Laplacians of Covering Complexes

Richard Gustavson[1]
Cornell University

Abstract

The Laplace operator on a simplicial complex encodes information about the adjacencies between simplices. A relationship between simplicial complexes does not always translate to a relationship between their Laplacians. In this paper we look at the case of covering complexes. A covering of a simplicial complex is built from many copies of simplices of the original complex, maintaining the adjacency relationships between simplices. We show that the Laplacian spectrum of a simplicial complex is contained inside the Laplacian spectrum of any of its covering complexes.

1 Introduction

The combinatorial Laplacian of a simplicial complex has been extensively studied both in geometry and combinatorics. See [?Duval1] for some history of the combinatorial Laplacian. The Laplace operator is not a topological invariant; thus even simplicial maps that preserve the underlying topological structure of a simplicial complex might change the Laplacian. Like topological spaces, complexes can have coverings. Our goal in this paper is to determine the relationship between the Laplacian of a simplicial complex and the Laplacians of its coverings. Our main theorem is the following relationship between the spectra of the two Laplacians.

Theorem. *Let $(\widetilde{K}, p)$ be a covering complex of simplicial complex K, and let $\widetilde{\Delta}_d$ and Δ_d be the Laplacian matrices of $\widetilde{K}$ and K, respectively. Then* $\mathrm{Spec}(\Delta_d) \subseteq \mathrm{Spec}(\widetilde{\Delta}_d)$.

Our goal in this paper is to prove this theorem. In Section 2, we give the basic definition of a simplicial complex and some terms associated with it. We introduce the boundary operator and its adjoint and provide a formulaic

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construction of the adjoint. We then define the Laplace operator. As a linear operator, the Laplacian has a spectrum, which is its multiset of eigenvalues. We introduce the notion of a covering complex in Section 3. After proving some simple relationships between a complex and its coverings, we prove our main theorem, that the Laplacian spectrum of a covering complex contains the Laplacian spectrum of the original complex.

2 Simplicial Complexes and the Laplacian

This section is devoted to definitions and basic facts about simplicial complexes. The definitions here will be used throughout the paper. See [?Kozlov1, ?Munkres1] for more information about abstract simplicial complexes.

Definition 2.1. *An* ***abstract simplicial complex*** K *is a collection of finite sets that is closed under set inclusion, i.e. if* $\sigma \in K$ *and* $\tau \subseteq \sigma$, *then* $\tau \in K$.

We will usually drop the word "abstract" and just use the term "simplicial complex." In addition, in this paper we will only deal with finite abstract simplicial complexes, i.e. only the case where $|K| < \infty$. A set $\sigma \in K$ is called a *simplex* of K. The *dimension* of a simplex σ is one less than the number of elements of σ. The *dimension* of K is the largest dimension of all of the simplices in K, or is infinite if there is no largest simplex. Since we will only discuss finite complexes, all complexes in this paper will have finite dimension. We call σ a *d-simplex* if it has dimension d. We write $S_d(K)$ for the set of all d-simplices of K. The *p-skeleton* of K, written $K^{(p)}$, is the set of all simplices of K of dimension less than or equal to p. The elements of the set $K^{(0)}$ are called the *vertices* of K.

We need to be able to speak about maps between simplicial complexes. We do this as follows.

Definition 2.2. *Let* K *and* L *be two abstract simplicial complexes. A map* $f : K^{(0)} \to L^{(0)}$ *is called a* ***simplicial map*** *when* $\{v_0, \ldots, v_n\}$ *is a simplex in* K *implies* $\{f(v_0), \ldots, f(v_n)\}$ *is a simplex in* L.

While f maps the vertices of K to the vertices of L, we will often speak of f as mapping K to L and write $f : K \to L$; thus if $\sigma \in K$ is a simplex, we will write $f(\sigma)$. Notice that if σ is a d-simplex in K, then $f(\sigma)$ need not be a d-simplex in L.

