Harmonic Maps on Cayley Graphs and Compactification

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Abstract

We introduce the analogue of harmonic maps on graphs and give some parallel results from Complex Analysis. On Cayley graphs, we compare general harmonic maps to those induced by group homomorphisms. In particular, we prove that the free group and abelian group on more than one generator admit harmonic functions that are not induced by homomorphisms. Furthermore, we introduce the Floyd compactification of a geodesic space, particularly that of a graph. Our goal is to compare the Floyd boundary to other graph boundaries up to homeomorphism. In this context, the Dirichlet Problem naturally arises. It asks for conditions for extending a continuous function on the boundary to a harmonic map on the entire space. The results of Anders Karlsson state that the solvability of the Dirichlet problem depends on the size of the Floyd boundary. We examine the connections of this result to properties of the differentials of harmonic maps and give applications to the hyperbolic groups of M. Gromov.

1 Introduction

The subject of harmonic maps appears in many physical contexts, ranging from the heat equation to Electrostatics. The main motivation for studying harmonic maps on graphs arises from problems involving the analogue of the heat equation on graphs, particularly, the heat kernel. While our results do not directly address the heat kernel, they do make new connections between harmonic maps and algebraic group properties, useful in the context of Cayley

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Graphs. Harmonic maps also appear in Complex Analysis in conjunction with the Cauchy-Riemann Equations. An expository paper by Lovasz extends a few of these results to the complex lattice graph of the Gaussian Integers. Our results extend the discussion to include Louivelle's theorem along and consider graphs different from the integral lattice or four-tree.

Any discussion of Harmonic Maps merits at least a brief reference to the Dirichlet Problem. Since harmonic maps can only be globally defined on infinite graphs, however, we need to introduce compactification in order to properly consider the Dirichlet Problem at Infinity. A few methods of compactification are discussed in papers of Anders Karlsson, in particular Floyd compactification. We have explicitly calculated the Floyd Boundary for two different Cayley Graphs in order to observe their behavior and potential connections with group theory. Another article summarizes two other boundary types, the Gromov and Ideal, in addition to conditions for comparing them up to homeomorphism. We tie harmonic maps into the discussion of boundary classes through the definition of harmonic isometries.

1.1 Basic Definitions and Results

Given a graph G, a function $f : V(G) \to C$ is called *harmonic* at vertex v if

$\sum_{w \sim v} f(w) - f(v) = 0.$

Note that such a function is a solution of Laplace's equation

$\Delta f = 0$, where $\Delta = D - A$, with D being the Degree matrix and A being the adjancency matrix, is the combinatorial laplacian.

In the case where we also have a weights $w : E(X) \to \mathbb{R}$ with $w(u, v) > 0$, for all $u, v \in V(X)$ the condition that a map be harmonic becomes $\sum_{u \sim v} \frac{f(u)-f(v)}{w(u,v)} = 0 \Leftrightarrow w(u, v)$ Note that this is equivalent to the laplacian simply changing to $\Delta f = W - I$, where the matrix W is given by $W(u, v) = \frac{1}{y(v)w(u,v)}$, when $u \sim v$ and zero otherwise. where $y(v) = \sum_{u \sim v} \frac{1}{w(u,v)}$.

Both of these are special cases of $P - harmonic$ maps, which are defined as maps the operator P, which acts on a map $f : V(X) \to \mathbb{C}$ with $Pf(v) = \sum_{u \sim v} P(u, v)f(v)$, leaves invariant. Obviously, in the weighted case, $P = W$ and in the Combinatorial case these are just special cases of the weighted case with $w(u, v) = 1$ for $u \sim v$. Also, in both of the above cases we have that, for any $v \in V(X)$, $\sum_{u \sim v} P(u, v) = 1$. For this reason, we may call P a *stochastic transition operator* or Markov Chain with probabilities that are associated to a random walk.

We say that f is a harmonic *at a point* $v \in X$ if $Pf(v) = f(v)$. If f is not harmonic at the vertex v, then f has a *pole* at vertex v.

Proposition 1. *(Lovasz, proposition 3.1 [1, p. 5]) Every complex valued non-constant function on a finite graph has at least two poles.*

However, this is not true for infinite graphs. As a counter - example, consider the infinite line lattice, G, with its vertices identified with $\mathbb{Z}$, and define the function $f : G \to \mathbb{Q},\ f(x) := \begin{cases} x/2 & \text{if } x \geq 0 \\ -x/2 & \text{if } x < \ 0 \end{cases}.$

Then f is harmonic on the entire line and contains no poles.

The result also need not hold for harmonic maps that have their values in a ring other than $\mathbb{Z}$. In particular, the homomorphisms $f : X = Cay(C_n, \{s, s^{-1})\} \to \mathbb{Z}/n\mathbb{Z}$ induce harmonic maps on X that have no poles, as we shall see in the next section.

A further result states that

Proposition 2. *(Lovasz, Proposition 3.2 [1, p.5]) For any set $S_0 \subseteq X$, every complex-valued function $f_0 \to \mathbb{C}$ has a unique extension to a $f : X \to \mathbb{C}$ so that f is harmonic on $X - S_0$.*

In [1] Lovasz gives examples of various contructions, particularly one involving random walks that easily be extended to the infinite case. It follows that, unlike propostion 1, proposition 2 does hold for infinite graphs, provided the starting set S_0 is finite.

Another interesting analogue of this in the infinite case would be *whether each f_0 defined on the a set $S_0 \subseteq X$ that is also harmonic at every node $u \in X$ with $N(u) \subset S_0 \neq \emptyset$ has a (unique) extension to X so that it is harmonic everywhere.*

Uniqueness, in fact, cannot hold for all graphs, since for any finite subset, A of $Cay(F_2 : S)$ where $S = \{a^{\pm 1}, b^{\pm 1}\}$ is the standard generating set, we can always pick a variety of different values of f outside of S so as to keep the function harmonic at every $u \in A$ with $N(u) - A \neq \emptyset$ and all $u \in X - A$. Also, consider the infinite path or $Cay(\mathbb{Z} : \{\pm 1\}$. A harmonic map on this graph completely determined by its values at 0 and 1. Thus, a map on the set consisting only of 0 or only of 1 does not have a unique extension to the entire line.

1.2 Liouville's Theorem

Another interesting result for harmonic maps in the continuous plane that fails to hold in the discrete case of infinite graphs is Liouville's Theorem, which claims that any harmonic map bounded from above or below on the entire space must be constant [7].

As a counterexample, consider the the three regular tree, and let the graph X be the three regular graph with center z. Set $f(z) = 1$, and, letting z_1, z_2, and z_3 denote the three vertices adjancent vertices, set $f(z_1) = 1$, $f(z_2) = 1/2$, and $f(z_3) = 3/2$. Now, set all the values of f on the vertices on the branch stemming from z_1 to 1. On the branch stemming from z_2, set the values of f on each consecutive pair of vertices equal to $\frac{1}{2^n}$, where n increases only when the vertices move further away from the center. Finally, on the branch stemming from z_3 set the values on on each consecutive pair of vertices equal to $\frac{a_n}{2^n}$ where a_n is a sequence defined iteratively by $a_n = 2(a_{n-1}) + 1$ and $a_1 = 3$. Then $f(x) \geq 0$ for all $x \in X$, so that f is bounded from below, but f is still a nonconstant harmonic map.

