Box Products, Box Exponentials, Isospectrality and Graph Covering Maps

Rachel Hunter *

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Abstract

This paper discusses the box product also known as the Cartesian product, and its relationship to covering maps and the spectrum of graphs. Further we define the box exponential and discuss its covering maps. Additionally we consider coverings and isospectrality from a geometric as well as algebraic point of view.

1 Introduction

A graph X is a set of vertices $V(X) = \{v_1, \dots v_n\}$ and the set of edge $E(G)$ such that $(v_i, v_j) \in E(X)$, is an edge connecting vertices v_i and v_j. In this paper we consider edges to be undirected (equivalently bidirected), that is, if $(v_i, v_j) \in E(X)$, then $(v_j, v_i) \in E(X)$.

The box product, frequently called the Cartesian product, is a commonly studied operation of graphs.

Definition 1. *If X and Y are graphs then the box product, $X\Box Y$, is the graph with*
$V(X\Box Y) = \{[x_i, y_j] \mid x_i \in V(X), y_j \in V(Y)\}$ *and*
$E(X\Box Y) = \{([x_i, y_j], [x_i, y_k]) \mid (y_j, y_k) \in E(Y)\} \cup \{([x_i, y_j], [x_k, y_j]) \mid (x_i, x_k) \in E(X)\}$.

In $X\Box Y$ there are $|V(Y)|$ copies of X and $|V(X)|$ copies of Y, labeled X_i, $i = 1, \dots m$ and Y_j, $j = 1, \dots n$ giving $|V(X\Box Y)| = |V(X)||V(Y)|$. Further, given a vertex $[x_i, y_i]$, then $deg([x_i, y_i]) = deg(x_i) + deg(y_i)$. In general the box product gives geometrically interesting graphs, and is used to produce several useful families of graphs. For example, grids ($P_n\Box P_m$), toroidal meshes ($C_n\Box C_m$), cylindrical meshes ($C_n\Box P_m$), and cubes ($P_2\Box \dots \Box P_2$) are all box product graphs.

*Cornell University, rah66@cornell.edu. Supported by NSF Grant at the Canisius College REU 2008, Geometry and Physics on Graphs

Many properties including crossing numbers, cutwidths, and colorings have been studied on these examples and on other product graphs.

After considering the box product, we introduce the box exponential, motivated by the ideas of set theory and category theory. In set theory, Cantor discovered that there was a natural exponential of sets, associated with the product of sets. Similarly, in graph theory, there is a categorical product of graphs, with an associated natural exponentiation of graphs [6]. Simlarly, though surprisingly, the box product also has an associated exponential, though little has been studied about it, except for a brief discussion by Doctermann, where he calls it the internal homomorphism [3]. Here we consider the box exponential and its relation to covering maps.

The idea of covering maps is a commonly studied concept in topology which has been applied to graph theory. Graph covering maps are important because they provide a way of relating the spectrum of graphs, which proves useful in considering families of large graphs. Additionally covering maps are often described in terms of groups, with the associated group being called the Galois group. This approach has been studied by Stark and others [11]. Skogman [10] gave a construction for the Galois covers of a graph which provides some of the inspiration for the treatment of coverings in this paper. However we also seek to add a geometrical perpective on covering maps. For this reason, it is natural to consider covering maps of box product graphs, as the two ideas combine to produce geometrically interesting graphs and both serve as a way of producing and relating useful families of large graphs.

As noted above covering maps relate the spectrum of graphs, since it is well known that if Y covers X then the characteristic polynomial of X divides that of Y. Likewise it is known that the spectrum of a graph $X\Box Y$ is all sums $\lambda_i+\mu_j$ where λ_i is an eigenvalue of X and μ_j is an eigenvalue of Y [1]. Also it has previously been shown algebraically that box products preserve isospectrality, however, we present an alternative geometric prove, adding further motivation for the geometric view of box products and covering maps.