The d-chains of K form a vector space $C_d(K)$ with scalars in $\mathbb{R}$. See [?Munkres1] for a formal construction of the d-chains. By choosing an ordering of the vertices of a d-simplex, we give it an orientation. An *oriented simplicial complex* K is one for which we have chosen an orientation for each of its simplices. The vector space $C_d(K)$ thus has as basis elements the oriented d-simplices. As a result, we can define linear functions acting on $C_d(K)$ by defining how they act on oriented d-simplices. In addition, the basis determines an inner product on $C_d(K)$, written $\langle\rangle_d$. We will now define perhaps the most important linear function on $C_d(K)$, the boundary operator.

Definition 2.3. *The* ***boundary operator*** $\partial_d : C_d(K) \to C_{d-1}(K)$ *is the linear function defined for each oriented d-simplex* $\sigma = [v_0, \ldots, v_d]$ *by*

$$\partial_d(\sigma) = \partial_d[v_0, \ldots, v_d] = \sum_{i=0}^{d} [v_0, \ldots, \widehat{v}_i, \ldots, v_d]$$

where $[v_0, \ldots, \widehat{v}_i, \ldots, v_d]$ *is the subset of* $[v_0, \ldots, v_d]$ *obtained by removing vertex* v_i.

If $f : K \to L$ is a simplicial map, we define a homomorphism $f_\# : C_d(K) \to C_d(L)$ by defining it on basis elements (i.e. oriented simplices) as follows:

$$\begin{aligned} f_\#(\sigma) &= f_\#([v_0, \ldots, v_d]) \\ &= \begin{cases} [f(v_0), \ldots, f(v_d)], & \text{if } f(v_0), \ldots, f(v_d) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

We call the family $\{f_\#\}$ the *chain map induced by the simplicial map* f.

Technically speaking, each $f_\#$ acts only on one d-chain C_d. When we want to specify which dimension we are working with, we shall write f_d instead of $f_\#$. Chain maps have the special property that they commute with the boundary operator. See [?Munkres1] for the proof of the following lemma.

Lemma 2.1. *The homomorphism* $f_\#$ *commutes with the boundary operator* ∂, *i.e.*

$$f_{d-1} \circ \partial_d = \partial_d \circ f_d.$$

Since each boundary operator $\partial_d : C_d \to C_{d-1}$ is a linear map, we can associate to it its adjoint operator $\partial_d^* : C_{d-1} \to C_d$ as the unique linear operator that satisfies

$$\langle \partial_d(\sigma), \tau \rangle_{d-1} = \langle \sigma, \partial_d^*(\tau) \rangle_d$$

where $\langle \rangle_{d-1}$ and $\langle \rangle_d$ are the inner products on C_{d-1} and C_d, respectively. Since ∂_d and ∂_d^* are both linear, they both have associated matrices, which we call $\mathcal{B}_d$ and $\mathcal{B}_d^T$, respectively (where here, $\mathcal{B}_d^T$ is the transpose of $\mathcal{B}_d$, as ∂_d^* is the adjoint of ∂_d).

We now give a way of calculating ∂_d^*. First we define two sets. Recall that $S_d(K)$ is the set of all d-simplices of the simplicial complex K, and let $\tau \in S_{d-1}(K)$. Then define the two sets

$$S_d^+(K, \tau) = \{\sigma \in S_d(K) | \text{the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } +1\}$$
$$S_d^-(K, \tau) = \{\sigma \in S_d(K) | \text{the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } -1\}.$$

Now we give an explicit formula for calculating ∂_d^*, which we will use later on in proving Theorem **??**.