Proposition 3. *For any infinite $k-regular$ tree, i.e. the Cayley graph of the free group F_k on k - generators, Liouville's theorem does not hold, and we can construct a harmonic map with a lower bound of zero using a technique similar to the one above.*

Note: this example *does not* contradict the Minimum principle in [9, p.5] since this f, while being harmonic, and therefore, superharmonic, it does not attain its minimum on X.

Also, note that Louville's theorem *does not* hold if we simply attach a finite ray perpendicular to the line, since we may choose the function whose values are all 1 on the right side of this line, all 0 values on the perpendicular line - segment, and the consecutive integers exceeding 1 on the left.

However, one can show that Liouville's theorem holds for $Cay(\mathbb{Z} : \pm 1)$.

1.3 Liftings

If $f : \Gamma \to (R, +)$ is a group homomorphism, and $\phi : Cay(\Gamma)/(kerf) \to (R, +)$, a harmonic function, then ϕ lifts to another harmonic function $\hat{\phi} : Cay(\Gamma) \to (R, +)$. This follows from the fact that there is a bijection between $N(\gamma)$ and $N(\gamma(\ker f))$, where $N(\gamma)$ denotes the neighborhood of γ for any $\gamma \in \Gamma$.

However, the problem of finding a harmonic function $\phi : Cay(\Gamma) \to (R, +)$ given a harmonic map $\hat{\phi} : Cay(\Gamma/(kerf)) \to (R, +)$ is virtually impossible.

The above is a special case of the much more general result:

Proposition 4. *Let $\tilde{X}$ be a cover of the graph X. Given a a harmonic function $f : X \to (R, +)$ we can define a lifting $\tilde{f} : \tilde{X} \to (R, +)$ that is harmonic.*

Proof. Let $p : \tilde{X} \to X$ be the covering map of X. Define $\tilde{f}$ as $\tilde{f} = f \circ p$.

We know that $\sum_{y \in N(x)} f(y) - f(x) = 0$ $for\ all$ $x \in X$. Also, since p is a covering, for each $x \in X$, $y \in N(x) \Rightarrow p(y) \in N(p(x))$ or $p(y) = x$ and furthermore, $p(N(x)) = N(p(x))$.

Therefore,

$\sum_{y \in N(x)} f(p(y)) - f(p(x)) = \sum_{z \in N(p(x))} f(z) - f(p(x)) = 0$,

so that $\tilde{f}$ is harmonic, as desired. □

While trying to define a harmonic map from G to $(R, +)$ given a harmonic function on $V(\tilde{G})$ is generally difficult. A specific case of complex-valued functions makes the problem pretty simple to solve:

Throughout, assume that the cover, p has finite, fixed index m :

$|p^{-1}(v)||N(x) \cap p^{-1}(u)| = m$. for all $u, v \in X$, $x \in p^{-1}(v)$.

Lemma 5. *Suppose that $x \in p^{-1}(v)$ Then $d_x = \frac{m}{|p^{-1}(v)|} d_v$*

Proof.

$$\begin{array}{rcl} d_x & = & |N(x)| \\ & = & \sum_{u\sim v} |N_u(x)| \\ & = & \sum_{u\sim v} |p^{-1}(u)| \\ & = & \sum_{u\sim v} \frac{m}{|p^{-1}(v)|} \\ & = & \frac{m}{|p^{-1}(v)|} |N(v)| \\ & = & \frac{m}{|p^{-1}(v)|} d_v \end{array}$$

□

Proposition 6. *Suppose that $|p^{-1}(v)|$ is finite for all v. If $\tilde{f}$ is a harmonic map from $\tilde{X}$ into $\mathbb{C}$, then $\tilde{f}$ descends to a map $f : X \to \mathbb{C}$ that is harmonic.*

Proof. Following the guidelines of Chung and Yau [10, p. 6], define $f = (p\tilde{f}) : V(G) \to (R, +)$ by the following: $(p\tilde{f})(v) = \sum_{x\in p^{-1}(v)} (f(x)\frac{d_x}{d_v})$

Then

$$\begin{array}{rcl} \sum_{u\sim v} f(u) - f(v) & = & \sum_{u\sim v} (\sum_{y\in p^{-1}(u)} \tilde{f}(y)\tilde{f}\frac{d_y}{d_u}) - d_v(p\tilde{f})(v) \\ & = & \sum_{y\in V(\tilde{G})} \tilde{f}(y)\frac{d_y}{d_{p(y)}} - d_v(p\tilde{f})(v) \\ & = & \sum_{y\in V(\tilde{G})} \tilde{f}(y)\frac{m}{|p^{-1}(u)| d_{p(y)}} d_u - d_v(p\tilde{f})(v) \\ & = & \sum_{y\in V(\tilde{G})} \tilde{f}(y)\frac{m}{|p^{-1}(u)|} - \sum_{x\in p^{-1}(v)} \tilde{f}(x)d_x \\ & = & \sum_{y\in V(\tilde{G})} \tilde{f}(y)|N(y)\cap p^{-1}(v)| - \sum_{x\in p^{-1}(v)} \tilde{f}(x)d_x \\ & = & \sum_{y\sim x}\sum_{z\in p^{-1}(v)} \tilde{f}(y) - \sum_{x\in p^{-1}(v)} \tilde{f}(x)d_x \\ & = & \sum_{x\in p^{-1}(v)} (\sum_{y\sim x} \tilde{f}(y) - d_x\tilde{f}(x)) \\ & = & \sum_{x\in p^{-1}(v)} (\sum_{y\sim x} (\tilde{f}(y) - \tilde{f}(x))) = 0 \end{array},$$

so that, indeed f is harmonic at every $v \in V(G)$. □

We also have the result from Chung and Yau [10] that, provided the covering p has index m, with the nonweighted cased corresponding to $w(u, v) = 1$ (and dealing with a complex-valued map).

Theorem 7. *(1) Any eigenvalue of G is also an eigenvalue of $\tilde{G}$*

(2) An eigenvalue of the Combinatorial Laplacian $\tilde{G}$ corresponding to an eigenmap with nontrivial image in G. Then is also an eigenvalue of G.

(3) If the covering is strong regular, then $Spec_\Delta(\tilde{G}) = Spec_\Delta G$, where the set $Spec_\Delta(X)$ is defined as the spectrum of the Combinatorial Laplacian Δ on the graph X

1.4 Harmonic Maps on Cayley Graphs and Homomorphisms

Let R be a ring. If $f : \Gamma \to (R, +)$ is a homomorphism on a finitely generated group Γ with generating set S, then it naturally induces to a function, also denoted by f, on $V(Cay(G : S))$, as each element of Γ is also a vertex.