2 Isospectrality

We say two graphs are isospectral when their adjacency matrices have the same characteristic polynomial. We denote this by $X \cong X'$.

It is well known that two graphs are isospectral if and only if they have the same number of closed paths of length k, for all $k \in \mathbb{N}$ (see Godsil [4] for one proof).

Theorem 1. *If $X \cong X'$ and $Y \cong Y'$ then $X\Box Y \cong X'\Box Y'$.*

Proof. Since $X \cong X'$ and $Y \cong Y'$, there is a bijection between the paths of length n in X and the paths of length n in X' and between the paths of length m in Y and the paths of length m in Y'.

Consider a path of length k in $X \Box Y$. Such a path is composed of paths contained in the X_i and Y_i subgraphs. Let γ_i denote a path in some X_i and η_i a path in some Y_j. Then a path of length k in $X \Box Y$ is of the form $\gamma_1, \eta_1, \gamma_2, \eta_2, \dots \gamma_r, \eta_r$. Since each γ_i remains in the same X_i, take α_i to be the path in X isomorphic to γ_i. Likewise take β_i to be the path in Y isomorphic to $\eta_i \in Y_i$ Then the path $\alpha_1, \dots \alpha_r$ is a path of legnth n in X and likewise $\beta_1, \dots, \beta_r$ is a path of length m in Y where $k = n + m$. Then we can find a path of lenth n in X' of the form $\alpha'_1, \dots \alpha'_r$ where α'_i is the path corresponding to α_i given by te bijection of paths in X and X'. Likewise there is a path $\beta'_1, \dots, \beta'_r$ of length m in Y'. Therefore we can create a path of length n in $X' \Box Y'$ of the form $\gamma'_1, \eta'_1, \gamma'_2, \eta'_2, \dots \gamma'_r, \eta'_r$ by inserting the paths η_i after each γ_i. Hence for a given closed path of length n in $X \Box Y$ there is a corresponding path of length n in $X' \Box Y'$. Similarly given a path in $X' \Box Y'$ we could find the corresponding path in $X \Box Y$. Hence there is a bijection between closed paths of length n in $X \Box Y$ and closed paths of length n in $X' \Box Y'$. Therefore if $X \cong X'$ and $Y \cong Y'$, then $X \Box Y \cong X' \Box Y'$. $\Box$

3 Coverings

Definition 2. *Given graphs X and Y, then $Y \xrightarrow{f} X$ is a morphism if for every $(a, b) \in E(Y)$ there is an edge $(f(a), f(b)) \in E(X)$.*

Definition 3. *A morphism $Y \xrightarrow{f} X$ is a covering if it is locally bijective. That is, if $v \in V(Y)$ is adjacent to the vertices $v_1, \dots, v_m \in V(Y)$, then $f(v) \in V(X)$ is adjacent to the vertices $f(v_1), \dots, f(v_m) \in V(X)$.*

We describe a covering as n sheeted if for every $a \in X$ there are n vertices, a^s such that $f^{-1}(a) = \{a^s \mid s = 1, \dots, n\}$. Note that the definition of a covering requires $deg(a) = deg(a^s)$.

When X is connected it follows that $|V(X)|$ divides $|V(Y)|$.

3.1 Examples

Let Q_n denote the n-dimensional cube. Then the vertices of Q_n are all ordered n-tuples and the edges connect vertices whose n-tuples differ in exactly one place. Note that Q_1 is the edge $(0, 1)$, $Q_2 = Q_1 \Box Q_1$ and in general $Q_n = Q_m \Box Q_k$ where $n = m + k$. The degree of every vertex in Q_n is n, and Q_n has 2^n vertices.