Theorem 2.1. *Let ∂_d^* be the adjoint of the boundary operator ∂_d. Then*

$$\partial_d^*(\tau) = \sum_{\sigma' \in S_d^+(K,\tau)} \sigma' - \sum_{\sigma'' \in S_d^-(K,\tau)} \sigma'' .$$

Proof. Let $f : C_{d-1} \to C_d$ be defined by

$$f(\tau) = \sum_{\sigma' \in S_d^+(K,\tau)} \sigma' - \sum_{\sigma'' \in S_d^-(K,\tau)} \sigma'' .$$

We show that $\langle \partial_d(\sigma), \tau \rangle_{d-1} = \langle \sigma, f(\tau) \rangle_d$, for $\sigma \in C_d, \tau \in C_{d-1}$, for then the function f will satisfy the requirements for the adjoint operator, and since the adjoint is unique, we will have $f = \partial_d^*$. First observe that f is linear, so (since ∂_d is also linear) we only need to show $\langle \partial_d(\sigma), \tau \rangle_{d-1} = \langle \sigma, f(\tau) \rangle_d$ for σ, τ basis elements, i.e. $\sigma \in S_d(K), \tau \in S_{d-1}(K)$.

Look at the term $\langle \partial_d(\sigma), \tau \rangle_{d-1}$. As τ is a single simplex, $\langle \partial_d(\sigma), \tau \rangle_{d-1} \neq 0$ if and only if τ is in the sum $\partial_d(\sigma)$, that is, if and only if $\tau \subseteq \sigma$. Since the coefficient of every term of ∂_d is either ± 1, we see that

$$\langle \partial_d(\sigma), \tau \rangle_{d-1} = \begin{cases} 1, & \tau \subseteq \sigma \text{ and the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } +1 \\ -1, & \tau \subseteq \sigma \text{ and the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } -1 \\ 0, & \tau \nsubseteq \sigma \end{cases}$$

Now look at the term $\langle \sigma, f(\tau) \rangle_d$. Since σ is a single simplex, $\langle \sigma, f(\tau) \rangle_d \neq 0$ if and only if σ is in the sum $f(\tau)$, that is, if and only if $\sigma \supseteq \tau$. Since the coefficient of every term of $f(\tau)$ is either ± 1, we see that

$$\langle \sigma, f(\tau) \rangle_d = \begin{cases} 1, & \sigma \supseteq \tau \text{ and the coefficient of } \sigma \text{ in } f(\tau) \text{ is } +1 \\ -1 & \sigma \supseteq \tau, \text{ and the coefficient of } \sigma \text{ in } f(\tau) \text{ is } -1 \\ 0, & \sigma \nsupseteq \tau \end{cases}$$

By the definition of f, however, we have that the coefficient of τ in $\partial_d(\sigma)$ is $+1$ if and only if the coefficient of σ in $f(\tau)$ is $+1$, and the coefficient of τ in $\partial_d(\sigma)$ is -1 if and only if the coefficient of σ in $f(\tau)$ is -1. Thus we have that $\langle \partial_d(\sigma), \tau \rangle_{d-1} = \langle \sigma, f(\tau) \rangle_d$, so by definition of the adjoint, $f = \partial_d^*$. $\square$

We now define the combinatorial Laplace operator and the Laplacian spectrum for a simplicial complex.

Definition 2.4. *Let K be a finite oriented complex. The* ***d^{th} combinatorial Laplacian*** *is the linear operator $\Delta_d : C_d \to C_d$ given by*

$$\Delta_d = \partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d.$$

The d^{th} *Laplacian matrix* of K, denoted $\mathcal{L}_d$, with respect to the standard bases for C_d and C_{d-1}, is the matrix representation of Δ_d, given by

$$\mathcal{L}_d = \mathcal{B}_{d+1}\mathcal{B}_{d+1}^T + \mathcal{B}_d^T \mathcal{B}_d.$$

Note that the combinatorial Laplacian is actually a set of operators, one for each d-chain in the complex. The Laplacian is a linear operator, and thus has eigenvalues. The d^{th} *Laplacian spectrum* of a finite oriented simplicial complex K, denoted $\mathrm{Spec}(\Delta_d(K))$, is the multiset of eigenvalues of the Laplacian $\Delta_d(K)$.