Fact 8. *If the order of every generator is greater than two, the function so induced is harmonic on at every vertex.*

Proof. By definition, the neighbors of a vertex $x \in G$ of the Cayley Graph are $xs,\ xs^{-1},\ where\ s \in S$, and since f is a homomorphism

$f(xs) - f(x) + f(xs^{-1}) - f(x) = f(s) - f(x) + f(x) - f(xs^{-1}) - f(x) = f(s) + f(s^{-1}) = f(s) - f(s) = 0$, so that, indeed, f is a harmonic: $\sum_{y \sim x} f(y) - f(x) = 0$. ☐

An example where $|s| \leq 2$ for some $s \in S$ and a group homomorphism does not induce a harmonic map on the Cayley graph given as follows:

Let $\Gamma = \mathbb{Z}/(2\mathbb{Z}) \oplus \mathbb{Z}/(2\mathbb{Z}) \oplus \mathbb{Z}/(2\mathbb{Z})$ and define a group homomorphism, $\phi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ by $\phi(x, y, z) = z$. Then ϕ induces a map on $Cay(\Gamma : S)$ where $S = \{(0,0,1),(0,1,0),(1,0,0)\})$ that is not harmonic at $(0,0,0)$ since $\sum_{s \in S} f(s) = 1 \neq 0$.

In fact, the condition that a homomorphism be harmonic when this is the case is the following:

$\sum_{|s|=2} f(s) = 0$.

Fact 9. *Conversely if f is a harmonic function on a Cayley graph with a symmetric generating set S, then it is a homomorphism if $\sum_{s \neq s_i, s \in S} f(xs) - f(x) = f(s) = -f(s_i)$ $for\ all\ s_i \in S\ and\ all\ x \in \Gamma = V(G)$.*

Proof. In this case, the fact that f is harmonic means that $0 = \sum_{s \in S} f(xs) - f(x) = f(xs_i) - f(x) - f(s_i) \Rightarrow f(xs_i) = f(x) + f(s_i) \forall s_i \in S\ and\ x \in \Gamma$.

Thus, since every $\gamma \in \Gamma$ can be written as $xs,\ for\ some\ s \in S,\ f$ must be a homomorphism. ☐

2 Harmonic Maps that are induced by Homomorphisms

2.1 Indtroduction

Let R be a ring such that, when viewed as a group under addition, there is an element $r \in R$ whose order is greater than 2.

Analyzing which harmonic functions on a Cayley graph are induced by group homomorphisms to the ring can provide insight into the algebra of the group and its generators.

I will begin with a lemma about abelian groups.

Lemma 10. *If the Cayley graph of a finitely generated abelian group G_2 on its standard generators admits a harmonic map f, we can extend f to any abelian group G_n containing G_2 as a subgroup.*

Proof. If $|S| > 4$, let $C \subset S$ such that C is symmetric and $|C| = 4$. I will denote by $< C >$ the subgroup of Γ generated by "C." Let $A =< s \in S - C >$, that is, the group generated by all the generators of Γ not contained in C. Since the group is abelian, C is normal, and thus, $\Gamma- < C >= \coprod_{a \in A, a \neq 1} a < C >$, so that any element $\gamma \in \Gamma- < C >$ may be written as $\gamma = \; ac$ for a "unique" pair $a \in A$. and $c \in < C >$.

By assumption, we know that f extends to a harmonic function on the free subgroup of Γ generated by C. I will then define $f(ac) = f(c)$. This function is well-defined because, due to the disjoint union of $ac = a'c'$ if and only if $a = a' \Rightarrow c = c'$.

Now we need only check that the function so defined is harmonic. Indeed, for each $\gamma = ac \in \Gamma- < C >$,

$\sum_{s \in S} f(\gamma s) - f(\gamma)$
$= \sum_{s \in S} f((ac)s) - f(ac)$
$= \sum_{s \in S} f(a(cs)) - f(c)$
$= \sum_{s \in C} f(cs) - f(c) + \sum_{s \notin C} f((as)c) - f(c)$

But, f is harmonic at c so that

$\sum_{s \in C} f(cs) - f(c) = 0$

Thus, the sum reduces to

$\sum_{s \notin C} f((as)c) - f(c)$.

Now, since $as \in A$ is one of the coset representatives, $f((as)c) = \; f(c)$, so that this last sum vanishes.

If $\gamma \in < C >$, then for every $s \in S - C$, $f(\gamma s) = f(\gamma)$, so that

$\sum_{s \in S} f(\gamma s) - f(\gamma) = \sum_{s \in C} f(\gamma s) - f(\gamma) = 0$ *(by assumption)* □

2.2 Free Groups and Abelian Groups

To make calculations more concise, I will define $n * *q = q + q + ...(n - times)$ and $(-n) * *q = -q - q - q...(n - times)$ for any $n \in \mathbb{N}$. First, notice that, if, $\Gamma = \mathbb{Z}$, and $S = \left\{s, s^{-1}\right\}$, it is easily seen that $Cay(\mathbb{Z} : S)$ then, for any harmonic function with $f(1) = 0$, and $f(s) = x \neq 0$, we must have $f(s^{-1}) = -f(s) = -x$, $f(s^k) = k * *f(s) = k * *x(k > 0)$, and $f(s^{-k}) = k * *f(s^{-k}) = (-k) * *f(s) = (-k) * *x$, $(k > 0)$ so that, indeed f is a homomorphism.

Also, the function defined by $g(s^x) = x * *q \; (x \neq 0) \; and \; g(1) = 0$, where $q \in R$ denotes the element of order infinity, satisfies the conditions for the desired non-constant harmonic function that also defines a homomorphism and has $\ker g = 1$..

Lemma 11. *A free group or abelian group on more than one generator admits a map that is harmonic at every generator and defined at every element of word length not exceeding 2.*

Proof. Pick a generator $a \in S$ and define a function f by $f(1) = f(a^2) = 0$, $f(a) = r$. Obviously f is not a homomorphism, for $f(a) + f(a) = 2 * *r \neq f(a^2) = 0$, and f is non-constant as $r \neq 0$. I will now extend f to a harmonic function on the generators keeping its values the same at 1, a, $and \; a^2$.

Let four of the generators be a, a^{-1}, b, $and \; b^{-1}$, let $k = |S|$, and denote the group so generated by $< S >= \Gamma$.

Defining,

$f(s) = f(as) = f(a^{-1}s) = f(bs) = f(b^{-1}s) = f(s) = f(s_1 s_2) = 0$ for $s \neq a,\ b,\ a^{-1},\ or\ b^{-1}$, and $s_1,\ s_2$ are not simultaneously in the set $\{a, b, a^{-1}, b^{-1}\}$,
$f(b^{-1}) = f(a^{-1}) = f(ab^{-1}) = f(a^{-1}b) = f(a^{-2}) = f(a^{-1}b^{-1}) = 0$ $f(b^{-1}a^{-1}) = f(ba^{-1}) = f(a^{-1}b) = f(b^{-1}a) = f(ab^{-1}) = f(b^{-2}) = 0$ $f(b) = -r,\ f(ab) = k**r, f(a^{-1}b) = (-k)**r,\ and\ f(b^2) = (-2k)**r$

the function is harmonic at the identity and at every generator, but *defined* on words of length 2 and is symmetric, so that it is consistent with the abelian case.

□

2.3 Main Theorem

Suppose Γ is an *infinite* finitely generated free group or abelian group with standard symmetric generating set S. That is, $S = \{x_i, x^{-1}\}$ with absolutely no relations on the x_i or $S = \{x_i, x^{-1}\}$ with any relations so long as they include $x_i x_j = x_j x_i$. Let X denote the Cayley graph of $\Gamma : S$ graph, $X = Cay(\Gamma : S)$, and assume that it admits a harmonic function, i.e. a function that is harmonic at a every node, that vanishes at the identity. Also, let R be a ring such that, when viewed as a group under addition, there is an element $r \in R$ whose order is greater than 2.

