,y,z,d)^{-1} = (-x,-y,-z+xy,-d-2(x^2y - 2xz + 9yz - 3xy^2))$.

The following theorem is arguably DuPre's most important result.

Theorem 3.11. *Any two non-trivial, one-dimensional central extension of the Heisenberg group H_1 yield isomorphic groups.*

4 The Center of Heisenberg Extensions

Example 4.1. *Let's examine the center of the group E from Example 2.10 denoted by $Z(E)$. Let $(x_0, y_0, z_0, d_0) \in Z(E)$. Then for all $x, y, z, d \in \mathbb{R}$ we have*

$$\begin{aligned}
&(x, y, z, d)(x_0, y_0, z_0, d_0) = (x_0, y_0, z_0, d_0)(x, y, z, d)\\
&\Rightarrow (x + x_0, y + y_0, z + z_0 + xy_0, d + d_0 + \beta((x, y, z)(x_0, y_0, z_0)))\\
&= (x_0 + x, y_0 + y, z_0 + z + x_0 y, d_0 + d + \beta((x_0, y_0, z_0)(x, y, z)))\\
&\Rightarrow xy_0 = x_0 y\\
&\Rightarrow (x_0, y_0) = (0, 0).
\end{aligned}$$

So we also have that

$$\begin{aligned}
&\beta((x, y, z)(x_0, y_0, z_0)) = \beta((x_0, y_0, z_0)(x, y, z))\\
&\Rightarrow 2(x^2 y_0 + 2xz_0) + 3(y^2 x_0 - x(y_0)^2 + 4yx_0 y_0 - 2xyy_0 - 6yz_0)\\
&= 2(x^2 y_0 + 2xz_0) + 3(y^2 x_0 - x(y_0)^2 + 4yx_0 y_0 - 2xyy_0 - 6yz_0)\\
&\Rightarrow 4xz_0 - 18yz_0 = 0\\
&\Rightarrow z_0 = 0.
\end{aligned}$$

Therefore, the center of E is everything of the form $(0, 0, 0, d)$ for all $d \in \mathbb{R}$. This makes sense because we are working under the assumption that E is a central extension.

Proposition 4.2. *The center of every 2-cocycle extension of the Heisenberg group is one-dimensional.*

Proof. We know from Theorem 2.9 that every cocycle of the Heisenberg group is consisting of elements of the form $\lambda_1(x^2y' + 2xz') + \lambda_2(y^2x' - x(y')^2 + 4yx'y' - 2xyy' - 6yz')$. This means that every extension is of the form

$$(x, y, z, d)(x', y', z', d') =$$
$$(x+x', y+y', z+z'+xy', d+d'+\lambda_1(x^2y'+2xz')+\lambda_2(y^2x'-x(y')^2+4yx'y'-2xyy'-6yz')).$$

Let (x, y, z, d) be an element of an arbitrary extension of the Heisenberg group denoted by D. Let let $(x_0, y_0, z_0, d_0) \in Z(D)$. Then for every $x, y, z, d \in \mathbb{R}$ we have

$$
\begin{aligned}
&(x, y, z, d)(x_0, y_0, z_0, d_0) = (x_0, y_0, z_0, d_0)(x, y, z, d) \\
&\Rightarrow (x + x_0, y + y_0, z + z_0 + xy_0, d + d_0 + \beta((x, y, z)(x_0, y_0, z_0))) \\
&= (x_0 + x, y_0 + y, z_0 + z + x_0 y, d_0 + d + \beta((x_0, y_0, z_0)(x, y, z))) \\
&\Rightarrow xy_0 = x_0 y \\
&\Rightarrow (x_0, y_0) = (0, 0).
\end{aligned}
$$

So we also have that

$$
\begin{aligned}
&\beta((x, y, z)(x_0, y_0, z_0)) = \beta((x_0, y_0, z_0)(x, y, z)) \\
&\Rightarrow 2(x^2 y_0 + 2xz_0) + 3(y^2 x_0 - x(y_0)^2 + 4yx_0 y_0 - 2xyy_0 - 6yz_0) \\
&= 2(x^2 y_0 + 2xz_0) + 3(y^2 x_0 - x(y_0)^2 + 4yx_0 y_0 - 2xyy_0 - 6yz_0) \\
&\Rightarrow 4xz_0 - 18yz_0 = 0
\end{aligned}
$$

Then whenever $(\lambda_1, \lambda_2) \neq (0, 0)$ (i.e. whenever the cocycle is non-trivial), we have that z_0 must be 0. Therefore, the center of every extension is one-dimensional as desired. $\square$

Lemma 9.6 and its proof from [?] states the following.

Proposition 4.3. *If $n \geq 2$ and $1 \to \mathbb{R} \to E \to H_n \to 1$ is a central extension, then the center of E has at least dimension 2.*

Proof. Let β be the alternating form defining the extension. Since $n \geq 2$, β is defined modulo the center of H_n (i.e. it is defined on $H_n/\mathbb{R} \cong \mathbb{R}^{2n}$). Since we may write the group multiplication in E as

$$(s, t, y)(s', t', y') = (s + s' + \beta(y, y'), t + t' + \alpha(y, y'), y + y'),$$

we see that

$$(s, t, 0)(s', t', y') = (s + s', t + t', y') = (s', t', y')(s, t, 0).$$

$\square$

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