The Laplacian acts on an oriented simplicial complex. However, simplicial complexes are not naturally oriented. Notice that when we constructed the boundary operator, and thus the Laplacian, we gave the simplicial complex an arbitrary orientation. This might lead one to believe that the same simplicial complex could produce different Laplacian spectra for different orientations of its simplices. However, this is not the case, as is shown in the following theorem. See [?Goldberg1] for the proof.

Theorem 2.2. *Let K be a finite simplicial complex. Then Spec$(\Delta_d(K))$ is independent of the choice of orientation of the d-simplices of K.*

As a result, we can speak of the Laplacian spectrum of a simplicial complex without regard to its orientation.

An important characteristic of the Laplacian is that it is not a topological invariant. We show this with an example. Let K_1 be the 1-complex with vertices v_1, v_2, v_3 and edges $\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}$, and let K_2 be the 1-complex with vertices u_1, u_2, u_3, u_4 and edges $\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}$, and $\{u_1, u_4\}$. Clearly K_1 and K_2 are topologically equivalent; they are both homeomorphic to the circle. However, it is easily seen that

$$\mathcal{L}_1(K_1) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \qquad \mathcal{L}_1(K_2) = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix}.$$

Even the spectra of the two Laplacians are not the same, as $\text{Spec}(\Delta_1(K_1)) = \{0, 3, 3\}$ and $\text{Spec}(\Delta_1(K_2)) = \{0, 2, 2, 4\}$. It should be noted, however, that the kernel of the Laplacian is a topological invariant; it can be shown that

$$\text{Ker}(\Delta_d(K)) \cong H_d(K)$$

where H_d is the d^{th} homology group. This theorem is known as Combinatorial Hodge Theory; a proof can be found in [?YanLin1].

3 Covering Complexes

We can think of simplicial complexes as topological spaces. As we have just shown, however, the Laplacian is not a topological invariant. Thus, while two complexes might be topologically homeomorphic, they could have very different Laplacian spectra. Our goal in this section is to show that if two simplicial complexes are related by a covering map, then their Laplacian spectra are also related. We begin with the definition of a covering complex. This definition comes from [`?Rotman1`], and is similar to the definition of a topological covering space.

Definition 3.1. *Let K be a complex. A pair $(\widetilde{K}, p)$ is a* ***covering complex*** *of K if:*

1. *$p : \widetilde{K} \to K$ is a map.*

2. *$\widetilde{K}$ is connected.*

3. *For every simplex $\sigma \in K$, $p^{-1}(\sigma)$ is a union of pairwise disjoint simplices, $p^{-1}(\sigma) = \bigcup \widetilde{\sigma}_i$, with $p|_{\widetilde{\sigma}_i} : \widetilde{\sigma}_i \to \sigma$ a bijection for each i.*

Since a covering is a simplicial map, there is a chain map associated to it. Let $(\widetilde{K}, p)$ be a covering of an oriented complex K. Define the *chain covering map* $p_\# : C_d(\widetilde{K}) \to C_d(K)$ to be the chain map induced by the covering map p. Notice that by definition of p, if $\{v_0, \ldots, v_d\} \in S_d(\widetilde{K})$, then $\{p(v_0), \ldots, p(v_d)\} \in S_d(K)$ (i.e. the $p(v_i)$ are distinct), so we can define $p_\#$ by

$$p_\#(\sigma) = p_\#([v_0, \ldots, v_d]) = [p(v_0), \ldots, p(v_d)].$$

Again, if we want to specify which dimension the chain covering acts on, we will write p_d instead of $p_\#$. By Lemma **??**, we see that $p_\#$ commutes with the boundary operator ∂. Normally, a chain map will not commute with the adjoint boundary operator. We now show, however, that for the chain covering this is the case. We will use the following lemma to show that the Laplacian Δ commutes with the chain covering $p_\#$.