Theorem 12. *If every non-constant harmonic function $f : Cay(\Gamma : S) \to (R, +)$ (where R is a ring with addition operation $+$) with $f(1) = 0$ is induced by a homomorphism $f : \Gamma \to (R, +)$, then $|S| = 2$, so that $\Gamma = \mathbb{Z}$. Moreover, if the ring contains an element of infinite order, then there exists a non-constant homomorphism $g : \Gamma \to (R, +)$ such that $\ker g = 1$. Note that, according to the above introductory section, g also induces a harmonic function on the Cayley graph.*

The last statement follows from the properties of $\mathbb{Z}$ shown in the last section. We need only prove that if the Cayley graph satisfies the hypothesis of the theorem, then $|S| = 2,\ i.e.\ S = \{s, s^{-1}\}$.

Proof. I will proceed with a proof by contrapositive. Suppose $|S| > 2$. I will extend the function introduced in Lemma 5 to the entire group in each of the following two cases:

Case 1: The group is free (non - abelian)

I will prove that the function can be extended to the whole group by induction on the word length in Γ. That is, I will show that, if f is harmonic at every node of a particular length n and defined on words of length $n+1$, it can be extended so that it is harmonic on every node of length $n+1$ (and defined at every node of length $n+2$). All the words of length one are just the generators and the words of length two are just a^2, a^{-2}, b^2, b^{-2}, ab, ba, ab^{-1}, $a^{-1}b$, $a^{-1}b^{-1}$, $b^{-1}a^{-1}$, and $s_2 s_3$ (where s_2, s_3 are not simultaneously in the set a, b, a^{-1}, b^{-1}).

Note that we have already defined the values of the function f at all of the nodes just listed such that f is harmonic at each of the generators and is symmetric about the generators, so that it is consistent in the case when the generators commute. Thus, the case works for $n = 1$.

Now, assume that the hypothesis is true for some $n \geq 1$. Then we need to find the values of $f(\gamma)$ for $|\gamma| = n + 2$, so that f is harmonic for each node of length $n + 1$. The condition that f be harmonic at x with $|x| = n + 1$ is

$\sum_{s\in S} f(xs) - f(s) = k * *f(x)$

Since the group is free (*not* free abelian), only one of each of the values, xs will reduce to a word of length n. For the sake of brevity, I will denote this word by y. Moving $f(y)$ to the right hand side of the equation we obtain $\sum_{s\neq x^{-1}y,\ s\in S} f(xs) = (\sum_{j=1}^{k-1} f(s_j)+f(s_j^{-1}))+f(s_2) = (4{*}{*}f(x))-f(y)$ (where $\{s_j\}$ is a rearrangement of the generators and $s_j \neq s_i$). Since $|y| = n,\ |x| = n+1$, the right hand side has definite, defined value in the ring R. As Γ is free, given any two $x,\ x' \in \Gamma$ such that $|x| = |x'| = n + 1$, and $s,\ s' \in S$. such that xs does not reduce to a shorter word length, $xs = x's'$ if and only if $x = x'$ and $s = s'$. Thus, in each equation, we may set $f(s_j) = 0$ for each j, leaving $f(s_2) = 4 * *f(x) - f(y)$.

Since every word of length $n + 2$ may be written as xs where $|x| = n + 1$ and $s \in S$, we have just extended the domain of the function to every word of length $n + 2$ so that f is harmonic at every word of the length $n + 1$. The result thus follows by induction.

Case 2: The Group is Abelian

From lemma 4, it suffices to show that the extension is harmonic only for the case when $|S| = 4$.

Every element of $< a,\ b,\ a^{-1},\ b^{-1} >$ can be written as $a^n b^m$ for $n \in \mathbb{Z}/k\mathbb{Z}$ and $m \in \mathbb{Z}/w\mathbb{Z}$. The equations that f be harmonic all at nodes $x = a^p b^q$ with $p,\ q \geq 0$ and $|x| = p + q = n + 1$ are

$f(a^{n+2}) + f(a^{n+1}b) + f(a^{n+1}b^{-1}) = c_1$

$f(a^{n+1}b) + f(a^n b^2) = c_2$

....

$f(a^2 b^n) + f(ab^{n+1}) = c_{n+1}$

$f(ab^{n+1}) + f(b^{n+2}) + f(a^{-1}b^{n+1}) = c_{n+2}$,

where $c_1 = (4 * *f(a^{n+1})) - f(a^n) \in R(\ as\ |a^n| = n)$,

$c_i = (4{*}{*}f(a^{n+2-i}b^{i-1}))-f(a^{n+1-i}b^{i-1})-f(a^{n+2-i}b^{i-2}) \in R(\ as\ |a^{n+1-i}b^{i-1}| = |a^{n+2-i}b^{i-2}| = n)$, for $n + 1 \geq i > 1$,

and $c_{n+2} = (4 * *f(b^{n+1})) - f(b^n) \in R(\ since\ |b^n| = n)$

It is easily seen that, setting $f(a^{n+1}b) = f(a^{n+1}b^{-1}) = f(a^{-1}b^{n+1}) = 0$, so that $f(a^{n+2}) = c_1$, the rest of the values in the system will determined from solving simple linear equations (simple linear equations are solvable in a ring because of the existence of additive inverses), e.g. $f(a^n b^2) = c_2$ so that $f(a^{n-1}b^3) = c_3 - f(a^n b^2) = c_3 - c_2$. Now, the other three [[systems of equations]] for the cases when $x = a^p b^q$ with $p \leq 0,\ p \leq 0, q \leq 0,\ \ p \geq 0,\ q \leq 0,\ and\ p \leq 0,\ q \leq 0$. are obtained from the above system simply by changing the sign of the powers of a only, the powers of b only, and the powers of both a and b respectively. Therefore, all three are solvable by the method just described.

Since the nodes $a^{n+2-i}b^i$, $a^{-(n+2-i)}b^i$, $a^{n+2-i}b^{-i}$, and $a^{-(n+2-i)}b^{-i}$, for $0 \leq i \leq n+2$

are all the distinct nodes of length $n+2$, we have extended the domain of the function to all words of length $n+2$ so that words of length $n+1$ are harmonic. The result thus follows by induction.

We have therefore successfully extended the function f to the entire group $< S >$ so that it is harmonic everywhere and does not define a homomorphism on Γ, contradicting the proposition's hypothesis.

□

However, even if $\Gamma = \mathbb{Z}$, then the the existence of a 1-1 homomorphism does not hold for a ring containing all elements of finite order. In fact, in this case, assuming that $f(s)$ has order n, any homomorphism f must have $f(s^n) = n * *f(s) = 0$. Thus, $\ker f = n(\mathbb{Z})$.

Also note that, if the condition that Γ be infinite were dropped, then $|S| = 2$ would just mean that the group was cyclic, and with this minor modification, our result would still hold. Of course, to deal harmonic on Cayley of finite groups that are harmonic at every node, we need to disclude subrings of $\mathbb{C}$.

