# Harmonic Maps and The Dirichlet Problem

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# Outline

# What Is a Harmonic Map?

- A *harmonic map* is a map $f$ with $\Delta f = 0$
  $\Delta$ is the Laplacian
- Two types of $\Delta$:
  Laplacian with respect to an operator
  Combinatorial Laplacian

# Harmonic Maps with Respect to an Operator

- $X$ a Graph
- $P = p(x, y),\ x, y \in X$: *stochastic transition operator*
- $P(h)(x) = \sum_{y \in X} p(x, y)h(y)$
- Laplacian with respect to $P$: $\Delta = P - I$
- $\Delta h = 0 \Rightarrow Ph = h$ iff $h$ is $P$ - harmonic

# Harmonic Maps on Graphs

- Combinatorial Laplacian on a Graph: $\Delta = D - A$
- $f$ on a graph $G$ is harmonic at $v \in V(G)$ if $\Delta f(v) = 0$.
- A point where a function is not harmonic is a *pole*

# The Number of Poles on Finite and Infinite Graphs

- Every nonconstant, complex-valued harmonic function on a finite graph has at least two poles (Lovasz).
- Not true for inifinite graphs, e.g. $Cay(\mathbb{Z} : \{\pm 1\})$

# Homomorphism and Cayley Graphs

- $\Gamma$ a group generated by the finite set $S$
- $R$ a ring
- $f : \Gamma \to (R, +)$ group homomorphism.
- $X = Cay(\Gamma : S)$
- $f$ naturally defines an $R$-valued *harmonic* function on $V(X)$

# Preliminaries

- $\Gamma$ a free group or free abelian group with natural finite symmetric generating set $S$.
- $X = Cay(\Gamma : S)$
- $R$ a ring with an element of additive order exceeding 2.

# Harmonic Functions that are Homomorphisms

## Theorem

*Let $\Gamma$ be a free group or free abelian group. If every non-constant harmonic function $f : Cay(\Gamma : S) \to (R, +)$ with $f(1) = 0$ also defines a homomorphism $f : \Gamma \to (R, +)$, then $|S| = 2$, so that $\Gamma = \mathbb{Z}$. Moreover, if the ring contains an element of infinite order, then there exists a non-constant monomorphism $g : \Gamma \to (R, +)$.*

# Corollary

## Theorem

*The Cayley graph of any free group or free abelian group not equal to $\mathbb{Z}$ on its standard generators admits a nonconstant harmonic function that is not a homomorphism.*

# What is the Dirichlet Problem?

Extend a continous function on $\partial X$ to $\overline{X}$ so that it is harmonic.

In the Following,

$X = Cay(\Gamma, S),\ |\Gamma| = \infty$ & $|S| < \infty.$

# The Floyd Metric

- $d$ = induced path metric, $z$ = the center of graph
  $|x| = d(z,\ x),\ \ x \in X$ and $|A| = \inf_{x \in A},\ d(z, x),\ \ A \subseteq X$

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- $(x, y) \notin E(X)$:
  $\alpha$ = path from $x$ to $y$, $L_\alpha = \sum_i d_F(x_i,\ x_{i+1}),\ (x_i, x_{i+1}) \in \alpha$,
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  $d_F(x, y) = \inf_\alpha L_\alpha$
- $\overline{X_F}$ completion of $X_F$ and $\partial X_F = \overline{X_F} - X_F$ Floyd Boundary

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- Cayley Graph of $\mathbb{Z}/2 * \mathbb{Z}/3 =< s, t | s^2 = t^3 = 1 >$ and same $F(r)$
- $\partial X_F$ = 3-Cantor Set: continually chop up $[0, 1]$ into 3 pieces

# Solvability of the Dirichlet Problem

**Theorem**

*(Karlsson)*
*If $|\partial X_F| = \infty$, then the Dirichlet problem is solvable with respect to $\partial X_F$. However, if $|\partial X_F| < \infty$ then for every nonconstant harmonic function, h on $X_F$, $\sum_{r>0} sup_{|x|\geq r}|dh(x)| = \infty$*

$x$ an edge

$dh(x) = h(xs) - h(x)$

# Group Cohomology

## Theorem

*(Karlsson)*
$|\partial X_F| \leq 1 + dim\, \overline{H}^1(\Gamma, L^1 C_0)$

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- $\overline{L^1 C_0} = \{h \in C_0(X_F),\ dh \in L^1 C_0\}$
- $\overline{H}^1(\Gamma, L^1 C_0) \simeq \{g : \Gamma \to \mathbb{R},\ dg \in L^1 C_0\} / (\mathbb{R} + \overline{L^1 C_0})$

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- Using compactification, the Dirichlet problem relates to summability of the differential of harmonic functions
- Differentials of functions can define a cohomology group and estimate the cardinality of the Floyd Boundary

# Outlook

- What else can harmonic functions on Cayley graphs tell us about the algebraic structure of groups?

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- What else can harmonic functions on Cayley graphs tell us about the algebraic structure of groups?
- How does the Dirichlet Problem's relationship to cohomology change for $P$ - harmonic functions?

# For Further Reading I


- Anders Karlsson
  *Harmonic Functions on Homogeneous Graphs*
  Discrete Mathematics and Theoretical Computer Science AC(2003), 137-144.
- Wolfgang Woess
  *Dirichlet problem at infinity for harmonic functions on graphs*
  31C12, 60J15 (1991).
- Lovasz Laszlo
  *Discrete Analytic Functions: An Exposition*