Lemma 3.1. *The adjoint boundary operator ∂^* commutes with the chain covering $p_\#$, i.e.*

$$p_d \circ \partial_d^* = \partial_d^* \circ p_{d-1}.$$

Proof. Since both ∂^* and $p_\#$ are linear, we only need to look at one basis element $\tau \in C_{d-1}(\widetilde{K})$, i.e. we must show

$$p_d \circ \partial_d^*(\tau) = \partial_d^* \circ p_{d-1}(\tau).$$

Using the formula for ∂^* from Theorem **??**, we see that

$$
\begin{aligned}
p_d \circ \partial_d^*(\tau) &= p_d \left(\sum_{\sigma' \in S_d^+(\widetilde{K},\tau)} \sigma' - \sum_{\sigma'' \in S_d^-(\widetilde{K},\tau)} \sigma'' \right) \\
&= \sum_{\sigma' \in S_d^+(\widetilde{K},\tau)} p_d(\sigma') - \sum_{\sigma'' \in S_d^-(\widetilde{K},\tau)} p_d(\sigma''),
\end{aligned}
$$

the last step because $p_\#$ is linear. In addition, we see that

$$
\partial_d^* \circ p_{d-1} = \sum_{\eta' \in S_d^+(K, p_{d-1}(\tau))} \eta' - \sum_{\eta'' \in S_d^-(K, p_{d-1}(\tau))} \eta''.
$$

First we show that for every $\sigma \in S_d(\widetilde{K})$ such that $\sigma \supseteq \tau$, there is exactly one $\eta \in S_d(K)$ such that $\eta \supseteq p(\tau)$ and $p(\sigma) = \eta$, and conversely (i.e. for every $\eta \in S_d(K)$ with $\eta \supseteq p(\tau)$ there is exactly one $\sigma \in S_d(\widetilde{K})$ with $\sigma \supseteq \tau$ and $p(\sigma) = \eta$).

Pick a $\sigma \in S_d(\widetilde{K})$ with $\sigma \supseteq \tau$. Since $p|_\sigma$ is a bijection, $p(\sigma)$ is unique and is in $S_d(K)$. But $\tau \subseteq \sigma$, so $p(\tau) \subseteq p(\sigma)$, so $\eta = p(\sigma)$ is the unique d-simplex satisfying the requirements.

Now pick an $\eta \in S_d(K)$ with $\eta \supseteq p(\tau)$. Look at $p^{-1}(\eta) = \bigcup \sigma_i$ with $\sigma_i \cap \sigma_j = \emptyset$ if $i \neq j$ and $p|_{\sigma_i}$ a bijection. Since $p(\tau) \subseteq \eta$, $p^{-1}(p(\tau)) \subseteq p^{-1}(\eta)$. But $\tau \subseteq p^{-1}(p(\tau))$, so $\tau \subseteq p^{-1}(\eta)$. Since τ is a simplex, it is connected, so it lies in exactly one of the σ_i in the inverse image of η. Call this unique simplex $\sigma \in S_d(\widetilde{K})$. Then $p(\sigma) = \eta$ and $\tau \subseteq \sigma$, with σ clearly unique by construction.

Now observe that since $p_\#$ simply assigns an orientation to each simplex in addition to performing the action of p, the above statement also holds for $p_\#$, i.e. if $\tau \in C_{d-1}(\widetilde{K})$ a basis element, then for every $\sigma \in C_d(\widetilde{K})$ a basis element such that $\sigma \supseteq \tau$, there is exactly one $\eta \in C_d(K)$ a basis element such that $\eta \supseteq p_{d-1}(\tau)$ and $p_d(\sigma) = \eta$, and for every $\eta \in C_d(K)$ a basis element with $\eta \supseteq p_{d-1}(\tau)$ there is exactly one $\sigma \in C_d(\widetilde{K})$ a basis element with $\sigma \supseteq \tau$ and $p_d(\sigma) = \eta$. Thus we see that for every term in $p_d \circ \partial_d^*(\tau)$, there is exactly one term in $\partial_d^* \circ p_{d-1}(\tau)$, and vice-versa; we now show that these terms are equal.