2.4 Consequences of the Theorem

In this section, give some basics results that directly follow and extend from the abvoe theorem, compare our result with the work of Cheeger and Gromov, and fit the result into a more general theory about harmonic maps on Cayley graphs.

Corollary 13. *The Cayley graph of any free group or free abelian group with standard generating set of order exceeding 2 admits a function that is harmonic at every node.*

For any harmonic function $f : X \to (R, +)$, define

$ker f = \{x \in X,\ f(x) = 0\}$. Note that, unlike the kernel in group theory, we can have $ker f = \emptyset$

Corollary 14. *Let R be a ring containing no element of finite order. The Cayley graph X of any free of group or group abelian group on its standard generators admits a non-constant harmonic function f such that supp $h = \Gamma = V(X)$, or equivalently $ker f = \emptyset$*

The proof again uses induction on the word length in Γ to construct a non-constant function on the entire graph, all of whose values are nonzero.

Note that Cheeger's and Gromov's result that the Cayley graph of an amenable group admits no non-constant Dirichlet harmonic functions [6, p.1], so that

Fact 15. *If $Cay(\Gamma : S)$ admits a non-constant harmonc function that is l_2 -summable, then Γ cannot be amenable.*

Our result does not contradict the fact that abelian groups are amenable because the harmonic maps we are considering are not the same as Dirichlet harmonic maps on free groups and free abelian groups. Since every Dirichlet harmonic map is also a harmonic map, we have the following:

Fact 16. *Every harmonic map on the Cayley Graph of an amenable group is has a differential that is not l_2 - summable.*

In fact, Elek and Tardos [6] have extended the result about Cayley graphs of amenable groups to general, roughly transitive, amenable graphs.

Corollary 17. *The harmonic maps defined on the abelian groups have differentials that are not l_2 - summable.*

2.5 Generalized Results

We can divide the harmonic maps on a given Cayley graph into three sets H_1, H_2 and H_3 where H_1 denotes the ones induced by homomorphisms from Γ to $(R, +)$ H_2 denotes the set of harmonic maps preserving group identities, $f(1) = 0$, that are not induced by homomorhisms, and H_3 denotes the set of harmonic maps with $f(1) \neq 0$. Now since, for any harmonic map, $u \in H_3$, $u - u(1)$ is also harmonic on the graph, so that $u - u(1) \in H_1$ *or* H_2. For this reason, it suffices only to consider the sets H_1 and H_2 since the other harmonic maps differe from them only by a constant.

Since every homomorphism induces a harmonic map on a Cayley graph of group all of whose elements have order exceeding 2, H_1 is just the set of all homomorphisms from Γ to $(R, +)$.

The above theorem can thus be restated as

For a free group or infinite abelian group on its standard symmetric generators $|H_2| = 0$, if and only if $\Gamma \simeq \mathbb{Z}$.

With minor modifications modifications to include finite graphs, we have

Fact 18. *For an abelian group on its standard symmetric generators, $|H_2| = 0$ if and only if Γ is cyclic.*

A counterexample for non-abelian groups would be the Dihedral Group, D_{2n}, which has $|H_2| = 0$, but is obviously non-cyclic.

Lemma 19. *The only groups whose cayley graph have the form of a series of polygons contained inside of a larger, similar one and joined symmetrically to each other pairwise at each vertex are, up to isomorphism, $D_{2n}, \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$, and $< x, r | x^n = r^2 = 1, R_i(x, r),\ 0 < m < n >$, where R_i defines the relations on the two generators.*

Proof. I will proceed with a proof by contrapositive. Note: obviously, $|\Gamma| = mn$.

If Γ is not isomorphic to any of the groups listed above, then there is only one possibility: $\Gamma$is generated by two or more distinct elements besides x. I will show that in each of the following cases, the the corresponding Cayley graph is *not* $Cyc_n \times 2$:

Assume that S contains two or more distinct elements in addition to x. Then, if $1 \in \Gamma = V(X)$ denotes the identity, $deg\ 1 > 3$, so that the graph cannot be the one desired since every vertex on the inner most polygon has degree 3. $\square$

Conjecture: Let $R = \mathbb{Z}/n\mathbb{Z}$, and suppose Γ with the order of the group satisfying $n < |\Gamma| < \infty$ contians a cyclic subgroup of order n. Let $H_{C_n)}$ denote the harmonic maps on $X = Cay(\Gamma : S)$ (where S is a standard, symmetric generating set of minimal order) such that the restriction of any $f \in H_{C_n}$ to C_n is a homomorphism. Then $|H_2 \cap H_{C_n}| = 0$ if and only if there is a $s \in \Gamma$ for which $< x, s >= \Gamma$, where $< x >= C_n$.

Proof: If Γ has to be generated by more than two elements, then $deg 1 > 3$, so that we can assign an element other than 0 in $\mathbb{Z}/n\mathbb{Z}$ to a vertex neighboring 1 while keeping the function harmonic at 1. Furthermore, we we can extend the function defined on C_n to the whole graph so that it is harmonic since the graph must have more than $2n$ vertices. To ensure that it is not a homomorphism, we merely need to set $f(r) = 1$ and $f(xr) \neq 2$; we may assume that $f(x) = 1$ withouth loss of generality.

We also know that for any free group and free abelian group on more than one standard generator and for a ring containing an element of infinite additive order, $|H_2| = \infty$.

What does the size of $|H_2|$ tell us about the structure of the group or its standard generators? In particular, if R contains an element of infinite order and $|H_2| < \infty$, then we know that Γ cannot be free or free abelian, but what else can we deduce about Γ?

3 Floyd Compactification

3.1 Definitions

Another application of harmonic functions involves studying group cohomology and exploring the Dirichlet Problem using compactification [2, p. 138].

Consider a geodesic space (X, d), where X is a graph and d is the path-metric induced from identifying each edge with the unit interval and gluing them together at the edges. Fixing a base point or center, $z \in X$ we also define $|x| = d(z,\ x), x \in X\ and\ |A| = \inf_{x \in A} d(z, x), A \subseteq X$

Now, given a function
$F : \mathbb{N} \to \mathbb{R}_+$, such that
$\sum_r |F(r)| < \infty$
called a *Floyd Admissible Function*, we may define the *Floyd Metric*, d_F, with the following equations:

(1) If $x \sim y$, then $d_F(x, y) = F(|x, y|)$

(2) Otherwise, let α be a path from x to y (permissible because X is a geodesic space) consisting of the vertices x_i, with x_i and x_{i+1} adjacent. Setting $L_\alpha = \sum_i d_F(x_i,\ x_{i+1})$, we can define $d_F(x, y) = \inf_\alpha L_\alpha$ where this infimum is taken over all paths α from x to y.

This new metric has a finite diameter on the new metric space X_F, so that the fact that $\bar{X_F}$, the completion of X_F as a metric (i.e. the union of X_F with all of its limit points) does not come as a big surprise. We are now in a position to define the "Floyd Boundary" as a $\partial X_F = \bar{X_F} - X_F$.

Note that, in general, the Floyd Compactification depends on the Floyd Admissible function F, and that this way defining the boundary is in now way unique. Other boundaries include the ideal, Gromov, and hyperbolic.