Observe that Q_3 covers K_4 by taking the top square (denoted by those tuples with a 1 in the first entry) rotating it by 180 degrees and identifying the two copies. Then the edges between $(0, 0, 0)$ and $(1, 0, 0)$ along with the edge between $(0, 1, 1)$ and $(1, 1, 1)$ give the edge between $(1, 3) \in K_4$ and the edges between

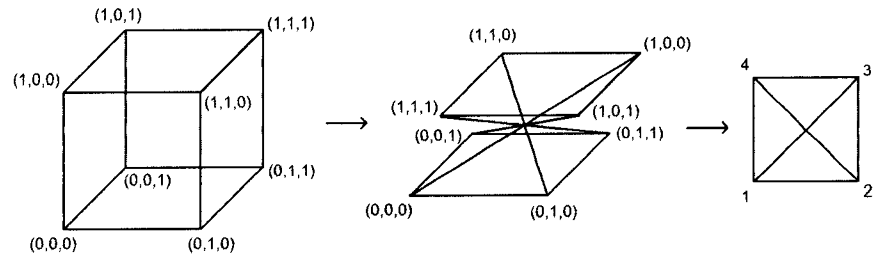

Figure 1: A covering map from Q_3 to K_4

$(0,1,0)$ and $(1,1,0)$ along with the edge between $(0,0,1)$ and $(1,0,1)$ give the edge between $(2,4) \in K_4$ (see figure 1). In general, Q_{2^n-1} covers K_{2^n}, which we can see geometrically just as for Q_3 covering K_4. We can think of producing the covering by choosing a set of Q_n subgraphs of Q_{2^n-1} rotating the Q_n subgraphs until vertex j is above vertex i, and collapsing the copies of Q_n together. This gives edges between vertex i and all vertices j not already adjacent to i in Q_n, resulting in a covering of K_{2^n} by Q_{2^n-1}. We show this formally below, but first must develop some additional ideas.

Definition 4. *A Grey Code is an ordering of all binary n-tuples such that the $i+1st$ item in the ordering differs in one place from the ith.*

The reflected binary Grey Code is created by iterating the last possible entry in an n tuple such that it is distinct from previous items and differs in only one place from the previous item. Any Grey Code gives a Hamiltonian cycle for Q_n. One can divide a Grey code into seqments of length 2^k and within each segment, switch the first 2^{k-1} numbers with the last. That is, given $(0,0),(0,1),(1,1),(1,0)$ one can reorder this as $(1,1),(1,0),(0,0),(0,1)$ or, $(0,1),(0,0),(1,0),(1,1)$ or $(1,0),(1,1),(0,1),(0,0)$. These operations give 2^n different arrangements of the n tuples into a Grey code.

Theorem 2. *$Q_{2^n-1} \xrightarrow{f} K_{2^n}$ is a covering map.*

Proof. There are 2^{2^n-1-n} disjoint copies of Q_n as subgraphs of Q_{2^n-1} since each time we takes $Q_k \Box Q_1$ we double the number of disjoint copies of Q_m, $m \leq k$. Since Q_n has a Hamiltonian cycle, we can represent Q_n with its vertices on an 2^n cycle. Label the vertices in order clockwise according to the Grey code, labeling the ith vertex in the Grey code as vertex i. Then to complete the cube, add the edges $(i, 2^k-i+1)$ for $k=2,\dots n$ and $(4k+1, 4(k+1))$ for $k=0,\dots,2^{n-2}-1$.

We can now create 2^n distinct labelings by dividing the Grey codes into segments of length 2^k and switching the first 2^{k-1} numbers within each block with the last. We now have 2^n labelings, such that each vertex is labeled exactly once by each number $1,\dots,2^n$.

We now have 2^{2^n-1-n} disjoint copies of Q_n, which we must connect together to form Q_{2^n-1}. Since in Q_n each vertex has degree n, each vertex is adjacent to n distinct vertices, each of which is labeled by a different number. Since $Q_{2^n-1} = Q_n \square Q_{2^n-n-1}$ there are 2^n disjoint copies of Q_{2^n-n-1} such that each vertex in each Q_n is also part of a Q_{2^n-n-1}. These Q_{2^n-n-1} subgraphs are labeled such that for a given Q_n, each i is adjacent in Q_{2^n-n-1} to a vertex labeled by every number in the set of $2^n - n - 1$ numbers not equal to i or adjacent to i in Q_n.