First, pick a $\sigma \in S_d^+(\widetilde{K}, \tau)$. We know that $p(\sigma) = \eta$ for some $\eta \in C_d(K)$. There are two cases, corresponding to $p_\#$ either preserving the orientation of σ or reversing it:

1. $p_d(\sigma) = \eta$
2. $p_d(\sigma) = -\eta$

First we show the case for (1). By definition, $\sigma \in S_d^+(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^+(K, p_{d-1}(\tau))$. Now η is a basis element of $C_d(K)$, so $p_d(\sigma)$ is also. Thus the coefficient of the basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is $+1$, and the coefficient of basis element $p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is $+1$, proving case (1).

Now we prove the case for (2). By definition, $\sigma \in S_d^+(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $-\eta \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^-(K, p_{d-1}(\tau))$ (this last equivalence is because switching the orientation switches the sign). Now η is a basis element of $C_d(K)$, so $-p_d(\sigma)$ is a basis element of $C_d(K)$. Thus the coefficient of basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is -1, and the coefficient of non-basis element $p_d(\sigma)$ in $p_d \circ \partial_d^*(\tau)$ is $+1$. But we want everything in terms of basis elements, so the coefficient of basis element $-p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is -1, proving case (2).

Now pick a $\sigma \in S_d^-(\widetilde{K}, \tau)$. We know that $p(\sigma) = \eta$ for some $\eta \in C_d(K)$. There are two cases, corresponding to $p_\#$ either preserving the orientation of σ or reversing it:

1. $p_d(\sigma) = \eta$
2. $p_d(\sigma) = -\eta$

First we show the case for (1). By definition, $\sigma \in S_d^-(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^-(K, p_{d-1}(\tau))$. Now η is a basis element of $C_d(K)$, so $p_d(\sigma)$ is also. Thus the coefficient of the basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is -1, and the coefficient of basis element $p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is -1, proving case (1).

Now we prove the case for (2). By definition, $\sigma \in S_d^-(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $-\eta \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^+(K, p_{d-1}(\tau))$ (this last equivalence is because switching the orientation switches the sign). Now η is a basis element of $C_d(K)$, so $-p_d(\sigma)$ is a basis element of $C_d(K)$. Thus the coefficient of basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is $+1$, and the coefficient of non-basis element $p_d(\sigma)$ in $p_d \circ \partial_d^*(\tau)$ is -1. But we want everything in terms of basis elements, so the coefficient of basis element $-p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is $+1$, proving case (2).

Thus we have that $p_d \circ \partial_d^*(\tau) = \partial_d^* \circ p_{d-1}(\tau)$, completing the proof. □

With Lemmas **??** and **??**, we can now prove that the Laplacian and the chain covering commute.

Theorem 3.1. *The Laplacian of a complex Δ commutes with the chain covering $p_\#$, i.e.*

$$\Delta_d \circ p_d = p_d \circ \Delta_d.$$

Proof. By definition, we have that $\Delta_d = \partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d$. By Lemma **??**, we know that $\partial_d \circ p_d = p_{d-1} \circ \partial_d$, and by Lemma **??**, we know that $\partial_d^* \circ p_{d-1} = p_d \circ \partial_d^*$. Thus we have that