3.2 Examples of the Floyd Boundary

Below, we show that the Cayley graph of two fairly different groups have the Cantor set as a Floyd Boundary. We then introduce the relationship of the order of this boudnary type to amenability and formulate questions using H_2.

Example 20. *Take X to be the 4 - regular tree and let $F(r) = 1/(2^{r+1})$, and define the Floyd metric as above. Notice that, under the Floyd Metric, the graph has a diameter bounded by 1. Then the only limit points that X_F does not contain are the endpoints of the geodesic rays originating at the center. Notice how these geodesic rays define Cauchy sequences because $1/(2^{r+1})$ is summable. It follows that the Floyd Boundary is, in this case, the 4-Cantor set, obtained by continually chopping up the interval $[0, 1]$ into 4 different pieces.*

Example 21. *Similarly, keeping the same Floyd admissible function, but taking F to be the Cayley graph of the group $\mathbb{Z}/2 * \mathbb{Z}/3 =< s, t|s^2 = t^3 = 1 >$.. Then, again, the graph has a diameter bounded by 1, and the limit points consist of the endpoints of the geodesic rays, which also happen to be geodesic rays on the 3 - regular tree, so that the limit points not contained in X_F is the 3- Cantor set, obtained again by continually chopping up the unit interval $[0, 1]$, but this time into 3 different pieces.*

If the metric were different (not obtained from a Floyd Compactification), so that some the new geodesics trace the lines on the triangular subgraphs not traced in the geodesics obtained from the Floyd metric in a zig-zag way that consecutively up and then down the tree-like structure of this graph. Assuming that the geodesics still include the geodesics on the 3-regular tree, the boundary just consists of the 3-Cantor set plus another two points.

From [5, p. 10] the Floyd boundary of a finitely generated amenable group is trivial, i.e. contains no more than 2 points. Consequently, we an determine if a group is non-amenable by analyzing its Floyd Boundary. For example, since the Cantor set is uncountably infinite, it follows that both the free product $\mathbb{Z}/2 * \mathbb{Z}/3$ and the free group F_2 are non-amenable. In fact, any free group F_n is non-amenable since it is non-commutative.

Groups that have trivial Floyd boundaries of 0 or 2 points are in general easy to come up with: Any finite group has an empty set as a the Floyd Boundary and $\mathbb{Z}$ has its two points at infinity as the Floyd boundary. However, groups with Floyd boundary consisting of only one point are a little more interesting, and examples include the fundamental group of compact negatively curved manifolds.

An interesting question arising from studying Floyd boundaries is *what does H_2 tell us about the Floyd Boundary?* In fact, since Karlsson already states that $|\partial X_F| = 2 \Rightarrow \Gamma = \mathbb{Z}$ we have the following:

$|H_2| = 0$ *if and only if* $|\partial X_F| = 2$.

3.3 The Dirichlet Problem

Generally, the Dirichlet problem involves finding a harmonic function on the completion $\bar{X_F}$ of the geodesic space X_F given a Floyd Compactification F and a continuous function f defined on the boundary ∂X_F [2, p. 139].

The Dirichlet problem can also be considered with respect to a stochastic transition operator, P, where harmonic functions become invariants of P, or rather, solutions to the Laplacian $\Delta = P - I$, instead of $\Delta = D - A$, as in the case just described. Not surprisingly, this problem has applications to random walks and is closely interrelated to Markov chains.

An important result about the solvability of the Dirichlet Problem and Harmonic functions on a Cayley graph $X = Cay(\Gamma, S)$ for $|S| < \infty$ is given in the following main theorem from [2].

Theorem 22. *Karlsson [2, p. 138] If the the geodesic space X admits an infinite Floyd Boundary, i.e. there is a compactification F such that $|\partial X_F| = \infty$, then the Dirichlet problem is solvable with respect to ∂X_F. On the other hand, if every Floyd compactification, F we have $|\partial_F X| < \infty$, then for every non-constant harmonic function h, $\sum_{r>0} sup_{|x|\geq} |dh(x)| = \infty$, where x denotes an edge of X viewed as a subset of the graph and with $|x|$ defined from the induced path metric in the section above describing Floyd compactification.*

Note that if Γ is amenable, then, according to [6], dh cannot even be l_2 -summable, for any nonconstant harmonic map h, on the Cayley Graph, so that this also satisfies the last equation of the above theorem, i.e.

Fact 23. Γ *amenable* $\Rightarrow \sum_{r>0} sup_{|x|\geq} |dh(x)| = \infty$,

3.4 Group Cohomology

Let L^1C_0 denote functions on the edges whose suprema are summable, i.e. $f : E(X) \to \mathbb{R} \in L^1C_0$, if and only if $\sum_{r>0} sup_{|x|\geq r} |f(x)| < \infty$.

Now, let $C_0(X_F)$ denote the space of all continuous functions on the vertices of X_F with respect to the Floyd Metric. Then I will define $\overline{L^1C_0} = \{h \in C_0(X_F),\ dh \in L^1C_0\}$.

and the first Cohomology group, $H^1(X_F : L^1C_0)$, as $H^1(\Gamma, L^1C_0) \simeq \{g : \Gamma \to \mathbb{R},\ dg \in L^1C_0)\} / (\mathbb{R} + \overline{L^1C_0})$.

Note that the elements of H^1 are just cosets of $(\mathbb{R} + \overline{L^1C_0})$ with representatives consisting of functions on the edges of X_F that cannot be broken down into a continuous function (with respect to Floyd) on the edges plus a constant.

Here, *number* may refer to both countably infinite or uncountably infinite, and the bound holds in the infinite case provided $|\partial X_F|$ and $|H^1|$ are not countable and uncountable, respectively.

The following result relates the first cohomology group just defined to the Floyd Boundary.

Theorem 24. *(Karlsson [2, p. 143]) The number of points in the Floyd Boundary is bounded by the number of elements in H^1, i.e. $|\partial X_F| \leq |H^1|$."*

A further result gives the better result of equality between the order of a similar group and the number of elements in the Floyd Boundary:

Theorem 25. *(Karlsson [2, p.143])* $|\partial X_F| = 1 + dim\bar{H}^1(\Gamma, C_c)$

Here, $C_c = \{h : E(X) \to \mathbb{R},\ |supp\ h| < \infty\}$
(where $supp\ h = \{x \in E(X),\ h(x) \neq 0\}$),
so that $\overline{C_c} = \{h \in C_0(X_F),\ dh \in C_c\}$,
and $\bar{H}^1 \simeq \left\{g : E(X_F) \Rightarrow \mathbb{R},\ dg \in L^1C_0)\right\}/(\mathbb{R} + \overline{C_c})$.

From the above results about the Floyd boundary, we may thus conclude the following:

Fact 26. *(1) For $\Gamma = \mathbb{Z}$, dim $\overline{H}^1(\Gamma, C_c) = 1$*

(2) For Γ the fundamental group of any negatively curved manifold, e.g. a surface, dim $\overline{H}^1(\Gamma, C_c) = 0$

(3) For $\Gamma = F_n$ the free group on n generators, dim $\overline{H}^1(\Gamma, C_c) = \infty$.