This gives Q_{2^n-1} where each vertex is assigned some number $i = 1, \ldots, 2^n$, and each i is adjacent to vertices labeled by all numbers $1, \ldots, 2^n$ not equal to i. Then defining f by $f(i) = i$, this gives a covering map $Q_{2^n} \xrightarrow{f} K_n$. □

Theorem 3. $Q_{2^n} \xrightarrow{f} K_{2^n,2^n}$ *is a covering map.*

Proof. Create and label the 2^{2^n-n-1} copies of $Q_{2^{n+1}}$ with their vertices on $C_{2^{n+1}}$ as in Theorem 2. Then each vertex is degree n, and each vertex labeled by an even number is connected only to vertices labeled with odd numbers. Then for a given even vertex i, it is contained in a Q_{2^n-n-1} subgraph. Then label the vertices adjacent to i in Q_{2^n-n-1} by the odd number not already adjacent to i and vice versa for odd i. This gives Q_{2^n} such that each vertex labeled by an even number is adjacent to a vertex labeled by every odd number and vice versa. Then defining f to be $f(i) = i$ gives a covering $Q_{2^n} \xrightarrow{f} K_{2^n,2^n}$. □

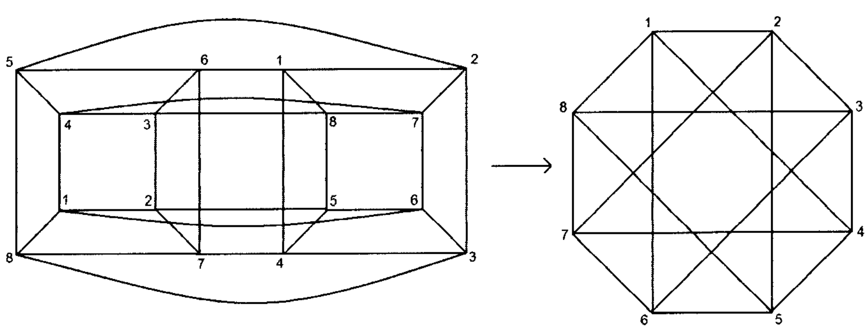

Figure 2: A covering map from Q_4 to $K_4, 4$

3.2 Box Products

Since $Y \square Z$ contains Y and Z as subgraphs, it seems natural that if $Y \xrightarrow{f} X$ and $Z \xrightarrow{g} W$ then $Y \square Z \xrightarrow{k} X \square W$. As we show in this section, this indeed holds true, but first we must define some terminology.

Definition 5. *Given a graph X, a spanning tree T of X is a tree that contains all the vertices of X and is a subgraph of X.*

Definition 6. *Given a graph X and a spanning tree T, the cycle edges of X are those edges in $X \backslash T$.*

Theorem 4. *Let X, Y, Z, W be graphs such that $Y \xrightarrow{g} X$ is a covering map and $Z \xrightarrow{h} W$ is a covering map. Then $Y \square Z \xrightarrow{k} X \square W$ is a covering map.*

Proof. Let $|V(X)| = x$, and $|V(W)| = w$ so $|V(X \square W)| = xw$. Also $|V(Y)| = mx$ and $|V(Z)| = nw$ so $|V(Z \square X)| = mnxw$. Consider a spanning tree in $X \square W$ composed of x spanning trees of the X plus $w - 1$ edges from the W_i that connect W_i to W_{i+1}. Form mn copies of this tree. Consider the cycle edges from one X_i in the first spanning tree and connect them to the first m spanning trees according to f. Repeat this for the remaining $n-1$ spanning trees to give n copies of Y. Repeat for the remaining $x-1$ copies of X_i. This gives nx copies of Y. Now consider the edges from each W_i and repeat the process adding the edges according the g. This gives mw copies of Z. The resulting graph, U, is clearly a cover of $X \square W$. Since U is constructed by taking $mx = |V(Y)|$ copies of Y and connecting them with $nw = |V(Z)|$ copies of Z, then $U = Y \square Z$. Hence $Y \square Z \xrightarrow{k} X \square W$, where k results from applying f to Y and g to Z is a covering map. □

A covering morphisms $Y \to X$ can be given in terms of a group Γ in addition to a function f.