$$\begin{aligned}
(\partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d) \circ p_d &= (\partial_{d+1} \circ \partial_{d+1}^*) \circ p_d + (\partial_d^* \circ \partial_d) \circ p_d \\
&= \partial_{d+1} \circ (\partial_{d+1}^* \circ p_d) + \partial_d^* \circ (\partial_d \circ p_d) \\
&= \partial_{d+1} \circ (p_{d+1} \circ \partial_{d+1}^*) + \partial_d^* \circ (p_{d-1} \circ \partial_d) \\
&= (\partial_{d+1} \circ p_{d+1}) \circ \partial_{d+1}^* + (\partial_d^* \circ p_{d-1}) \circ \partial_d \\
&= (p_d \circ \partial_{d+1}) \circ \partial_{d+1}^* + (p_d \circ \partial_d^*) \circ \partial_d \\
&= p_d \circ (\partial_{d+1} \circ \partial_{d+1}^*) + p_d \circ (\partial_d^* \circ \partial_d) \\
&= p_d \circ (\partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d).
\end{aligned}$$

Thus $\Delta_d \circ p_d = p_d \circ \Delta_d$, completing the proof. □

We can now prove that the spectrum of a covering complex contains the spectrum of the original complex. Recall that the d^{th} *Laplacian spectrum* of a simplicial complex K, denoted $\mathrm{Spec}(\Delta_d(K))$, is the multiset of eigenvalues of the Laplacian $\Delta_d(K)$.

Theorem 3.2. *Let $(\widetilde{K}, p)$ be a covering complex of simplicial complex K, and let $\widetilde{\Delta}_d$ and Δ_d be the Laplacian matrices of $\widetilde{K}$ and K, respectively. Then $Spec(\Delta_d) \subseteq Spec(\widetilde{\Delta}_d)$.*

Proof. By definition the map $p_\#$ is surjective. Let $\mathrm{Ker}(p_\#)$ be the kernel of $p_\#$. Then $\widetilde{\Delta}_d$ carries $\mathrm{Ker}(p_\#)$ to itself, for if $\sigma \in \mathrm{Ker}(p_\#)$, then $p_\#(\sigma) = 0$, which implies $\Delta_d(p_\#(\sigma)) = 0$, which by Theorem **??** implies that $p_\#(\widetilde{\Delta}_d(\sigma)) = 0$, which implies that $\widetilde{\Delta}_d \in \mathrm{Ker}(p_\#)$.

Choose a basis $v_1, \ldots, v_k$ for $\mathrm{Ker}(p_\#)$, and choose $u_1, \ldots, u_j \in C_d(\widetilde{K})$ so that $v_1, \ldots, v_k, u_1, \ldots, u_j$ is a basis for $C_d(\widetilde{K})$. Then $p_\#(u_1), \ldots, p_\#(u_j)$ is a basis for $C_d(K)$. (To see why, suppose not; then we can write $\sum \alpha_i p_\#(u_i) = 0$ with the α_i not all 0. But then since $p_\#$ is linear, that means $p_\#(\sum \alpha_i u_i) = 0$, which means $\sum \alpha_i u_i \in \mathrm{Ker}(p_\#)$, a contradiction, since we chose the u_i so that this would not be true.) Let M be the matrix for Δ_d with respect to the basis $p_\#(u_1), \ldots, p_\#(u_j)$, and let N be the basis for $\widetilde{\Delta}_d$ restricted to $\mathrm{Ker}(p_\#)$, with respect to $v_1, \ldots, v_k$. Then (by [?Lang1] Theorem 15.4.12) the matrix for $\widetilde{\Delta}_d$ with respect to the basis $v_1, \ldots, v_k, u_1, \ldots, u_j$ is in the block form

$$\Gamma = \begin{bmatrix} N & * \\ 0 & M \end{bmatrix}.$$

We then see that

$$\det(\Gamma - \lambda I) = \det \begin{bmatrix} N - \lambda I & * \\ 0 & M - \lambda I \end{bmatrix} = \det(N - \lambda I)\det(M - \lambda I).$$

Thus the characteristic polynomial of M divides the characteristic polynomial of Γ, so $\mathrm{Spec}(\Delta_d) \subseteq \mathrm{Spec}(\widetilde{\Delta}_d)$. □

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