An intersting question that arises is to impose conditions, e.g. properties of harmonic maps on the Cayley Graph of, Γ with respect to its standard generating set in order to determine whether the size of $\overline{H}^1(\Gamma, C_c)$. In fact, from the above results, we can conclude that $|H_2| = 0 \Leftrightarrow dim\overline{H}^1(\Gamma, C_c) = 1$. Now, in order to further consider the effect of the set H_2 on the set $\overline{H}^1(\Gamma, C_c)$, it makes sense to compare homomorphisms to continous functions plus a constant and harmonic maps not induced by homomorphisms to non-continous functions. Note that continuity is defined in terms of the Floyd metric. In this case, remember that we are only considering maps whose support is finite.

4 Comparing Boundaries up to Homeomorphism

4.1 Floyd, Ideal and Gromov Boundaries

Throughout the following, let

∂X_F : Floyd boundary with respect to Floyd Admissible Function F

∂X_G : Gromov boundary

∂X_I : Ideal boundary

All three of these boundaries have topologies either iduced by or defined using the original metric d, e.g. the path-metric. We also define a length space metric as a rectifabley connected metric space for which $d(x, y) = inf_{\gamma} len_d(\gamma)$ where this infimum is taken over all rectifiable paths $\gamma : [0, T] \to \overline{X_d}$ with $\gamma(0) = x$ and $\gamma(T) = y$ parametrized by arc length under the d metric.

Properties of Metric and Geodesic Spaces

Before giving the comparison results, I will define δ *hyperbolic* and $CAT(0)$ space and then use these to define the ideal and Gromov boundaries (we have already defined the Floyd boundary in the above section).

In order to define a $CAT(0)$ space, I will define the notion of a triangle map, $f : T - \mathbb{R}^2$, where T is a geodesic triangle. A triangle map has the property that it preserves the length of the sides of T and the restriction of f to each geodesic side in T is an an isometry. Now, a $CAT(0)$ is a geodesic space for which every triangle map has $d(u, v) \leq |f(u) - f(v)|$, for all $u, v \in T$.

To define $\delta - hyperbolic$ I first need to define the *Gromov inner product* with respect to a vertex $p \in X$:

$\langle x, y \rangle_p = (d(x, p) + d(p, y) - d(x, y))/2, \ x, \ y \in X.$

We then say that the metric space X is $\delta \ - \ hyperbolic$ if

$\langle x, z \rangle_p \geq min(\langle x, y \rangle_p ,) \, \langle y, z \rangle_p - \delta$

which, unlike the triangle inequality, gives a *lower* bound to the distance between x and z.

Buckely and Kokendoff also introduce the notion of a δ - thin triangle, which simply means that, letting T_1, T_2, and T_3 be the sides of a triangle, for any $u \in T_1$ and $v \in T_2 \cup \ T_3$, we have $d(u, v) \leq \delta$. They also note that $X \ is \ \delta \ - \ hyperbolic \Rightarrow X \ has \ 3\delta - thin \ triangles$ and $X \ having \ \delta - \ thin \ tringles \Rightarrow X \ is \ 3\delta \ - \ hyperbolic.$

Ideal Boundary:

To define, the ideal boundary, we first need to introduce the notion of the Hausdorff distance, d_H of a given metric d. This metric assigns distance between two geodesic rays given via the equation,

$d_H = max(sup_{x \in \gamma} dist_d(x, \nu), sup_{y \in \nu} dist_d(x, \gamma)).$

Two rays, $\gamma, \nu \in \partial X = GR(X)$, where $GR(X)$ denotes the set of all geodesic rays parametrized by $d - arclength$, are equivalent if and only if $d_H(\gamma, \nu) < \infty$, or, equivalenlty, $sup_t d(\gamma(t), \nu(t)) < \infty$. The ideal Boundary is then defined simply as the free group

To the space $\overline{X}_I = X \cup \partial X_I$, we attach what is called the *cone topology*, provided that the space is both complete and CAT(0). Most basically, the cone topology is defined using $X_r = \partial X_I \cup (X - \overline{B}_d(c, r))$, where $B_d(c, r)$ is a ball of radius r from a point $c \in \overline{X}_I$ and the projection map $p_r : X_r \to S_d(c, r)$, (where $S_d(c, r)$ denotes the sphere of radius r centered at $c \in X$, or, $S_d(c, r) = \partial B_d(c, r)$) defined by $p_r(x) = \gamma_x(r)$, where $\gamma_x(r)$ is the unique geodesic (parametrized by arclength under the d - metric) joining c to x. The topology consists of all sets $U(a, r, s)$, with $r, s > 0$ defined as $x \in X_r | d(p_r(x), p_r(a)) < s$ with the local base $a \in \partial X_I$.

Gromov Boundary:

Let $<,>$ denote the Gromov metric with respect to an arbitrary base point.

A Gromov sequence is in some sense the opposite of a Cauchy sequence in the sense that $\gamma = (\gamma_i) \in X$ is Gromov if and only if $< \gamma_i, \gamma_j > \to \infty$ as $i, j \to \infty$.

Now, we can readily define the Gromov boundary, as the set of all equivalence classes of sequences $\gamma = (\gamma_i) \in G$ under the equivalence relation $\gamma \sim \eta \Leftrightarrow$ $there\ is\ a\ set\ of\ k+1\ sequences\ \gamma = x^0, \eta = x^{k_0},\ and x^{k-1} E x^k$ where

$zEw\ if\ and\ only\ if \Leftrightarrow \liminf_{i,j\to\infty} < w_i, z_j >= \infty$. The set of equivalence classes of such geodesic rays, ∂X_G is called the Gromov boundary.

Using this boundary, we can extend the Gromov metric to all of ¡math¿ ∂X_G using $< a, b >= \inf \liminf_{i,j\to\infty} < x_i, y_j >\ for\ [x] = a\ and\ [y] = b$, where $a, b \in \partial X_G$

and $\inf \liminf_{i,j\to\infty} < x_i, y_j >\ for\ [x] = a$ where $a \in \partial X_G$, and $b \in X$.

Now, the metric, or rather, *pseudo-metric* used in comparing the boundary is defined with respect to a number $\epsilon > 0$ as

$\rho_\epsilon(a, b) = exp(-\epsilon < a, b >)$ $\delta_\epsilon(a, b) = inf(\sum_{j=1}^{n} \rho_\epsilon(a_{j-1}, a_j)), a,\ b \in \partial X_G$, where this infimum is taken over all finite sequences starting at a and ending at b.

Now, if X is $\delta\ -\ Hyperbolic$ and $\epsilon\delta \leq 5$, then d_ϵ is in fact a real metric.

From [4, p. 8], the following relationships among the boundaries occur:

Theorem 27. *(Kokendoff and Buckley) Assume that the space (graph)* X *is* $\delta - Hyperbolic$ *under the* $l - metric$ *and nonempty. Also, let* ϵ *and* δ *be such that* $\epsilon\delta \leq \frac{1}{5}$

Then

(1) Since there is a $K > 0$ *and a* $\epsilon_0(\delta) > 0$ *for which* $F(t)exp(\epsilon_0 t) \geq K$, $\partial X_G \simeq \partial X_F$ *under the* d_ϵ *dometric. Furthermore, if* X *is geodesic, complete and* $CAT(0)$ *under the* $l - metric$*, then* $\partial X_I \simeq \partial X_F$.