Definition 7. *A group Γ is the Galois group of Y over X, denoted $Gal(Y/X)$, if Γ is the group of graph automorphisms of Y such that given $f : Y \to X$ then $f(\gamma(y)) = f(y),\ \gamma \in \Gamma,\ \forall y \in Y,$.*

The Galois cover of a group can be generated by making $|\Gamma|$ copies of some spanning tree $T \subset X$. Then choose some set of generators of Γ. For each generator, α, there is at least one edge $(v, u) \in X$ that is covered by the edges $(v_i, u_{\alpha \cdot i}) \in Y,\ \forall v_i = f^{-1}(v)$. The remaining edges $(w, z) \in X$ are covered by the edges $(w_i, z_{\beta \cdot i}) \in Y$ for any $\beta \in \Gamma$.

Now we get the following corollary of Theorem 2 by describing the covering morphisms in terms of groups.

Corollary 1. *Let $|Gal(Y/X) = G$ where $|G| = m$ and $|Gal(Z/W) = H$ where $|H| = n$. Then $Gal(Y \square Z / X \square W) = G \times H$.*

Proof. If the maps g and h are defined according to the groups G and H then k corresponds to $G \times H$. □

3.3 Box Exponential

For the remaining section we consider all graphs to be simple, that is, containing no loops or multi-edges.

We now define another operation on graphs, known as the box exponential, and denoted $[Z, X]$.

Consider $Z\Box K_2 \to X$. Then $Z\Box K_2$ contains two copies of Z labeled Z_1 and Z_2 each of which is mapped into X.

Definition 8. *If X and Z are graphs, define $W = [Z, X]$ to be the graph where $V(W)$ is the set of morphisms $Z \xrightarrow{f} X$ and $E(W)$ is the set of morphisms $Z\Box K_2 \xrightarrow{f} X$ connecting the two vertices correponding to each morphism, $Z_i \xrightarrow{f} X$ where $Z_i \subset Z\Box K_2$.*

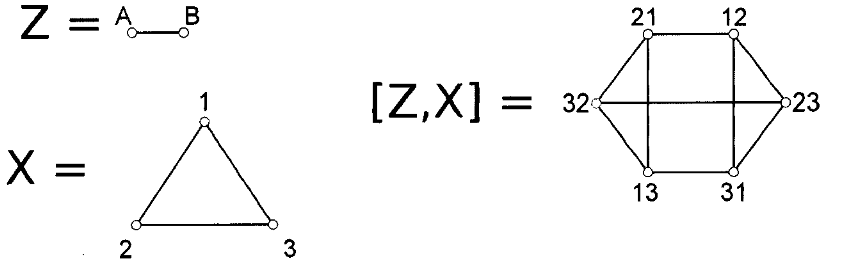

Figure 3: $[K_2, C_3]$

This parallels a situation in set theory. For sets R, S, the set theoretic exponential, denoted S^r or $[R, S]$, is the set of all maps of R into S. The set theoretical exponent is adjoint to the set theoretic product, meaning there is a natural bijection between $T \to [R, S]$ and $T \times R, S$. Dochterman shows that for graphs the box exponential is also adjoint to the box product, meaning there is a natural bijection between the maps $Z \to [Y, X]$ and $Z\Box Y, X$ for graphs X, Y, Z.

Definition 9. *Let $C_4(X)$ denote the set of injective maps of C_4 into X. That is, all maps of C_4 into X such that all the vertices of C_4 map to distinct vertices of X.*

Theorem 5. *Let $Y \xrightarrow{f} X$ be an n sheeted cover. Then $[K_2, Y] \xrightarrow{f} [K_2, X]$ is a covering map if and only if $C_4(Y) \to C_4(X)$ is n to 1.*

To prove sufficiency, assume $C_4(Y) \to C_4(X)$ is n to 1.
Since we are considering only simple graphs, then $C_{4k} \to C_4$ can be a covering map, but there does not exist an A such that $C_4 \to A$ is a covering map. Therefore if $C_4 \in Y$ then $C_4(Y) = f^{-1}(C_4(X))$ and since by assumption there is an n to 1 map between $C_4(Y)$ and $C_4(X)$, so $f(C_4(Y)) = C_4(X)$.

Now consider the set of vertices and edges of $[K_2, X]$ and $[K_2, Y]$.
$V([K_2, X]) = \vec{E}(X)$ and

Given $v \in V([K_2, X])$ then $v = (a, b) \in \vec{E}(X)$. The set of vertices adjacent to v is the set $\{(a, c_i), (b, d_i), (b, a), (a', b') \mid (a, c_i), (b, d_i) \in \vec{E}(X), C_4 \to (a, b, a', b') \in X\}$

Then $V([K_2, Y]) = \vec{E}(Y) = \{(u, v) \mid (f(u), f(v)) \in \vec{E}(X)\}$

Given $w \in V([K_2, X])$ then $w = (a^s, b^r) \in \vec{E}(Y)$ such that $f(a^i) = a, f(b^i) = b$. Then the set of vertices adjacent to w is the set

$\{(a^s, c_i^s), (b^r, d_i^r), (b^r, a^s), (a'^s, b'^r) \mid (a^s, c_i^s), (b^r, d_i^r) \in \vec{E}(X), C_4 \to (a^s, b^s, a'^r, b'^r) \in Y\}$ where a^s represents a particular $f^{-1}(a)$ and c^s is the particular $f^{-1}(c)$ adjacent to a^s and likewise for b^r, d^r.

This indeed gives a covering map since by the above $w \in V([K_2, X])$ is adjacent to vertices $w_1, \ldots w_k$ if and only if $f(w)$ is adjacent to $f(w_1), \ldots f(w_k)$.

It is necessary that $C_4(Y) \to C_4(X)$ be n to 1 since if not then there will be some vertex $w \in V([K_2, Y])$ such that $deg(w) < deg(f(w))$. We see this since for $v \in Y$, then $deg(v) = deg(f(v))$, so an edge $(u, v) \in E(Y)$ and $(f(u), f(v)) \in E(X)$ has the same number of morphisms to adjacent edges which gives the same contribution to the degree of $w = (u, v)$ and $f(w) = (f(u), f(v))$. Then if $(u, v) \notin C_4(Y)$ but $(f(u), f(v)) \in C_4(X)$, this gives an aditional edge incident to $f(w)$ but not to w, giving $deg(w) < deg(f(w))$. Hence $[K_2, Y] \xrightarrow{f} [K_2, X]$ is not a covering.

Corollary 2. *If $Gal(Y/X) = G$ then $Gal([K_2, Y]/[K_2, X]) = G$.*

4 Conclusion

While much work has been done on the box product of graphs, little consideration has been given to the box exponential. Further work, both on the covering maps of the box exponential with a general exponent, and on other properties of the graph exponential are intersting topics worth exploring. Additionally there are many open questions about graph covering maps, such as enumerating all graphs that are covered by a given graph. Perhaps some of these questions are easier to answer for box product graphs than general graphs. Further, due to the geometric nature of graphs, it is of interest to consider whether there are other algebraic ideas besides the graph specrum that are better approached from a geometric viewpoint.

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