(2) If X *is complete and* $CAT(0)$*, then* $\partial X_I \simeq \partial X_G$ *under the cone topology* τ_C *and the Gromov* d_ϵ *metric respectively.*

A result about comparing Floyd and Hyperbolic Boundaries comes from [5, p. 8] and states that if the geodesic space is the Cayley graph of a finitely generated word hyperbolic group, then ∂X_F for $F(r) = (r+1)^{-2}$ coincides with the standard Hyperbolic boundary. Note: Karlsson also compares the Floyd boundary of certain groups to a particular limit set: $|L_s| \leq |\partial X_F| = |\partial\Gamma|$

where Γ is the fundamental group a hyperbolic 3-manifold and L_s is the limit set of this manifold.

The Poisson Boundary

The Poisson boundary also merits comparison, particularly because of its applications in solving the Dirichlet Problem. To define the Poisson boundary, I first need to defined the notion of a Markov Chain induced by a probability measure μ, ergodic components of Γ, the time Shift T, the path space, and also the μ- boundary, respectively.

Given, a probability measure μ on a group, the induced random walk is defined as

$p(x,y) = \mu(x^{-1}y)$. Note that $p(hx,hy) = p(x,y)$.

First, we need to introudce Kaimanovich's conditions (CP), (CS), and (CG):

(CP) or "Projectivity" Any two sequences remaining a bounded distance from each other must converge to one and the same boundary point. Equivalently, for a group G, if the sequence $g_n \in G$ has $g_n \to g \in \partial G$ then $g_n x \to g$ for all $x \in G$.

(CS) "Separated by strips" For any two $\gamma, \eta \in \partial\Gamma$, $S(\gamma,\eta)$ be the strip bounded by the geodesics rays eminating from infinity and converging to γ and η respectively. Then the condition states that for any three points, $x_i \in \partial X$, there are neighborhoods U_i about each such that $S(U_1,U_2) \cap U_3 = \emptyset$.

(CG) d is left-invariant, i.e. $d(x,y) = d(gx,gy)$ for all $g,h \in \Gamma$. and the gauge is temperate. According to Kaimonich, a gauge is a sequence of sets $\mathcal{G}_k, (k \geq 1)$ that approximate the entire group. Such a gauge is temperate if $sup_k \frac{1}{k} log|\mathcal{G}_k| < \infty$.

Relating these to Floyd Boundaries, in [5, p.9] the author proves the following:

Proposition 28. *(Karlsson) If $\partial\Gamma$ as a Floyd type contains at least three points, then $\overline{\Gamma}$ satisfies (CP),(CS), and (CG).*

4.2 Harmonic Isometries

Studying these properties about metric and geodesic spaces raises the question: When is a harmonic map on a subset of $\overline{X}$ an isometry, and what does this say about the geometry of X, properties of the metric space, or the algebraic structure of the group of which X is Cayley Graph.

From [2, p.9], a proposition claims that $\partial\Gamma$, of a group Γ under Floyd Compactification is a geodesic space, so one would expect that harmonic isometries be related to the Dirichlet Problem.

By definition an isometry into $\mathbb{R}$ has the following property: $d(x,y) = |f(x) - f(y)|$, where d denotes the metric on the graph. As we have seen above, this metric could be the induced path metric, the Floyd metric, or the Gromov metric.

In any case, the condition that this isometry be harmonic is $\sum_{y\sim x} f(y) - f(x) = \sum_{y\sim x} sgn_f(y,x)d(y,x) = 0$, where $sgn_f(y,x) = sgn(f(y) - f(x))$. Now, in the case that d is the induced path metric, we have have the somewhat conciser requirement that $\sum_{y\sim x} sgn_f(y,x) = 0$.

I will call a map that is simultaneously a harmonic and isometric a *harmonic isometry*

The following two results show that harmonic isometries are related to the parity of the degree of the vertices of a graph:

Proposition 29. *Let X be a graph containing a vertex of odd degree and let d denote the induced path metric. Then X does not admit any harmonic isometries to $\mathbb{R}$.*

Proof. Let $v \in X$ be the vertex of odd degree and let f denote any isometry from X to $\mathbb{R}$. Then for any $y \in N(v)$, $|f(y) - f(v)| = 1$. Therefore, $\sum_{y \in N(v)} sgn\ d(y,v)$ will only be zero if there are an equal number of $+1$ s and -1 s. Since $|N(v)|$ is odd, this is not possible. □

Theorem 30. *Let k be an even integer. Then every k - regular graph admits a harmonic isometry under the induced path metric.*

Proof. We can easily define an isometry on a k - regular graph simply by sending each vertex to its distance from a point chosen to be the center (all of these values will be in $\mathbb{Z}$. Denote this isometry by f.

Proceeding by induction on the distance from the center, we see that we can easily assure that f is harmonic is the sphere of radius one around the center z by negating half of the values. Since the center was taken as the starting point, this map indeeds remains an isometry. Now, assume the function can be extended to a sphere of radius $m \geq 1$ about the center such that, when restricted to this sphere, it is an isometry and also harmonic at every vertex whose neighborhood lies within the sphere. The sphere of radius $m+1$ can be formed by taking the union of all the neighbors of vertices in S_m whose neighbors lay outside of S_m. We need only to make sure that f remain an isometry at these vertices that that its restriction to S_{m+1} is an isomety. Now, let $v \in S_m$ such that $N(v) \in S_{m+1} - S_m$. For a regular graph this is actually just the set of vertices for which $d(z,v) = m$, where z is the center of the graph. Since the restriction of f to S_m is an isometry, the values at the neighbors $N(v)$ differ from $f(v)$ 1. Since the graph is regular of even degree, there must be as many neighbors within S_m as there are in $S_{m+1} - S_m$. Therefore, we may choose each value $f(w)$ of the neighbors $w \in S_{m+1} - S_m$ so that they correspond to a $w* \in S_m$ and satisfy $f(w) + f(w*) - 2f(v) = 0$. Since each neighbor in $S_{m+1} - S_m$ has distance $m+1$ from the center, and each other neighbor has distance $m-1$ from the center (since the graph is regular) it follows that this map remains an isometry at vertices lying at a distance of $m+1$ from the center is harmonic for all vertices lying at a distance of m from the center. The result thus follows by induction.

□

As an immediate consequence of these two theorems applied to Cayley Graphs, we have that

Fact 31. *(1) The cayley graph of F_k on its standard symmetry generators admits a harmonic isometry, provided that k is even.*

(2) If the Cayley graph X admits a harmonic isometry, then $|S|$ must be divisible by 4.

(3) If the Cayley graph X admits a harmonic isometry and $|\partial X_F| < \infty$, then $|\partial X_F|$ must be even.

5 Future Outlook

In the above paper, we have only considered harmonic maps on unweighted Cayley Graphs. A natural way to continue our studies would be to extend some of our results to Weighted Graphs, while tying Group Theory and perhaps including a thorough discussion of the Heat Kernel. More connections between different boundaries, harmonic maps, and group theory may also be considered. However, physics motivates another way of extending the specific theory of harmonic maps. Since many physical equations are given as differential equations, it seems natural to consider what other differential operators, besides the Laplacian, can be defined on infinite Cayley graphs. Connections with Group Theory and Compactification might then arise.

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